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Cohomological Dimension of Local Fields

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Introduction

Let K be a local field in the unequal characteristic case and let k be the residue-class field (char $k=p>0$). In this paper we want to show that there is a connection between the cohomological p-dimension of the Galois group G_K of K and the p-degree of the residue-class field k (i.e. the number of elements in a p-basis of k over k^p).

We first show that the cohomological q-dimension of G_K for any prime q depends only on the residue-class field k, using a decomposition theorem which tells that the canonical exact sequence of Galois groups

$$1 \to G_{K_{nr}} \to G_K \to \mathrm{Gal}(K_{nr}/K) \to 1$$

splits, where K_{nr} denotes the maximal unramified extension of K (2.1). For the cohomological p-dimension of G_K we then find (4.1)

$$\mathrm{cd}_p\, G_K \leqq 1 + \mathrm{cd}_p\, G_k + \text{p-deg}\, k$$

which is a generalisation of the well known formula for a perfect residue-class field k. For a special type of residue-class fields ("function-fields") we will have a stronger result (5.2), from which it follows that the inequality is in general not an equality. An application to C_r-questions is added in Section 6.

In Section 1 we summarize some facts on local fields, profinite groups and their cohomology.

I want to express my gratitude to J. P. Serre who filled a gap in the proof of Lemma 4.4 (concerning the h-equivariance of the map 4).

1. Notations and General Facts

In this section we want to summarize for further reference some results concerning local fields and profinite groups.

1.1. Throughout this paper under a local field K we understand a field of characteristic 0 which is complete in the topology defined by a discrete valuation v and which has a residue-class field k of characteristic

$p>0$. It is convenient to introduce the following terminology:

$\bar{K}$ = algebraic closure of K,
$\bar{k}, k_s, k^{p^{-\infty}}$ = algebraic, separable algebraic and perfect closure of k respectively,
$k^{p^\infty} = \bigcap_{n=0}^{\infty} k^{p^n}$ = maximal perfect subfield of k,
$A, \mathfrak{m}$ = ring of integers of K and its maximal ideal $\mathfrak{m}$,
G_K, G_k = Galois group of $\bar{K}/K$ and of k_s/k respectively.
p-deg k = p-degree of k/k^p = number of elements in a p-basis of k over k^p (see [11], tome I, Chap. II, § 17).

1.2. If L is an algebraic extension of a local field K we denote by v_L, A_L and k_L the (unique) extension of the discrete valuation v of K to L, the ring of integers of L and the residue-class field of L respectively. The ramification index of L/K is given by $e_{L/K} = v_L(\pi_K)$ where π_K is a prime element of K. The extension L/K is called totally ramified if $k_L = k$, and unramified if $e_{L/K} = 1$ and k_L/k separable algebraic.

If k'/k is any algebraic extension of the residue-class field of K there exists always an algebraic extension K'/K such that $e_{K'/K} = 1$ and $k_{K'} \xrightarrow{\sim} k'$, and K' is unique up to a unique isomorphism if k'/k is separable algebraic. In the case $k' = k_s$, K' is called the maximal unramified extension of K and is denoted by K_{nr}. One has the canonical isomorphism

$$\mathrm{Gal}(K_{nr}/K) \xrightarrow{\sim} \mathrm{Gal}(k_s/k) = G_k$$

and the exact sequence of Galois groups

$$1 \to G_{K_{nr}} \to G_K \to G_k \to 1.$$

1.3. Let L be any field of characteristic 0 with a discrete valuation v_L and let $\hat{L}$ denote the completion of L with respect to the topology defined by v_L. Then by the lemma of Krasner (see [9], Chap. II, § 2, Exercices 1), 2)) the compositum $\bar{L} \cdot \hat{L}$ is an algebraic closure of $\hat{L}$ and one has a canonical monomorphism

$$\varphi_L\colon\ G_{\hat{L}} \hookrightarrow G_L$$

of the Galois groups. In the special case where L is an algebraic extension of a local field K with $e_{L/K}$ finite L is algebraically closed in $\hat{L}$ and hence φ_L an isomorphism.

1.4. Let G be a profinite group (i.e. a projective limite of finite groups, see [8]) and q any prime number. By $\mathrm{cd}_q\, G$ we denote the cohomological q-dimension of G in the sense of Tate (for this definition and the following facts see [8], Chap. I). If $H \subset G$ is any closed subgroup of G the index

$[G:H]$ is a supernatural number $\prod_q q^{n_q}$ where q ranges over all primes and $0 \leqq n_q \leqq \infty$. We have the following results:

(i) *For every prime q there exists a q-Sylow subgroup G_q of G* (a subgroup with order a power of q and index prime to q) *and all q-Sylow subgroups are conjugated.*

(ii) *If $H \subset G$ is a closed subgroup of G we have for all q*

$$\mathrm{cd}_q H \leqq \mathrm{cd}_q G$$

with equality in following two cases:

a) *the index $[G:H]$ is prime to q,*

b) *H is an open subgroup and $\mathrm{cd}_q G$ is finite.*

(iii) *If $H \subset G$ is a closed normal subgroup of G there is a spectral sequence*

$$H^m(G/H, H^n(H, A)) \Rightarrow H^{m+n}(G, A)$$

for any G-module A (a G-module A is an abelian group with discrete topology and a continuous action of G compatible with the group structure).

