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Quotient Varieties

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0. INTRODUCTION

0.1. Let G be a reductive algebraic group acting algebraically on an affine variety X. (We assume the base field to be $\mathbf{C}$, the field of complex numbers.) The *algebraic quotient* $X//G$ is defined by identifying two points of X whenever their orbit closures have non-empty intersection. Let $p: X \to X//G$ be the canonical map. It turns out that $X//G$ is an affine algebraic variety whose coordinate ring is the ring of G-invariant regular functions on X and that p is a morphism, called the *quotient morphism.* If all orbits are closed (e.g., if G is finite) then $X//G$ is the usual orbit space. In general the quotient $X//G$ is in a certain way the best algebraic approximation to the orbit space X/G; it became a fundamental tool for the study of many algebraic classification problems (cf. [MF, Kr1]).

In the case of a compact Lie group K acting on a topological space X much is known about the orbit space X/K (see [Br2]). The aim of this paper is to extend some of the results from the topological setting to the algebraic setting by comparing X/K with $X//G$ when G is a reductive algebraic group acting algebraically on an affine variety X and $K \subset G$ a maximal compact subgroup. Here are two of our main results:

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THEOREM A. *The canonical map* $\bar{p}: X/K \to X/\!/G$ *induces an isomorphism in cohomology.*

THEOREM B. *If* X *is contractible* (*respectively acyclic*), *then so is* $X/\!/G$.

0.2. Theorem B is related to a rich history of material from the subject of smooth actions of compact Lie groups on contractible manifolds and takes its impetus from comparing these actions with representations. The activity in this area might be called the *Comparison Problem*: Compare the invariants of smooth actions of compact Lie groups on contractible manifolds with those on Euclidean space (or its unit disk) which arise from representations of these groups. Chief among these invariants are fixed point sets, orbit spaces, and isotropy representations. P. A. Smith's work showed that fixed sets of p-groups acting on contractible spaces are mod p acyclic. Montgomery many years ago asked whether the orbit space of an action of a compact Lie group on Euclidean space was contractible. This was proved by Conner for finite groups and the affirmative statement of Montgomery's question became better known as the *Conner Conjecture*. It was proved by Oliver [Ol2], and Oliver's theorem plays an important rôle in this paper. To be specific, here are statements of related theorems. Let K be a compact Lie group and X, Y paracompact K-spaces of finite cohomological dimension and with finitely many orbit types.

0.3. THEOREM. *Let* $f: X \to Y$ *be a* K*-equivariant map which induces an isomorphism in cohomology. Then the orbit map* $f/K: X/K \to Y/K$ *also induces an isomorphism in cohomology.*

0.4. THEOREM. *If* X *is acyclic* (*or contractible*), *then so is* X/K.

Oliver's theorem on the contractibility of the orbit space is the only result (known to us) in the Comparison Problem which holds for all compact Lie groups. For example, Oliver [Ol1] has shown that the fixed set of a smooth action on a disk can be quite arbitrary provided it is consistent with Smith's theorem and the Lefschetz Fixed Point Theorem. Petrie and Randall [PR1] have shown that the set of isotropy representations of an action on a disk can be quite arbitrary subject to Oliver's conditions on fixed sets. (Since the interior of a disk is diffeomorphic to Euclidean space, these results have implications for actions on Euclidean space.)

0.5. Our long-term goal is to determine the extent to which these results on the Comparison Problem for smooth actions of compact Lie groups carry over to algebraic actions of reductive groups on smooth contractible varieties. Very little is known about such actions. The basic problem is:

LINEARITY PROBLEM. Is every algebraic action of a reductive group G on the affine space $\mathbf{C}^n$ isomorphic to a representation? (See note added in proof.)

The Linearity Problem is known to have an affirmative answer only in some very special cases, e.g., for one fix pointed actions (1.1) with a fixed point set of dimension $\leqslant 2$, or for semisimple groups G and dimension $n \leqslant 4$ (see [Kr2]). Theorem B is the only general result in this area which holds for all reductive groups. In the spirit of the Comparison Problem we note that quotient spaces of representations are contractible, so an affirmative solution to the Linearity Problem would imply Theorem B.

0.6. One must be careful with the implications of the results from the smooth setting for the algebraic setting. In [PR2] it is shown that for finite groups, real algebraic actions on varieties diffeomorphic to $\mathbf{R}^n$ correspond to smooth actions on the unit disk in $\mathbf{R}^n$. This implies, for example, that an algebraic action of a cyclic group on $\mathbf{R}^n$ has a fixed point while a smooth action need not.

0.7. We note that if the quotient variety of a representation is smooth, it is isomorphic to $\mathbf{C}^k$, where k is the dimension of the quotient. (This is well known and due to the fact that the coordinate ring of the quotient is graded and its homogeneous maximal ideal is by assumption generated by k elements [Kr1, II:4.3 Lemma 1].) This leads to the following:

PROBLEM. If G is reductive, X a G-variety isomorphic to $\mathbf{C}^n$, and $X/\!/G$ smooth of dimension k, is $X/\!/G$ isomorphic to $\mathbf{C}^k$?

This has an affirmative answer for $k \leqslant 2$ (see [Kr2]) but is unknown for $k > 2$. We can, however, say something using Theorem B (see Corollary 4.4).

COROLLARY. *Let X be a G-variety diffeomorphic to $\mathbf{C}^n$. If $X/\!/G$ is smooth of dimension k, then $(X/\!/G) \times \mathbf{C}$ is diffeomorphic to $\mathbf{C}^{k+1}$.*

We note that $X/\!/G$ need not be diffeomorphic to $\mathbf{C}^k$ (see Remark 4.5).

0.8. An essential point in our approach is to understand the topology of *one fix pointed* G-varieties. By definition an affine G-variety is called *one fix pointed* if X^G is a single point x_0 and x_0 is in the closure of every G-orbit in X, or equivalently, if X^G is a single point and every G-invariant regular function on X is a constant. It follows from Luna's Slice Theorem that a *smooth* one fix pointed affine G-variety is G-isomorphic to a representation. In particular, it is contractible. We show that this is true in general.

THEOREM C. *A one fix pointed affine G-variety X is contractible.*

0.9. This paper is organized as follows. In the first section we prove Theorem C. In the second we introduce the concept of *homologically proper* maps and show that a quotient morphism $p: X \to X//G$ and the induced map $\bar{p}: X/K \to X//G$ are homologically proper. From this Theorem A follows easily. In Section 3 we study the fundamental group of a quotient variety. A result due to Bredon shows how to calculate it in the case of a finite group acting on a simply connected space. In Section 4 we present a proof of Theorem B. This result is the main application of our methods. Further applications and some remarks are given in the last section.

0.10. *Conventions and Notations.* In what follows all topological spaces are supposed to be *locally contractible* and of *finite cohomological dimension.* It is known that algebraic varieties have these properties, where we always use the ordinary **C**-topology. In fact algebraic varieties are triangulable as locally finite simplicial complexes and admit algebraic compactifications (see [KP, Appendix]). In particular we can work with singular or Čech cohomology (or Alexander–Spanier or sheaf cohomology, see [Br1, I.7 and III]) where cohomology with integral coefficients is to be understood. For some basic material from algebraic geometry and algebraic transformation groups we refer to the literature (see [Kr1, Kr2, MF]). We mainly work with affine varieties, i.e., zero-sets of polynomials in $\mathbf{C}^n$, and with reductive groups, i.e., algebraic groups all of whose rational representations are completely reducible.

Note. After having finished a first version of this paper we became aware of the work of A. Neeman [Ne], which has some overlap with ours. Using [Ne, Corollary 2.2] (in Proposition 2.2 here) allowed us to remove the assumption that X is smooth in several places. The introduction to [Ne] notes that [Ne, Conjecture 3.1] follows from an inequality of Łojasiewicz [Ł, Proposition 1, p. 92], and that means that [Ne, Theorem 2.1] can be improved to include $a \geqslant 0$. Using this improved version of [Ne, 2.1] would shorten some of our arguments and offers a different approach to our Theorems A and B. However, a proof of the Łojasiewicz inequality already is more involved than what is done directly here.

1. One Fix Pointed Varieties Are Contractible

Let G be a reductive group. The aim of this section is to prove Theorem C. More generally we show the following:

THEOREM C′. *Let X be an affine G-variety and let Y be a G-invariant* *subvariety which contains all closed orbits of X. Then the inclusion of Y into X is a homotopy equivalence. Equivalently,* $\pi_i(X, Y) = 0$ *for all i.*

The proof requires some preliminaries which we now develop.

1.1. DEFINITION. An affine G-variety X is called *fix pointed* if every closed orbit is a fixed point. It is called *one fix pointed* if in addition there is only one fixed point.

The following two results are due to Kempf.

1.2. LEMMA (cf. [Ke2, Sect. 2]). *Let X be a fix pointed* $\mathbf{C}^*$*-variety. Then the inclusion* $X^{\mathbf{C}^*} \subsetneq X$ *is a homotopy equivalence. In fact there exists a* $\mathbf{C}^*$*-equivariant deformation retraction of X onto* $X^{\mathbf{C}^*}$.

Proof. Let

$$X_\varepsilon := \{x \in X \mid \lim_{t \to 0} t^\varepsilon x \in X^{\mathbf{C}^*}\}$$

for $\varepsilon = \pm 1$. Then $X = X_1 \cup X_{-1}$ and $X^{\mathbf{C}^*} = X_1 \cap X_{-1}$, since X is fix pointed. The morphisms $\phi_\varepsilon \colon \mathbf{C}^* \times X_\varepsilon \to X_\varepsilon$ extend to morphisms $\bar{\phi}_\varepsilon \colon \mathbf{C} \times X_\varepsilon \to X_\varepsilon$, where $(0, x)$ maps to the fixed point in the orbit closure $\overline{\mathbf{C}^* x}$. This clearly implies the claim. ∎

Let G be connected, let X be an affine G-variety, and let $Y \subset X$ be a G-stable closed subvariety. In [Ke1] Kempf has introduced the concept of an *optimal one-parameter subgroup* λ for a point $x \in X$ with respect to Y. See [Ke1, Theorem 3.4]. Such a λ exists and $\lim_{t \to 0} \lambda(t) x \in Y$ if the orbit closure $\overline{Gx}$ meets Y. There can be several optimal one-parameter subgroups for x, but they all have the same associated parabolic subgroup P_λ. Furthermore, using [Ke1, Corollary 3.5(a)] we have the following result:

1.3. PROPOSITION. *Let* $x \in X$ *and* $g \in G$. *If the one-parameter subgroup* λ *is optimal for x and gx, then* $g \in P_\lambda$.