It follows that

$$\mathrm{cd}_q G \leqq \mathrm{cd}_q G/H + \mathrm{cd}_q H$$

and if $m_0 = \mathrm{cd}_q G/H$ and $n_0 = \mathrm{cd}_q H$ are both finite there is an isomorphism

$$H^{m_0}(G/H, H^{n_0}(H, A)) \xrightarrow{\sim} H^{m_0+n_0}(G, A).$$

(iv) *If G is a pro-p-group* (i.e. the order of G is a power of p) *we have the following equivalence:*

$$\mathrm{cd}_p G \leqq n \quad \textit{if and only if} \quad H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$$

($\mathbb{Z}/p\mathbb{Z}$ with trivial action of G).

1.5. The following results on the cohomological dimension of the Galois group of a field K can be found in [8], Chap. II.

(i) *Let K be any field of characteristic 0 and let L be a finitely generated extension of K. Then for any prime q we have*

$$\mathrm{cd}_q G_L \leqq \mathrm{trdeg}_K L + \mathrm{cd}_q G_k$$

(where $\mathrm{trdeg}_K L$ denotes the transcendence degree of L over K) *with equality if $\mathrm{cd}_q G_K$ is finite.*

(ii) *If k is a field of characteristic $p > 0$ then*

$$\mathrm{cd}_p G_k \leqq 1.$$

(iii) *If K is a local field with perfect residue-class field k then for any prime q*

$$\mathrm{cd}_q\, G_K \leqq 1 + \mathrm{cd}_q\, G_k$$

with equality if $\mathrm{cd}_q\, G_k$ is finite.

2. A Decomposition Theorem

The following theorem is given in [5], Appendice n° 2.1 for the case of a perfect residue-class field. (The version there is not correct and we want to give a complete proof here.)

2.1. **Theorem.** *Let K be a local field with residue-class field k and let L be a normal extension of K containing the maximal unramified extension K_{nr}. Then there exists an extension T of K and an isomorphism*

$$T \otimes_K K_{nr} \xrightarrow{\sim} L.$$

For T one can take any field T' contained in L with $k_{T'}/k$ purely inseparable and maximal under this conditions.

$$\begin{tikzcd} T \arrow[r, dash] \arrow[d, dash] & L \arrow[d, dash] \\ K \arrow[r, dash] & K_{nr}. \end{tikzcd}$$

As an immediate consequence we have

Corollary. *The exact sequence of Galois groups*

$$1 \to G_{K_{nr}} \to G_K \to \mathrm{Gal}(K_{nr}/K) \to 1$$

splits.

The proof of the theorem will be given in two steps. In 2.2 we prove the existence of such a field T under the additional assumption that L is a finite extension of K_{nr}. In 2.3 we show that any field T contained in L with k_T/k purely inseparable and maximal under these conditions induces an isomorphism

$$T \otimes_K K_{nr} \xrightarrow{\sim} L.$$

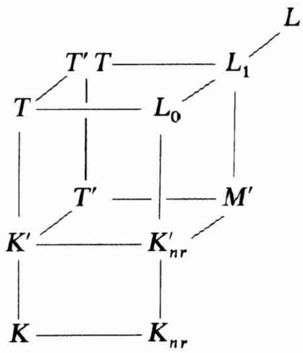
2.2. Let L/K_{nr} be a finite extension such that L/K is normal. Then the Galois group $\mathrm{Gal}(L/K_{nr})$ is solvable (see [11], Ch. II, § 10. Theorem 25 and use the fact that the residue-class field of K_{nr} is separably closed together with 1.3) and by induction on $[L:K_{nr}]$ one easily reduces to the case of an abelian extension L/K_{nr} with Galois group isomorphic to $(\mathbb{Z}/q\,\mathbb{Z})^n$ for some prime q. If q is different from p, the extension L/K_{nr} is totally ramified and therefore cyclic of order q and of the form

$$L = K_{nr}[x] \quad \text{with} \quad x^q = \pi$$

where π is a prime element of K (for any positive integer d prime to p there exists exactly one extension of K_{nr} of degree d and this extension is cyclic; compare [5], Appendice, Prop. 1.6). Then the field $T = K[x]$ has the required property. In the case $q = p$ the exact sequence

$$1 \longrightarrow \mathrm{Gal}(L/K_{nr}) \longrightarrow \mathrm{Gal}(L/K) \xrightarrow{\varphi} \mathrm{Gal}(K_{nr}/K) \longrightarrow 1$$

splits because $\mathrm{Gal}(K_{nr}/K) \xrightarrow{\sim} G_k$ has cohomological p-dimension $\leqq 1$ by 1.5.(i) (see [8], Chap. I, Prop. 16). If σ is a section of φ the field T of fixed elements of L under $\sigma(\mathrm{Gal}(K_{nr}/K))$ has the required property.



2.3. In the general case let T be a maximal extension contained in L such that k_T/k is purely inseparable and define $L_0 = T \cdot K_{nr}$ to be the compositum of T and K_{nr} in L. If L_0 is strictly contained in L there exists a non trivial finite extension L_1/L_0 contained in L such that L_1/T is normal. In this situation we can find a finite extension K'/K contained in T and a finite extension M' of $K'_{nr} = K' \cdot K_{nr}$ contained in L_1 with the following properties: M'/K' is normal, M' and L_0 are linearly disjoint over K'_{nr} and $L_1 = L_0 \cdot M'$ is the compositum. By 2.2 there exists a finite extension T' of K' contained in M' such that $T' \otimes_{K'} K'_{nr} \xrightarrow{\sim} M'$ is an isomorphism. It follows that $(T \otimes_{K'} T') \otimes_{K'} K'_{nr} \xrightarrow{\sim} L_1$ is an isomorphism, hence $T \otimes_{K'} T' \xrightarrow{\sim} T \cdot T'$ and K'_{nr} are linearly disjoint over K'. The residue-class field $k_{T \cdot T'}$ of $T \cdot T'$ is therefore a purely inseparable extension of k and this contradicts the maximality of T.