1.4. LEMMA. *Let G be a reductive group and let X be an irreducible G-variety such that the union of the closed orbits in X is not Zariski-dense. Then there exist a one-parameter subgroup* λ *and* P_λ*-stable closed subvarieties* $Y_0 \subset X_0$ *of X with the following properties:*

(a) X_0 *is irreducible and* $X = GX_0$;

(b) $Y = GY_0$ *is a proper closed subvariety of* X;

(c) $\lim_{t \to 0} \lambda(t) x \in Y_0$ *for all* $x \in X_0$;

(d) *The canonical map*

$$G \times_{P_\lambda} (X_0, Y_0) \to (X, Y)$$

is a proper relative homeomorphism (*i.e., a proper map which is a homeomorphism* $G \times_{P_\lambda} (X_0 - Y_0) \xrightarrow{\sim} X - Y$ *of the complements*).

Proof. Let X' be the Zariski closure of the union of closed orbits in X. Fix a maximal torus T of G and denote by Λ the set of one-parameter subgroups of T. For $\lambda \in \Lambda$ put

$$X(\lambda) := \{x \in X \mid \lambda \text{ is optimal for } x \text{ (w.r.t. } X')\}.$$

It follows from Proposition 1.3 and the definition of P_λ that $X(\lambda)$ is P_λ-stable and that $X = \bigcup_{\lambda \in \Lambda} G \cdot X(\lambda)$. Since Λ is countable this implies that $X = \overline{G \cdot X(\lambda_0)}$ for some $\lambda_0 \in \Lambda$. (Here $\bar{S}$ denotes the Zariski closure of the subset S in X.) We now use λ to stand for λ_0. $\overline{X(\lambda)}$ is P_λ-stable, hence $G \cdot \overline{X(\lambda)} = X$ (cf. [Kr1, III.2.5 Satz 2]). Therefore there is an irreducible component X_0 of $\overline{X(\lambda)}$ with $GX_0 = X$. Clearly X_0 is P_λ-stable too, and the G-equivariant morphism

$$\phi \colon G \times_{P_\lambda} X_0 \to X$$

is surjective and proper. (In fact, decompose ϕ as the composition

$$G \times_{P_\lambda} X_0 \xrightarrow{\iota} G \times_{P_\lambda} X \xrightarrow{\psi} G/P_\lambda \times X \xrightarrow{\text{pr}} X,$$

where $\iota \colon G \times_{P_\lambda} X_0 \hookrightarrow G \times_{P_\lambda} X$ is the inclusion, whose image is closed, and ψ is the isomorphism given by $\psi[g, x] = ([g], gx)$ for $g \in G$, $x \in X$. Since G/P_λ is compact the claim follows.) By 1.3, ϕ is injective on the dense subset $G \times_{P_\lambda} (X_0 \cap X(\lambda))$. This shows that ϕ is birational, hence there is a G-stable Zariski-open subset $U \subset X$ such that ϕ induces an isomorphism $\phi^{-1}(U) \xrightarrow{\sim} U$. Since X properly contains X' we may assume that $U \cap X' = \varnothing$. Define $Y = X - U$. Then the inverse image $\phi^{-1}(Y)$ is G-stable and Zariski-closed, hence of the form $G \times_{P_\lambda} Y_0$ with $Y_0 = Y \cap X_0 \supset X' \cap X_0$. Note that $\lim_{t \to 0} \lambda(t)x \in X_0 \cap X'$ for all $x \in X_0$, since $X(\lambda) \cap X_0$ is dense in X_0 and $\{x \in X_0 \mid \lim_{t \to 0} \lambda(t)x \in X'\}$ is Zariski-closed. Thus $\lim_{t \to 0} \lambda(t)x \in Y_0$ for all $x \in Y_0$. This shows that ϕ is a relative homeomorphism, and verifies (c). ∎

Now we give a result whose analog in cohomology is well known. A pair (X, Y) will be called *triangulable* if it is homeomorphic to a pair (A, B) of a locally finite triangulable simplicial complex A and a subcomplex B. It follows that Y has arbitrarily small *regular neighborhoods*, i.e., closed neighborhoods N of Y such that Y is a deformation retract of N (cf. [ES, Chap. II, Sect. 9, and Exercise F]).

1.5. Proposition. *Let* (X, Y) *and* (X', Y') *be triangulable and let* $f \colon (X', Y') \to (X, Y)$ *be a proper relative homeomorphism. Suppose* $\pi_i(X', Y') = 0$ *for all* i. *Then* $\pi_i(X, Y) = 0$ *for all* i.

Proof. By [Hi, VII, Theorem 1.7] we have $\pi_i(X, Y)=0$ for all i if and only if every map of a finite complex into X is homotopic to a map into Y; moreover, if a subcomplex is mapped into Y, the homotopy may be chosen to have this property also. We must therefore show that any map $\gamma: A \to X$, where A is a finite complex, is homotopic to a map into Y. Since f is a proper relative homeomorphism, it maps each open set in X' which contains Y' onto an open set in X. Hence there are regular neighborhoods $N_0 \subset M$ of Y and N' of Y' with $f^{-1}(N_0) \subset N' \subset f^{-1}(M)$. Let $N_1 := \overline{X - N_0}$, so that $X = N_0 \cup N_1$, and let $A_i := \gamma^{-1}(N_i)$ and $B := A_0 \cap A_1$. We have $A = A_0 \cup A_1$. By the Simplicial Approximation Theorem [ES, Chap. II, Theorem 7.3] we may assume that γ is simplicial and therefore that A_0 and A_1 are subcomplexes of A. Note that $\gamma(A_1) \cap Y = \varnothing$; so if $\gamma_1 = \gamma|_{A_1}$, then $\gamma_1: (A_1, B) \to (X, N_1)$ lifts to a map $\tilde{\gamma}_1: (A_1, B) \to (X', N')$ with $f\tilde{\gamma}_1 = \gamma_1$. Since Y' is a deformation retract of N' and $\pi_i(X', Y')=0$ for all i, there is a homotopy $\tilde{h}_t: (A_1, B) \to (X', N')$ with $\tilde{h}_0 = \tilde{\gamma}_1$ and $\tilde{h}_1(A_1) \subset N'$. Let