3. Cohomological Dimension and Residue-Class Field

In this section we want to show that the cohomological dimension of a local field depends only on the residue-class field. For a perfect residue-class field this is proved by Ax in [1] and it woulden't be difficult to deduce from this result that part of Theorem 3.1 which concerns the cohomological q-dimension for $q \neq p$.

3.1. **Theorem.** *Let K be a local field with residue-class field k. Then for any prime q the cohomological q-dimension $\mathrm{cd}_q\, G_K$ of the Galois group of K depends only on the residue-class field k and for $q \neq p =$ characteristic of k we have*

$$\mathrm{cd}_q\, G_K = 1 + \mathrm{cd}_q\, G_k.$$

The proof will be given in 3.2, 3.3 and 3.4.

3.2. We first consider the case $q \neq p$. By Theorem 2.1 there exists an embedding $G_k \hookrightarrow G_K$, hence $\mathrm{cd}_q G_k \leqq \mathrm{cd}_q G_K$ which proves our theorem in the case $\mathrm{cd}_q\, G_k = \infty$. We may therefore assume that $\mathrm{cd}_q G_k$ is finite. Let K' be any algebraic extension of K such that $e_{K'/K} = 1$ and $k_{K'} = k^{p^{-\infty}}$ the perfect closure of k (1.2). By 1.3 and 1.5.(iii) it follows

$$\mathrm{cd}_q\, G_{K'} = \mathrm{cd}_q\, G_{\hat{K}'} = 1 + \mathrm{cd}_q\, G_k.$$

By construction the index $[G_K : G_{K'}]$ is a power of p, hence

$$\mathrm{cd}_q\, G_K = \mathrm{cd}_q\, G_{K'}$$

which proves our theorem in the case $q \neq p$.

3.3. In the case $q = p$ it is enough to show that for any finite extension L/K with $k_L = k$ we have $\mathrm{cd}_p\, G_L = \mathrm{cd}_p\, G_K$. This is clear if $\mathrm{cd}_p\, G_K$ is finite. For the general case we use the following result of Serre [10]:

Lemma. *Let G be a profinite group without elements of order p. Then for any open subgroup $U \subset G$ we have*

$$\mathrm{cd}_p\, U = \mathrm{cd}_p\, G.$$

By Proposition 3.4 below the assumption of the lemma is fullfilled if G is the Galois group of a local field with residue-class field k of characteristic p. This completes the proof of Theorem 3.1.

3.4. **Proposition.** *Let K be a local field with residue-class field k of characteristic $p > 0$. Then the Galois group G_K of K contains no element of order p.*

We first construct a special algebraic extension K_∞ of K with ramification index $e_{K_\infty/K} = 1$ and residue-class field $k_{K_\infty} = k^{p^{-\infty}}$ the perfect closure of k. Take any p-basis $\{\beta_i\}_{i \in I}$ of k and representatives $\{b_i\}_{i \in I}$ in K. For any $i \in I$ let $(x_{i\nu})_{\nu=0}^{\infty}$ be a coherent system of (p^ν)-th roots of b_i, i.e. $x_{i\nu} \in \bar{K}$ with

$$x_{i0} = b_i \quad \text{and} \quad x_{i\nu+1}^p = x_{i\nu} \qquad \text{for } \nu = 0, 1, 2, \ldots.$$

Define

$$K_\nu = K[\{x_{i\nu}\}_{i\in I}].$$

We obtain a tower of fields

$$K = K_0 \subset K_1 \subset K_2 \subset \cdots$$

with

$$e_{K_i/K} = 1 \quad \text{and} \quad k_{K_i} = k^{p^{-i}}$$

and the union

$$K_\infty = \bigcup_{\nu=0}^{\infty} K_\nu$$

has the required properties.

Furthermore let T be the extension of K obtained by adjoining for all n the (p^n)-th roots of unity to K and let T_∞ denote the compositum

$$\begin{tikzcd}
 & \bar{K} \arrow[d, no head] \\
T \arrow[r, no head] & T_\infty \\
K \arrow[u, no head] \arrow[r, no head] & K_\infty \arrow[u, no head]
\end{tikzcd}$$

of T and K_∞ in $\bar{K}$: $T_\infty = T \cdot K_\infty$. By construction of K_∞, T_∞/T is an abelian extension with Galois group

$$\operatorname{Gal}(T_\infty/T) \xrightarrow{\sim} \hat{\mathbb{Z}}_p^I \tag{$*$}$$

and also T/K is abelian with Galois group

$$\operatorname{Gal}(T/K) \xrightarrow{\sim} F \oplus \hat{\mathbb{Z}}_p \tag{$**$}$$

where F is a finite cyclic group of order prime to p (a subgroup of $\mathbb{Z}/(p-1)\,\mathbb{Z}$). Now consider the following tower of fields

$$K \subset T \subset T_\infty \subset \bar{K}$$

and the corresponding normal series of Galois groups

$$G_K \supset G_T \supset G_{T_\infty} \supset (1). \tag{$***$}$$

In order to prove the theorem it is sufficient to show that no factor group of this normal series contains an element of order p. This follows from $(*)$ and $(**)$ for the factor groups G_T/G_{T_∞} and G_K/G_T respectively. Furthermore G_{T_∞} is a closed subgroup of $G_{K_\infty} \xrightarrow{\sim} G_{\hat{K}_\infty}$ (1.3) and by 1.5.

$$\mathrm{cd}_p\, G_{\hat{K}_\infty} = 1 + \mathrm{cd}_p\, G_{k^{p^{-\infty}}} \leqq 2,$$

hence G_{K_∞} contains no element of order p.