$$h_t = f\tilde{h}_t: (A_1, B) \to (X, M_0)$$

and let $H_t: (A, A_0) \to (X, M_0)$ be a homotopy of γ which extends h_t (use the Homotopy Extension Theorem). Then $H_1(A) \subset M_0$, so γ is homotopic to a map into M_0. Since M_0 has Y as a deformation retract, γ is homotopic to a map into Y. This completes the proof. ∎

1.6. Now we are ready to prove Theorem C′. We may suppose that G is connected. In fact the connected component G^o is of finite index in G, hence the union of closed G-orbits in X is equal to the union of closed G^o-orbits.

Proof of Theorem C′. By induction, we may assume that the claim has been proved for every pair (X', Y'), where X' is a proper G-stable subvariety of X and $Y' \subset X'$ is a G-stable closed subvariety containing all closed orbits of X'. It suffices therefore to prove the claim for one pair (X, Y) with $Y \neq X$ (in case such a pair exists: otherwise there is nothing to prove). If X is irreducible this follows from previous results. In fact in the notation of Lemma 1.4 the closed subvarieties $Y_0 \subset X_0$ are both fix pointed $\mathbf{C}^*$-varieties via the one-parameter subgroup λ and $X_0^{\mathbf{C}^*} \subset Y_0$ (1.4(c)). Therefore the inclusion $Y_0 \subset X_0$ is a homotopy equivalence by Lemma 1.2. But $G \times_{P_\lambda} Y_0$ and $G \times_{P_\lambda} X_0$ are bundles over G/P_λ with fibres X_0 and Y_0, hence the inclusion $G \times_{P_\lambda} Y_0 \subsetneq G \times_{P_\lambda} X_0$ is a homotopy equivalence too. Since algebraic varieties are triangulable (0.10) we can apply Proposition 1.5 to the proper relative homeomorphism $G \times_{P_\lambda} (X_0, Y_0) \to (X, Y)$ (1.4(d)). If X is reducible, let $Y \subset X$ be the closure of the union of the closed orbits in X and X' an irreducible component of X not contained in

Y. Let X'' be the union of the other irreducible components of X. Then the claim holds for (X', Y') with $Y' = (X' \cap X'') \cup (Y \cap X')$. Since the inclusion $(X', Y') \hookrightarrow (X, Y \cup X'')$ is a proper relative homeomorphism, the claim also holds for $(X, Y \cup X'')$. ∎

In the case $G = \mathbf{C}^*$ we have the following additional information about one fix pointed G-varieties:

1.7. PROPOSITION. *Let X be an irreducible one fix pointed $\mathbf{C}^*$-variety with fixed point s and assume $X \neq \{s\}$. Then $X - \{s\}$ is homeomorphic to $C \times \mathbf{R}$, where C is compact and connected. Consequently X has one end ∞ and*

$$\pi_1(X)_s \cong \pi_1(X - \{s\}) \cong \pi_1(C) \cong \pi_1(\infty),$$

where $\pi_1(X)_s$ denotes the local fundamental group at s and $\pi_1(\infty)$ the fundamental group at the end ∞.

Proof. It is not hard to see that the action of $\mathbf{C}^*$ on $\dot{X} := X - \{s\}$ is proper and that the orbit space $\dot{X}/\mathbf{C}^*$ is compact. (In fact one can always embed X as a closed subvariety in a $\mathbf{C}^*$-representation V with weights of only one sign. Now intersect X with the unit sphere in V.) It follows from [Ab, Theorem 2.1] that $\dot{X}$ is S^1-isomorphic to $C \times \mathbf{R}$ with a compact S^1-space C and trivial action on $\mathbf{R}$, from which the claims can be easily deduced. ∎

1.8. Theorem C and Proposition 1.7 give necessary topological conditions for a variety to support a one fix pointed G-action. The final answer to the problem of characterizing such varieties, however, must take into account the algebraic structure of the variety and not just its topology. To see this, we note that there is a smooth 3-dimensional affine variety V which is diffeomorphic to but not algebraically isomorphic to $\mathbf{C}^3$. In fact we can take $V = N \times \mathbf{C}$, where N is the Ramanujam affine surface [Ra] which is contractible but not isomorphic to $\mathbf{C}^2$. This V is not isomorphic to $\mathbf{C}^3$ by [Fu], but is diffeomorphic to $\mathbf{C}^3$ because it is contractible and 1-connected at infinity. (A theorem of Stallings [St] asserts that a contractible smooth manifold of dimension $n \geqslant 5$ which is simply connected at infinity is diffeomorphic to $\mathbf{R}^n$.) Even though V is diffeomorphic to $\mathbf{C}^3$ (which obviously supports a one fix pointed $\mathbf{C}^*$-action), V does not have a one fix pointed action because one fix pointed actions on smooth varieties are linear (0.8).