3.5. *Remark.* It follows from the proof above that we have the inequality

$$\mathrm{cd}_p\, G_K \leqq 2 + \mathrm{cd}_p\, G_k + \text{p-deg}\, k$$

(see 1.4.(iii) and use the fact that $\mathrm{cd}_p\, \hat{\mathbb{Z}}_p = 1$). We will show in the next section that the right hand side of the inequality can be replaced by

$$1 + \mathrm{cd}_p\, G_k + \text{p-deg}\, k.$$

3.6. *Remark.* One can show that the fields T and K_∞ constructed in 3.4 are always linearly disjoint over K. This is clear if K contains a p-th root of unity and if the absolute ramification index e_K of K is prime to p, because in this case T/K is totally ramified. We will need later the linear disjointness of T and K_∞ only in this special situation (see Lemma 4.4).

4. An Upper Bound for the Cohomological *p*-Dimension

4.1. **Theorem.** *Let K be a local field with residue-class field k of characteristic $p > 0$. Then we have the following inequality for the cohomological p-dimension of the Galois group G_K of K:*

$$\mathrm{cd}_p\, G_K \leqq 1 + \mathrm{cd}_p\, G_k + \text{p-deg}\, k.$$

After the reduction to the case of a separably closed residue-class field in 4.3 the proof of the Theorem is given in 4.5 using Lemma 4.4.

4.2. *Remark.* The inequality in the Theorem 4.1 is in general not an equality, as we will show in Section 5 (see Corollary of Theorem 5.2 and Remark 5.3). *But it seems probable that for a separably closed residue-class field k one always has:* $\mathrm{cd}_p\, G_K = 1 + \text{p-deg}\, k$. *This is true for* p-deg $k \leqq 1$, *because one has more generally* $\mathrm{cd}_p\, G_K \geqq 2$ *if the residue-class field is not perfect.* (In fact, the norm map of a totally ramified normal extension of degree p of K is not surjective, because the induced map on the residue-class field is not surjective.)

4.3. Let K_{nr} be the maximal unramified extension of K. From 1.4.(iii) and 1.3 we get

$$\mathrm{cd}_p\, G_K \leqq \mathrm{cd}_p\, G_k + \mathrm{cd}_p\, G_{K_{nr}} = \mathrm{cd}_p\, G_k + \mathrm{cd}_p\, G_{\hat{K}_{nr}}.$$

We may therefore assume that the residue-class field k is separably closed and that the p-degree of k is finite. In this case we already know (see Remark 3.5) that

$$\mathrm{cd}_p\, G_K \leqq 2 + \text{p-deg}\, k$$

and the following Lemma will be the crucial point of the proof of Theorem 4.1.

4.4. **Lemma.** *If in addition to the assumptions* 4.3 *the absolute ramification index e_K of K is prime to p and K contains the group μ_p of p-th roots of unity, then*

$$H^{d+2}(G_K, \mu_p) = 0 \quad \textit{with} \quad d = \text{p-deg}\, k.$$

Consider the extension fields K_∞ and T and the compositum $T_\infty = T \cdot K_\infty$ constructed in 3.4. The situation is pictured in the diagram (T and K_∞ are linearly disjoint over K, see Remark 3.6)

$$\begin{tikzcd}
 & \overline{K} \arrow[d, dash, "H"] \\
T \arrow[r, dash, "g"] \arrow[d, dash, "h"'] & T_\infty \arrow[d, dash] \\
K \arrow[r, dash] \arrow[ur, dash, "\bar{G}"] & K_\infty
\end{tikzcd}$$

where we use the following notations:

$$H = \mathrm{Gal}(\overline{K}/T_\infty)$$
$$g = \mathrm{Gal}(T_\infty/T) \xrightarrow{\sim} \hat{\mathbb{Z}}_p^d$$
$$h = \mathrm{Gal}(T/K) \xrightarrow{\sim} \hat{\mathbb{Z}}_p$$
$$\bar{G} = \mathrm{Gal}(T_\infty/K) \xrightarrow{\sim} G_K/H.$$

The exact sequence of G_K-modules

$$1 \longrightarrow \mu_p \longrightarrow \overline{K}^* \xrightarrow{?p} \overline{K}^* \longrightarrow 1$$

yields an isomorphism

$$H^{d+2}(G_K, \mu_p) \xrightarrow{\sim} H^{d+1}(\bar{G}, T_\infty^*/T_\infty^{*p})$$

(see 1.4.(iii) and use the following facts: $\mathrm{cd}_p \bar{G} \leqq d+1$, $\mathrm{cd}_p H \leqq 1$ and $T_\infty^*/T_\infty^{*p} \xrightarrow{\sim} H^1(H, \mu_p)$ is an isomorphism of $\bar{G}$-modules). A similar argument applied to the normal subgroup $g \subset \bar{G}$ yields an isomorphism

$$H^{d+1}(\bar{G}, T_\infty^*/T_\infty^{*p}) \xrightarrow{\sim} H^1(h, H^d(g, T_\infty^*/T_\infty^{*p})).$$

If z is a profinite group isomorphic to $\hat{\mathbb{Z}}_p$ and if A is any z-torsionmodule, it follows from the cohomology of $\hat{\mathbb{Z}}_p$ (see [9], Ch. XIII, §1) that the canonical epimorphism $A \to A_z$ to the largest quotient of A with trivial z-operation induces an isomorphism

$$H^1(z, A) \xrightarrow{\sim} H^1(z, A_z) = \operatorname{Hom}(z, A_z)$$

and hence a canonical epimorphism

$$\operatorname{Hom}(z, A) \longrightarrow H^1(z, A).$$

In the situation above, we can find a decomposition of g in h-invariant cyclic factors:

$$g = g_1 \times g_2 \times \cdots \times g_d.$$

We have therefore a canonical epimorphism

$$\operatorname{Hom}(g_1, T_\infty^*/T_\infty^{*p}) \to H^1(g_1, T_\infty^*/T_\infty^{*p})$$

which is a morphism of g- and h-modules. Since K contains μ_p, h operates trivially on g/g^p and the operation of h on

$$\operatorname{Hom}(g_1, T_\infty^*/T_\infty^{*p}) \xleftarrow{\sim} \operatorname{Hom}(g_1/g_1^p, T_\infty^*/T_\infty^{*p})$$

is given by the operation of h on T_∞^*/T_∞^{*p}. Hence any generator σ_1 of g_1 induces an isomorphism

$$\operatorname{Hom}(g_1, T_\infty^*/T_\infty^{*p}) \xrightarrow{\sim} T_\infty^*/T_\infty^{*p}$$

of g- and h-modules. This yields an epimorphism

$$T_\infty^*/T_\infty^{*p} \to H^1(g_1, T_\infty^*/T_\infty^{*p})$$

of g- and h-modules and by induction we find an epimorphism of h-modules

$$\varphi\colon T_\infty^*/T_\infty^{*p} \to H^d(g, T_\infty^*/T_\infty^{*p})$$

depending on generators σ_i of g_i, $i = 1, 2, \ldots, d$.

Now φ induces an epimorphism

$$H^1(\varphi)\colon H^1(h, T_\infty^*/T_\infty^{*p}) \to H^1(h, H^d(g, T_\infty^*/T_\infty^{*p}))$$

and hence an epimorphism

$$H^1(h, T_\infty^*/T_\infty^{*p}) \to H^{d+2}(G_K, \mu_p). \tag{$*$}$$

If we apply the arguments used at the beginning of this proof to the field K_∞, we get an isomorphism

$$H^2(G_{K_\infty}, \mu_p) \xrightarrow{\sim} H^1(h, T_\infty^*/T_\infty^{*p})$$

because $h \xrightarrow{\sim} \mathrm{Gal}(T_\infty/K_\infty)$ is a canonical isomorphism. But

$$H^2(G_{K_\infty}, \mu_p) = 0$$

because $\mathrm{cd}_p\, G_{K_\infty} \leqq 1$ (see 1.5.(iii); $G_{\hat{K}_\infty} \xrightarrow{\sim} G_{K_\infty}$ by 1.3), hence by $(*)$

$$H^{d+2}(G_K, \mu_p) = 0.$$

4.5. We now complete the proof of Theorem 4.1. By Theorem 3.1 we may without loss of generality assume that K contains the p-th roots of unity and that the absolute ramification index e_K is prime to p. Let $G_p \subset G_K$ be a p-Sylow subgroup and let K_p be the corresponding field of fixed elements. Then $K_p = \bigcup_{i \in I} L_i$ where L_i runs over all finite extensions of K contained in K_p. Clearly all L_i satisfy the assumptions of Lemma 4.4 and we get

$$H^{d+2}(G_p, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^{d+2}(G_p, \mu_p) \xrightarrow{\sim} \varinjlim_I H^{d+2}(G_{L_i}, \mu_p) = 0.$$

Hence

$$\mathrm{cd}_p\, G_K = \mathrm{cd}_p\, G_p \leqq d+1$$

by 1.4.(iv).

5. Local Fields with Separably Generated Residue-Class Field

In this section we want to refine the result of Theorem 4.1 for a special class of residue-class fields and also give some information on the structure of the Galois group.

5.1. **Definition.** *A field k of characteristic $p > 0$ is called separably generated if k is separably generated over its maximal perfect subfield k^{p^∞}, i.e. if there exists a transcendence basis $\{X_i\}_{i \in I}$ of k/k^{p^∞} such that k is separable algebraic over $k^{p^\infty}(\{X_i\}_{i \in I})$.*

This definition can also be expressed in the following way: *Every p-basis of k is a transcencence basis of k/k^{p^∞}.* (In fact: p-independent elements of k are always algebraically independent over k^{p^∞}.) In particular, *k is separably generated if*

$$p\text{-deg}\, k = \mathrm{trdeg}\, k/k^{p^\infty} < \infty.$$

5.2. **Theorem.** *Let K be a local field with separably generated residue-class field k. Then we have for the cohomological p-dimension of the*

Galois group G_K of K:

$$1+\text{p-deg}\, k \leqq \mathrm{cd}_p\, G_K \leqq 1+\text{p-deg}\, k+\mathrm{cd}_p\, G_{k^{p^\infty}}$$

with

$$\mathrm{cd}_p\, G_K = 1+\text{p-deg}\, k+\mathrm{cd}_p\, G_{k^{p^\infty}}$$

if k is finitely generated over k^{p^∞}.