2. Quotient Maps Are Homologically Proper

2.1. Definition. A continuous map $f: X \to Y$ is called *homologically proper* if for every $y \in Y$ the canonical homomorphisms

$$\varinjlim_{U \ni y} H^p(f^{-1}(U)) \xrightarrow{\sim} H^p(f^{-1}(y)),$$

where U runs over a system of neighborhoods of y, are isomorphisms for all p.

Clearly a proper map is homologically proper. Another way to express the definition is in terms of the Leray sheaf $\mathscr{H}^p(f)$ of the map f. Recall that $\mathscr{H}^p(f)$ is the sheaf associated to the presheaf $U \mapsto H^p(f^{-1}(U))$ on Y. If f is homologically proper we have canonical isomorphisms

$$\mathscr{H}^p(f)_y \xrightarrow{\sim} H^p(f^{-1}(y)) \qquad \text{for all} \quad y \in Y.$$

In particular it follows from the Leray spectral sequence that a *homologically proper map with acyclic fibres induces an isomorphism in cohomology.*

2.2. Proposition. *Let G be a reductive group, $K \subset G$ a maximal compact subgroup, and X an affine G-variety. Then the quotient morphism $p: X \to X/\!/G$ and the induced map $\bar{p}: X/K \to X/\!/G$ are both homologically proper.*

Proof. In case X is smooth it follows from Luna's Slice Theorem that every point $y \in Y := X/\!/G$ has a fundamental system of neighborhoods $\{U_i\}_{i \in I}$ such that the inclusions $p^{-1}(y) \hookrightarrow p^{-1}(U_i)$ are homotopy equivalences. Since $\bar{p}^{-1}(y) = p^{-1}(y)/K$ and $\bar{p}^{-1}(U_i) = p^{-1}(U_i)/K$, Oliver's Theorem 0.3 implies that the inclusions $\bar{p}^{-1}(y) \hookrightarrow \bar{p}^{-1}(U_i)$ induce isomorphisms in cohomology too. In general, by a result of Neeman [Ne, Corollary 2.2], there is a closed K-invariant subset $X' \subset X$ which is a deformation retract of X with a deformation along G-orbits such that $p' := p|_{X'}: X' \to Y$ is proper and surjective. In particular for every open subset $U \subset Y$ the inclusion $p^{-1}(U) \cap X' \hookrightarrow p^{-1}(U)$ is a homotopy equivalence. It follows again from Oliver's Theorem 0.3 that $\bar{p}^{-1}(U) \cap X'/K \hookrightarrow \bar{p}^{-1}(U)$ induces an isomorphism in cohomology too, since clearly $\bar{p}^{-1}(U) \cap X'/K = (p^{-1}(U) \cap X')/K$. ∎

Remark (Luna). For a smooth G-variety X the Leray sheaf $\mathscr{H}^*(p)$ of the quotient morphism $p: X \to X/\!/G$ is locally constant along the strata of $X/\!/G$ given by the slice representations.

For the proof of Theorem A we need the following lemma, which holds for every Lie group having a finite number of connected components.

2.3. LEMMA. *Let G be an algebraic group, $H \subset G$ a closed algebraic subgroup, and $K \subset G$ a maximal compact subgroup. Then the double coset space $K\backslash G/H$ is contractible.*

Proof. Replacing K by a conjugate if necessary, we can assume that $K' := K \cap H$ is a maximal compact subgroup of H. By a theorem of Mostow [Mo, 10, Theorem A] there is a closed Euclidean subspace F of G which is stable under conjugation by K' such that $G/H = K \times_{K'} F$. Hence $K\backslash G/H \cong F/K'$, which is contractible by Oliver's Theorem 0.3. ∎

Remark. We will only need that the double coset space $K\backslash G/H$ is *acyclic* for which we offer the following direct proof. The Iwasawa decomposition implies that the inclusions $K \hookrightarrow G$ and $K' \hookrightarrow H$ are homotopy equivalences. It follows from the exact homotopy sequences for the fibrations $K \to K/K'$ and $G \to G/H$ that the inclusion $K/K' \hookrightarrow G/H$ is a homotopy equivalence too. Now the claim follows from Theorem 0.3.

2.4. Now we are ready to prove our first main result.

THEOREM A. *Let G be reductive, $K \subset G$ a maximal compact subgroup, and X an affine G-variety. Then the canonical map $\bar{p}: X/K \to X//G$ induces an isomorphism in cohomology.*

Proof. We already know that $\bar{p}$ is homologically proper (Proposition 2.2). It remains to show that the fibres of $\bar{p}$ are acyclic (see 2.1). It follows from Luna's Slice Theorem that the fibres of the quotient morphism $p: X \to X//G$ are of the form $G \times_H Z$, where $H \subset G$ is a closed reductive subgroup and Z a one fix pointed affine H-variety. In particular the inclusion $G/H \hookrightarrow G \times_H Z$ is a homotopy equivalence by Theorem C. As a consequence of Theorem 0.2 the map $K\backslash G/H \hookrightarrow K\backslash G \times_H Z$ is a cohomology isomorphism, hence $K\backslash G \times_H Z$ is acyclic by Lemma 2.3. Clearly the fibres of $\bar{p}$ are of this form. ∎

Using Theorem 0.2 we get the following consequence which is the second part of Theorem B.