As an immediate consequence we have

Corollary. *If in addition to the assumption of the theorem the field k^{p^∞} is algebraically closed, then*

$$\mathrm{cd}_p\, G_K = 1+\text{p-deg}\, k.$$

5.3. *Remark.* The inequality $\mathrm{cd}_p\, G_{k^{p^{-\infty}}} \leqq \mathrm{cd}_p\, G_k$ holds for any field of characteristic $p>0$. It follows from this and Theorem 5.2. that the inequality of Theorem 4.1 is in general not an equality.

5.4. *Remark.* If K is a local field with separably generated residue-class field k of infinite p-degree, Theorem 5.2 tells that $\mathrm{cd}_p\, G_K$ is also infinite. But this can already be deduced from that part of Theorem 5.2. concerning only separably generated residue-class fields of finite p-degree. (In fact for any integer $n>0$ there exists an algebraic extension k'/k such that k' is separably generated of p-degree n so that we may apply Theorem 5.2 together with 1.1 and 1.4.(ii).) It is therefore enough to prove the theorem for separably generated residue-class fields of finite p-degree.

5.5. By the corollary of Theorem 2.1 (decomposition theorem) we know that the canonical exact sequence of Galois groups

$$1 \to G_{K_{nr}} \to G_K \to \mathrm{Gal}(K_{nr}/K) \to 1$$

splits. The following theorem gives some information on the structure of the Galois group $G_{K_{nr}}$, i.e. of the Galois group of a local field with separably closed residue-class field:

5.6. **Theorem.** *Let K be a local field with separably closed residue-class field k. Then the p-Sylow-subgroup $G_{(p)}$ of G_K is a normal subgroup and we have a split exact sequence:*

$$1 \to G_{(p)} \to G_K \to \prod_{q \neq p} \hat{\mathbb{Z}}_q \to 1.$$

Furthermore if k is separably generated with finite p-deg $k=d<\infty$ *there is a normal series of Galois groups contained in G_K of length $d+1$:*

$$G_{(p)} = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_d \supset G_{d+1} = (1)$$

such that the factor groups G_i/G_{i+1} are free pro-p-groups.

The first part of this theorem is well known: A local field with separably closed residue-class field k of characteristic $p>0$ has exactly one extension of degree n if n is prime to p and this extension is cyclic (compare [5], Prop. 1.6; this has already been used in 2.2).

The proof of the Theorems 5.2 and 5.6 will be given in the rest of this section. We always assume that the residue-class field has finite p-degree since this is allowed by Remark 5.4.

5.7. The proof of the theorems will use some induction argument on the p-degree of k. We need the following construction:

Let K be any local field with residue-class field k, ring of integers A and maximal ideal $\mathfrak{m}$, and let X be a transcendental element over K. Denote by A_X the localisation of the polynomial ring $A[X]$ at the prime ideal generated by $\mathfrak{m}$:

$$A_X = A[X]_{(\mathfrak{m}\cdot A[X])}$$

and let A' be the completion of A_X in the $\mathfrak{m}$-adic topology. A_X is a discrete valuation ring with quotient field $K(X)$ and residue-class field $k(X)$ and the quotient field K' of A' is a local field with residue-class field $k(X)$. We have the following diagram

$$\begin{tikzcd}
 & \overline{K(X)} \arrow[dash, d] \arrow[dash, dr] & \\
 & & K'\cdot\overline{K(X)} \arrow[dash, d, "g"] \\
\bar{K} \arrow[dash, r] \arrow[dash, d] & \bar{K}(X) \arrow[dash, r] \arrow[dash, d] & K'\cdot\bar{K}(X) \arrow[dash, d] \\
K \arrow[dash, r] & K(X) \arrow[dash, r] & K'
\end{tikzcd}$$

Lemma. (a) *$\bar{K}(X)$ and K' are linearly disjoint over $K(X)$.*

(b) *The compositum $K'\cdot\overline{K(X)}$ is an algebraic closure of K'.*

(c) *There is an exact sequence of Galois groups*

$$1 \to g \to G_{K'} \to G_K \to 1$$

where g is a closed subgroup of a free profinite group (in particular: $\operatorname{cd} g \leqq 1$).

The assertion (a) follows from the fact that the maximal ideal of A_X is not decomposed in any finite extension of $K(X)$ contained in $\bar{K}(X)$ and (b) follows from the lemma of Krasner (see 1.3). In order to prove (c) observe that $g = \mathrm{Gal}(K'\cdot\overline{K(X)}/K'\cdot\bar{K}(X))$ is canonically embedded in $G_{\bar{K}(X)}$ which is a free profinite group (see [3]).

5.8. With the notations of 5.7 let L be an algebraic extension of K' with finite ramification index $e_{L/K'}$ such that L and $\bar{K}$ are linearly disjoint over K:

$$
\begin{tikzcd}
 & \overline{K(X)} \arrow[dr, dash] \arrow[dd, dash] & & \\
 & & \bar{K}' \arrow[r, equal] \arrow[d, dash] & \bar{L} \arrow[d, dash, "H"] \\
\bar{K} \arrow[r, dash] \arrow[d, dash] & \bar{K}(X) \arrow[r, dash] \arrow[d, dash] & K' \cdot \bar{K}(X) \arrow[r, dash] \arrow[d, dash] & L \cdot \bar{K}(X) \arrow[d, dash] \\
K \arrow[r, dash] & K(X) \arrow[r, dash] & K' \arrow[r, dash] & L
\end{tikzcd}
$$

Lemma. *If in addition to the assumptions and notations above we have* $\mathrm{cd}_p\, G_K < \infty$ *then*

$$\mathrm{cd}_p\, G_L \leqq 1 + \mathrm{cd}_p\, G_K$$

with equality in the following two cases

(a) L/K' *is finite.*

(b) *The residue-class field k of K is separably closed and $k_L/k_{K'}$ is separable algebraic.*

Furthermore if the residue-class field k_L of L is separably closed, then the Galois group

$$H = \mathrm{Gal}(\bar{L}/L \cdot \bar{K}(X))$$

is a free pro-p-group.

Let $G_{(p)}$ be a p-Sylow subgroup of G_K and let K_p be the corresponding field of fixed elements. Then we have the following diagram:

$$
\begin{tikzcd}
 & \overline{K(X)} \arrow[dr, dash] \arrow[dd, dash] & \\
 & & \bar{L} \arrow[d, dash] \\
\bar{K} \arrow[r, dash] \arrow[d, dash, "G_{(p)}"'] & \bar{K}(X) \arrow[r, dash] \arrow[d, dash] & M \arrow[d, dash] \\
K_p \arrow[r, dash] \arrow[d, dash] & K_p(X) \arrow[r, dash] \arrow[d, dash] & L_p \arrow[d, dash] \\
K \arrow[r, dash] & K(X) \arrow[r, dash] & L
\end{tikzcd}
$$

with $L_p = K_p \cdot L$ and $M = \bar{K} \cdot L$. We already know from Lemma 5.7.(c) that $\mathrm{cd}_p\, G_L \leqq 1 + \mathrm{cd}_p\, G_K$ and we want to show that in the cases (a) and (b) we have $H^{d+1}(G_{L_p}, \mathbb{Z}/p\,\mathbb{Z}) \neq 0$ for $d = \mathrm{cd}_p\, G_K$, from which the first part of the lemma follows. As in the proof of Lemma 4.4 we have an isomorphism

$$H^{d+1}(G_{L_p}, \mathbb{Z}/p\,\mathbb{Z}) \xrightarrow{\sim} H^d(G_{(p)}, M^*/M^{*p})$$

where $G_{(p)}$ acts on M^* by the canonical isomorphism $G_{(p)} \xrightarrow{\sim} \mathrm{Gal}(M/L_p)$. Of course, M is a Henselian field with $e_M = \infty$ and residue-class field k_M containing $\bar{k}(X)$ and there is an epimorphism

$$\varphi\colon M^*/M^{*p} \to k_M^*/k_M^{*p}$$

induced by the epimorphism $U_M \to k_M^*$ (U_M = units of M) and the isomorphism $U_M/U_M^p \xrightarrow{\sim} M^*/M^{*p}$ (because the value group of M is p-divisible). φ is an epimorphism of $G_{(p)}$-modules and we want to show that there is a quotient of the $G_{(p)}$-module k_M^*/k_M^{*p} isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with trivial action. This is clear in case (b) because in this case k_M is not perfect ($k_M/\bar{k}(X)$ is separable algebraic, hence p-deg $k_M = 1$) and the action of $G_{(p)}$ on $\bar{k}$ and hence also on k_M is trivial. In case (a) we may assume without loss of generality that $L = K'$, hence $k_M = \bar{k}(X)$, and then the place $X = 0$ will induce the required quotient. In both cases this yields an epimorphism

$$H^d(G_{(p)}, M^*/M^{*p}) \to H^d(G_{(p)}, \mathbb{Z}/p\,\mathbb{Z})$$

and proves our assertion, because $H^d(G_{(p)}, \mathbb{Z}/p\,\mathbb{Z}) \neq 0$ by assumption. If now k_L is separably closed, the same is true for k by the linear disjointness of L and $\bar{K}$ over K, and there exists only one extension of L of degree n if n is prime to p, and this extension is contained in L_p (because it comes from an extension of K contained in K_p). The Galois group $H = \mathrm{Gal}(\bar{L}/M)$ is therefore a pro-p-group with $\mathrm{cd}_p H \leqq 1$, hence a free pro-p-group.

5.9. Now let K be a local field with separably generated residue-class field k of p-deg $k < \infty$. Then k is a separable algebraic extension of the field $k_0(X_1, \ldots, X_d)$ of rational functions in d = p-deg k variables over the perfect field $k_0 = k^{p^\infty}$. The ring of Witt-vectors $W(k_0)$ (see [9], Ch. II, § 6.) is embedded in K in a unique way such that it induces the embedding $k_0 \hookrightarrow k$ of the residue-class fields. Let $K^{(0)}$ denote the algebraic closure of the quotient field of $W(k_0)$ in K and consider a tower of local fields

$$K^{(0)} \subset K^{(1)} \subset \cdots \subset K^{(d)} = K$$

constructed in the following way: Take the completion $K^{(i)\prime}$ of the field $K^{(i)}(X_{i+1})$ constructed as in 5.7, embed it into K and denote by $K^{(i+1)}$ the completion of the algebraic closure of $K^{(i)\prime}$ in K. The fields $K^{(i)}$ are all local fields with residue-class fields