2.5. COROLLARY. *If X is acyclic, then so is $X//G$.*

More generally we find:

2.6. COROLLARY. *Let $f: X \to Y$ be a G-equivariant morphism of affine G-varieties. If f induces an isomorphism in cohomology, then so does $\bar{f}: X//G \to Y//G$.*

3. Fundamental Groups of Quotients

We first recall an important result from the theory of Kempf and Ness ([KN, cf. DK]).

3.1. Proposition. *Let G be a reductive group, $K \subset G$ a maximal compact subgroup, and X an affine G-variety. Then there is a real algebraic subvariety X_c with the following properties:*

(a) *X_c is contained in the set of closed orbits and meets every closed orbit of X;*

(b) *For all $x \in X_c$ we have $Gx \cap X_c = Kx$;*

(c) *For all $x \in X_c$, K_x is a maximal compact subgroup of G_x.*

Proof. If X is a representation of G, this is [PS, 2.4 Remark 2, and 2.1(4)]. In general we embed X as a G-invariant subvariety in a representation V of G and put $X_c = V_c \cap X$. ∎

Let $p_c\colon X_c \to X/\!/G$ and $\bar{p}_c\colon X_c/K \to X/\!/G$ be the obvious maps. Since the variety $X/\!/G$ carries the quotient topology from X [Lu, Théorème 2.7] we immediately get the following:

3.2. Corollary. *The canonical map $\bar{p}_c\colon X_c/K \to X/\!/G$ is a homeomorphism.*

The next two results follow from 3.2 and the fact that the map $X_c \to X_c/K$ has the path lifting property [Br2, Chap. II, Theorem 6.2 and Corollary 6.3].

3.3. Corollary. *The quotient map $p\colon X \to X/\!/G$ has the path lifting property.*

3.4. Corollary. *If X is connected and if $p\colon X \to X/\!/G$ has at least one connected fibre (e.g., if G is connected or $X^G \neq \varnothing$) then $p_*\colon \pi_1(X) \to \pi_1(X/\!/G)$ is surjective.*

The following proposition (due to Armstrong [Ar]) and its proof have been communicated to us by Bredon.

3.5. Proposition. *Let G be a finite group and X an arcwise connected and simply connected G-space. Then $\pi_1(X/G)$ is the quotient of G by the normal subgroup G_X generated by all isotropy groups G_x, $x \in X$.*

Proof. First we show that X/G_X is simply connected. Once this is established the claim follows since G/G_X acts freely on X/G_X, hence $\pi_1(X/G)$ is isomorphic to G/G_X.

In order to show that X/G_X is simply connected we may assume that $G=G_X$. Let $p: X \to X/G$ be the orbit map. Choose a base point $x_0 \in X$. Since X is simply connected, the map $\phi: G \to \pi_1(X/G)$ defined by $\phi(g) = p\gamma$ for $g \in G$, where γ is any path in X from x_0 to gx_0, is a well-defined homomorphism. The result on path lifting (Corollary 3.3) implies that ϕ is onto. We now show that each isotropy group G_x for $x \in X$ lies in the kernel of ϕ. Let $g \in G_x$, and choose a path ϑ from x_0 to x and a path γ from x_0 to gx_0. Then the composition of paths $\vartheta^{-1}(g\vartheta)\gamma$ defines a loop γ' in X which is null-homotopic. Thus $p\gamma'$ is a null-homotopic loop in X/G. But $p\gamma'$ is homotopic to $p\gamma = \phi(g)$. This argument shows that each isotropy group lies in the kernel of ϕ. Since G is generated by the isotropy subgroups of the action on X, the kernel of ϕ is G. ∎

3.6. *Remark.* Clearly G_X is generated by all subgroups $P \subset G$ of prime power order which have a non-empty fixed point set. Hence *if X is contractible then the orbit space X/G is simply connected*, since by Smith Theory every finite p-group has a fixed point.

More generally we claim the following:

3.7. COROLLARY. *Let G be a finite group and X, Y arcwise connected and simply connected G-spaces. If $f: X \to Y$ is a G-map which induces an isomorphism in cohomology then $G_X = G_Y$ and $\bar{f}_*: \pi_1(X/G) \xrightarrow{\sim} \pi_1(Y/G)$ is an isomorphism.*

Proof. Clearly $G_X \subset G_Y$. On the other hand, if p is a prime and $P \subset G$ is any subgroup of p-power order, then $f^*: H^*(Y^P, \mathbf{Z}_p) \to H^*(X^P, \mathbf{Z}_p)$ is an isomorphism by Smith Theory. (Use the mapping cone of f.) Thus X^P and Y^P are either both empty or both nonempty, so $G_X = G_Y$. The result then follows from 3.5. ∎

4. SOME QUOTIENTS ARE CONTRACTIBLE

We now want to prove Theorem B, that the quotient of a contractible affine G-variety is contractible. For the proof we need the following theorem of J. H. C. Whitehead (see [Hi, Chap. VII, Theorem 3.8]).

4.1. THEOREM. *Let X and Y be triangulable spaces and $f: X \to Y$ be a map which induces an isomorphism of fundamental groups and an isomorphism in cohomology on universal covering spaces. Then f is a homotopy equivalence.*

In particular a simply connected acyclic variety is contractible.

4.2. THEOREM B. *Let X be a contractible affine G-variety, G a reductive group. Then the quotient $X//G$ is contractible.*

Proof. Let G^o be the connected component of G. Then $X//G^o$ is acyclic by Corollary 2.5 and simply connected by Corollary 3.4. Hence $X//G^o$ is contractible. Therefore we are reduced to the case of a finite group G, where the claim follows from Theorem A and Remark 3.6. ▮

Theorem B is a special case of the following more general result. (Take Y to b a point.)

4.3. THEOREM B′. *Let X and Y be simply connected affine G-varieties and $f: X \to Y$ a G-morphism which induces an isomorphism in cohomology. Then $f: X//G \to Y//G$ is a homotopy equivalence.*

Proof. Again let G^o be the connected component of G. Both spaces $X//G^o$ and $Y//G^o$ are simply connected (3.4) and $\bar{f}_o: X//G^o \to Y//G^o$ is a cohomology equivalence (2.6). Hence we are reduced to the case of a finite group G. By Corollary 3.7, $G_X = G_Y$ and $X//G_X$ and $Y//G_Y$ are both simply connected, hence $X//G_X \to Y//G_Y$ induces an isomorphism in cohomology (2.6). So we may assume that the finite group G acts freely on X and Y, in which case the claim follows from Whitehead's Theorem 4.1. ▮

4.4. COROLLARY. *Let X be a G-variety diffeomorphic to $\mathbf{C}^n$. If $X//G$ is smooth of dimension k, then $(X//G) \times \mathbf{C}$ is diffeomorphic to $\mathbf{C}^{k+1}$.*

Proof. By Theorem B, $(X//G) \times \mathbf{C}$ is contractible. It is also simply connected at infinity, so by Stalling's theorem [St] it is diffeomorphic to $\mathbf{C}^{k+1}$. ▮

4.5. *Remark.* In general the assumption that the G-variety X is diffeomorphic to $\mathbf{C}^n$ and $X//G$ is smooth of dimension k does not imply that $X//G$ is isomorphic to $\mathbf{C}^k$. For example, take $X = N \times \mathbf{C}$ (see 1.8) and let $G = \mathbf{C}^*$ act trivially on N and by multiplication on $\mathbf{C}$. Then $X//G = N$ is not diffeomorphic to $\mathbf{C}^2$.

Here is another application of Whitehead's theorem.

4.6. PROPOSITION. *Let X, Y be triangulable spaces and $f: X \to Y$ a homologically proper map with acyclic fibres. If f induces an isomorphism $f_*: \pi_1(X) \xrightarrow{\sim} \pi_1(Y)$, then f is a homotopy equivalence.*

Proof. Let $\tilde{f}: \tilde{X} \to \tilde{Y}$ be the map induced by f on universal covering spaces. Since f induces an isomorphism on fundamental groups, for every open $U \subset \tilde{Y}$ which is mapped homeomorphically onto $\pi_Y(U) \subset Y$ under the covering map $\pi_Y: \tilde{Y} \to Y$, the inverse image $\tilde{f}^{-1}(U)$ is mapped homeomorphically onto $f^{-1}(\pi_Y(U))$ under $\pi_X: \tilde{X} \to X$. It follows that $\tilde{f}$ is homologically proper with acyclic fibres, hence a cohomology isomorphism by the Leray spectral sequence. Now the claim follows from Whitehead's theorem. ▮

4.7. With this result we get the following extension of Theorem A in the simply connected case.

THEOREM A′. *Let X be a simply connected G-variety. Then $\bar{p}: X/K \to X//G$ is a homotopy equivalence.*

Proof. We know that $\bar{p}$ is homologically proper (2.2). By Proposition 4.6, we must therefore show that p induces an isomorphism of fundamental groups. By 3.4, $X//G^o$ is simply connected, and X/K^o is simply connected too [Br2, Chap. II, Corollary 6.3]. Moreover $K/K^o = G/G^o$ acts on these spaces with orbit spaces X/K and $X//G$, respectively. Since $\bar{p}_o: X/K^o \to X//G^o$ is a homotopy equivalence by Theorem A and 4.1, Corollary 3.7 implies that $\bar{p}$ induces an isomorphism of fundamental groups. ▮

5. Some Remarks and More Applications

So far we have mainly worked with affine G-varieties and their quotients. There is an obvious generalization of this concept to any G-variety which is important for applications to moduli schemes (cf. [MF]). Let us recall that a variety X is a space with a sheaf of $\mathbf{C}$-valued functions, which has a finite open covering by affine varieties. As before G will always denote a reductive algebraic group.

5.1. DEFINITION. Let X be a G-variety. A morphism $p: X \to Y$ is called a *quotient* of X by G if p is constant on orbits and satisfies the following condition:

(Q) For every affine open subvariety $U \subset Y$ the inverse image $p^{-1}(U)$ is affine and $p: p^{-1}(U) \to U$ is a quotient (of affine varieties).

(One can show that the condition (Q) has only to be checked for an open affine covering of Y.)

Most of the properties of quotients of affine G-varieties hold in this morale general setting. In particular quotients are *unique up to isomorphism* (if they exist); *quotient morphisms $p: X \to Y$ are surjective and map G-stable closed subsets to closed subsets.*

EXAMPLES. (a) The canonical map $p: \mathbf{C}^{n+1} - \{0\} \to \mathbf{P}^n$ is a quotient by $\mathbf{C}^*$.

(b) Let $k<n$ and $M'_{k,n}$ be the set of $k \times n$ matrices of maximal rank k. Then the canonical map $p: M'_{k,n} \to \mathrm{Gr}_{k,n}$, where $\mathrm{Gr}_{k,n}$ is the Grassmannian and p sends a matrix to the span of its row vectors, is a quotient by GL_k.