$$k_i = \text{algebraic closure of } k_{i-1}(X_i) \text{ in } k,$$

and we have the following diagram

$$\begin{tikzcd}
 & & & & \overline{K^{(d)}} & = & \overline{K} \\
 & & & & K_d^{(d)} \arrow[u, dash] & = & K_d \arrow[u, dash] \\
 & & & & \vdots & & \vdots \\
 & & \overline{K^{(2)}} \arrow[r, dash] & \cdots \arrow[r, dash] & K_3^{(d)} & = & K_3 \\
 & \overline{K^{(1)}} \arrow[r, dash] & K_2^{(2)} \arrow[u, dash] \arrow[r, dash] & \cdots \arrow[r, dash] & K_2^{(d)} \arrow[u, dash] & = & K_2 \arrow[u, dash] \\
 \overline{K^{(0)}} \arrow[r, dash] & K_1^{(1)} \arrow[u, dash] \arrow[r, dash] & K_1^{(2)} \arrow[u, dash] \arrow[r, dash] & \cdots \arrow[r, dash] & K_1^{(d)} \arrow[u, dash] & = & K_1 \arrow[u, dash] \\
 K^{(0)} \arrow[u, dash] \arrow[r, dash] & K^{(1)} \arrow[u, dash] \arrow[r, dash] & K^{(2)} \arrow[u, dash] \arrow[r, dash] & \cdots \arrow[r, dash] & K^{(d)} \arrow[u, dash] & = & K \arrow[u, dash]
\end{tikzcd}$$

with obvious notations. It follows by induction from Lemma 5.8 that

$$\mathrm{cd}_p\, G_K \leqq \mathrm{cd}_p\, G_{K^{(0)}} + d = \mathrm{cd}_p\, G_{K^{(0)}} + \text{p-deg}\, k$$

with equality in the following two cases:

(a) k is finitely generated over $k_0 = k^{p^\infty}$,

(b) k is separably closed.

In case (a) we get

$$\mathrm{cd}_p\, G_K = 1 + \mathrm{cd}_p\, G_{k^{p^\infty}} + \text{p-deg}\, k$$

and in case (b)

$$\mathrm{cd}_p\, G_K = 1 + \text{p-deg}\, k.$$

If k is an arbitrary separably generated field, we have therefore

$$\mathrm{cd}_p\, G_K \geqq \mathrm{cd}_p\, G_{K_{nr}} = 1 + \text{p-deg}\, k$$

and this proves Theorem 5.2.

Now let us consider the situation of Theorem 5.6 where the residue-class field k is separably closed. In the above diagram all local fields $K^{(i)}$ have separably closed residue-class field and it follows from Lemma 5.8 that the Galois groups

$$\mathrm{Gal}(K_{i+1}/K_i) \xrightarrow{\sim} \mathrm{Gal}(\overline{K^{(i)}}/K_i^{(i)})$$

are free pro-p-groups. It follows that the normal series

$$G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_d \supset G_{d+1} = (1)$$

with

$$G_0 = G_{(p)} = p\text{-Sylow subgroup of } G_K$$

$$G_i = G_{K_i}$$

has the required property and this completes the proof of Theorem 5.6.

6. Application to (C_r)-Questions

6.1. *One says that a domain R has Tsen-level* $\mathrm{TS}(R) \leqq r$, *or has the property* (C_r), *if any homogenous form* $f(X_1, \ldots, X_n) \in R[X_1, \ldots, X_n]$ *of degree d such that $n > d^r$ has a nontrivial zero in R.*

By the paper [6] of Lang (completed by Nagata [7]) and the result [4] of Greenberg we have the following "transition properties":

(a) If L/K is an extension of fields then

$$\mathrm{TS}(L) \leqq \mathrm{TS}(K) + \operatorname{trdeg} L/K.$$

(b) If R is a discrete valuation ring and $\hat{R}$ the completion of R in the topology given by the discrete valuation then

$$\mathrm{TS}(\hat{R}) \leqq \mathrm{TS}(R).$$

6.2. **Proposition.** *Let K be a local field with separably generated residue-class field and assume that k^{p^∞} is algebraically closed. Then*

$$\mathrm{TS}(K) = 1 + \mathrm{TS}(k) = 1 + \text{p-deg}\, k = \mathrm{cd}_p\, G_K.$$

It follows from the construction in 5.7 and the transition properties (a) and (b) of 6.1 that

$$\mathrm{TS}(K) \leqq \mathrm{TS}(K^{(0)}) + d$$

$$\mathrm{TS}(k) \leqq d$$

with $d = \text{p-deg}\, k$. On the other hand one knows that a local field with algebraically closed residue-class field is (C_1) (see Lang [6]), hence

$$\mathrm{TS}(K) \leqq 1 + d.$$

Furthermore it is easy to see that for a field k of characteristic $p > 0$ we have

$$\mathrm{TS}(k) \geqq \text{p-deg}\, k$$

(consider a p-basis) and for a local field K with residue-class field k we get

$$\mathrm{TS}(K) \geqq 1 + \mathrm{TS}(k)$$

(see Lang [6], Part II: "Existence of normic forms"). Hence

$$1 + d \leqq 1 + \mathrm{TS}(k) \leqq \mathrm{TS}(K) \leqq 1 + d$$

which proves the proposition.

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