(c) If $p: X \to Y$ is a quotient by G and $Y' \subset Y$ a locally closed subvariety, then $p: p^{-1}(Y') \to Y'$ is a quotient too.

Many of our results in the previous sections remain true in this more general setting, in particular Theorems A, A′, B, and B′. (In fact a quotient morphism $p: X \to Y$ has the path lifting property (3.3) and p as well as $\bar{p}: X/K \to Y$ is homologically proper (2.2). From this the theorems follow with the same proofs as in the affine case.)

As an example let us point out the following result.

5.2. Proposition. *Let $\rho: G \to GL(V)$ be a representation of a semisimple group G. Then the open subvariety $(V//G)_{\text{reg}}$ of non-singular points of the quotient $V//G$ is simply connected.*

Proof. Let $p: V \to V//G$ be the quotient morphism. We first remark that every G-stable hypersurface $H \subset V$ is defined by an invariant polynomial f, since G has no characters. Hence its image $p(H) \subset V//G$ is also defined by the function f, and therefore has codimension 1 by Krull's Hauptidealsatz [Kr1, AI.3.4]. Now $V//G$ is normal, and so the closed subvariety S of singular points of $V//G$ has codimension $\geqslant 2$ (cf. [Kr1, II.3.3 Satz 1 and AI.6.1 Satz]). This implies that the inverse image $p^{-1}(S)$ has codimension at least 2. Therefore $V' := V - p^{-1}(S) = p^{-1}((V//G)_{\text{reg}})$ is simply connected. Since $p|_{V'} \to (V//G)_{\text{reg}}$ is a quotient map the claim follows from Corollary 3.4. ∎

As an easy consequence we get the following result due to Kempf [Ke2, Theorem 2.4].

5.3. Corollary. *If $V//G$ is of dimension 2 then it is isomorphic to $\mathbf{C}^2$.*

Proof. In fact $V//G$ is a one fix pointed $\mathbf{C}^*$-variety with fixed point $s := p(0)$, and s is the only possible singularity (since $V//G$ is normal; see proof of Proposition 5.2). It follows from Propositions 1.7 and 5.2 that the local fundamental group of $V//G$ at s is trivial. By Mumford's Smoothness Criterion [Mu], this shows that $V//G$ is smooth, hence isomorphic to $\mathbf{C}^2$ (being one fix pointed: see 0.8). ∎

5.4. *Remark.* In the case of a representation of a *finite* group H on a vector space W we have in contrast to Proposition 5.2 the following result: *If W/G is singular then $(W/G)_{\text{reg}}$ is not simply connected.*

In fact let us consider the image H' of H in $GL(W)$ and denote by H'_r the (normal) subgroup of H' generated by pseudo-reflections. By Chevalley's theorem the quotient $W/H = W/H'$ is smooth if and only if $H'_r = H'$ [BGAL, Chap. V.5.5, Théorème 4]. In particular W/H'_r is isomorphic to a vector space $\bar{W}$ on which $\bar{H} := H'/H'_r$ acts linearly with quotient $\bar{W}/\bar{H} = W/H$, and $\bar{H} \subset GL(\bar{W})$ contains no reflections. But then the singular points of $\bar{W}/\bar{H}$ are exactly the images of points with non-trivial isotropy group, which form a subvariety S of codimension $\geqslant 2$. Thus the complement $\bar{W} - S$ is simply connected with a free action of $\bar{H}$, hence $\pi_1((W/H)_{\text{reg}})$ is isomorphic to $\bar{H} = H'/H'_r$.

Combining 5.4 with Proposition 5.2 (and the fact that a smooth quotient of a representation is isomorphic to $\mathbf{C}^n$) we arrive at a result due to Panjushev [Pa].

5.5. COROLLARY. *Consider a representation of a semisimple group G on a vector space V and a representation of a finite group H on a vector space W. If $V//G$ is isomorphic to W/H then both are isomorphic to $\mathbf{C}^n$.*

Our methods also allow us to generalize some of the results obtained by Kraft, Luna, and Schwarz (unpublished) concerning the following:

Conjecture. Consider an action of a reductive group G on $\mathbf{C}^n$ with a one-dimensional quotient $\mathbf{C}^n//G$. Then the action is linearizable. (See note added in proof.)

In fact we believe that linearization holds more generally for any action of a reductive group on a *smooth affine acyclic variety* with one-dimensional quotient. A first observation is the following:

5.6. LEMMA. *Let X be a normal affine G-variety with one-dimensional quotient $X//G$. If X is acyclic then $X//G$ is isomorphic to* $\mathbf{C}$.

(In fact $X//G$ is normal, hence a smooth affine curve. By Theorem B it is acyclic and therefore isomorphic to $\mathbf{C}$.)

5.7. Now suppose that X is a *smooth affine acyclic G-variety with a one-dimensional quotient.* The following results can be obtained using the methods developed by Kraft, Luna, and Schwarz and Lemma 5.6 above:

(A) The fixed point set X^G is either a point or isomorphic to $\mathbf{C}$.

(B) X is rational.

(C) X is G-isomorphic to a representation in the following cases:

(1) There is more than one fixed point;

(2) G is simple;

(3) The generic orbit is closed with trivial stabilizer.

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Note added in proof. Gerald Schwartz has recently given examples of non-linearizable actions on affine space $\mathbf{C}^n$ with one-dimensional quotient.

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