Contemporary Mathematics
Volume **10**, 1989

G-VECTOR BUNDLES AND THE LINEARIZATION PROBLEM

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Abstract. The following is an expanded version of my talk at the Montreal Conference on "Group Actions and Invariant Theory" where I gave a report on some old and new results about G-vector bundles on algebraic varieties and their connection with the Linearization Problem (4.1). This problem asks whether every algebraic action of a reductive algebraic group G on affine space $\mathbb{A}^n$ is linearizable. A positive answer would imply that every G-vector bundle on $\mathbb{A}^n$ is trivial (4.2). At the time of the conference the Linearization Problem was completely open. Since then Schwarz has constructed non-trivial G-vector bundles on some representation spaces, thus giving non-linearizable actions on affine spaces (see 4.3).

§ 1. G-Vector Bundles

Throughout the paper the base field is the field of complex numbers $\mathbb{C}$. Of course, it could be replaced by any other algebraically closed field of characteristic zero. Let G be an algebraic group and X a variety with an algebraic G-action $(g,x) \mapsto g \cdot x$. Recall that this means that the map $G \times X \to X$, $(g,x) \mapsto g \cdot x$, is a morphism of complex algebraic varieties. We then call X a *G-variety*. Typical examples are *linear* actions on vector spaces obtained from *rational representations* $\rho : G \to \mathrm{GL}(V)$, where *rational* means that ρ is an algebraic homomorphism.

Definition. A *G-vector bundle* on X is a vector bundle $\mathcal{V}$ on X with an algebraic G-action such that the following holds:
(a) The projection $p : \mathcal{V} \to X$ is G-equivariant;
(b) The action is linear on the fibres $\mathcal{V}_x := p^{-1}(x)$, (i.e. for every $g \in G$ and $x \in X$ the map $v \mapsto gv : \mathcal{V}_x \to \mathcal{V}_{gx}$ is linear).

It follows from the definition that for every $x \in X$ we obtain a rational representation of the isotropy group $G_x := \{g \in G \mid g \cdot x = x\}$ on the fibre $\mathcal{V}_x$ of the G-vector bundle.

1991 *Mathematics Subject Classification.* Primary 14L30, 14F05; Secondary 14D25, 14L15.
Supported in part by SNF (Schweizerischer Nationalfonds)

The category of G-vector bundles on X will be denoted by $\mathrm{Vec}_G(X)$. Homomorphisms, isomorphisms, direct sums and tensor products are defined in the usual way.

EXAMPLE. Every *G-module* M (i.e. finite dimensional rational representation of G) determines a G-vector bundle $\mathcal{V} := M \times X \xrightarrow{\mathrm{pr}} X$. A G-vector bundle is called *trivial* if it is isomorphic to a G-vector bundle of this form.

REMARK 1. If X is *affine* with coordinate ring $\mathcal{O}(X)$ the category $\mathrm{Vec}_G(X)$ is equivalent to the category $\mathrm{Proj}_G(\mathcal{O}(X))$ of G-$\mathcal{O}(X)$-modules which are projective and of finite rank over $\mathcal{O}(X)$ and (locally finite and) rational as G-modules. The functor is given by taking global sections: $\mathcal{V} \mapsto \mathcal{V}(X)$. It is easy to see that any $P \in \mathrm{Proj}_G(\mathcal{O}(X))$ is also projective as a G-$\mathcal{O}(X)$-module, i.e. a direct summand of a free G-$\mathcal{O}(X)$-module $M \otimes_{\mathbb{C}} \mathcal{O}(X)$ (cf. [**BH85, 4.2**]).

§ 2. Some Examples

2.1. Family of representations. If the G-action on X is trivial, a G-vector bundle on X can be understood as an *algebraic family of representations* $(\mathcal{V}_x)_{x\in X}$ of G parametrized by X, in the following sense: Consider the set

$$\mathrm{Rep_n}(G) := \{\rho : G \to \mathrm{GL_n}(\mathbb{C}) \mid \rho \text{ a representation}\}$$

of n-dimensional representations of G. It is a subset of the set of all morphisms $G \to \mathrm{M_n}(\mathbb{C})$, which in turn can be identified with $\mathcal{O}(G) \otimes \mathrm{M_n}(\mathbb{C})$:

$$\mathrm{Rep_n}(G) \subset \mathrm{Mor}(G, \mathrm{GL_n}(\mathbb{C})) = \mathcal{O}(G) \otimes \mathrm{M_n}(\mathbb{C}).$$

A map $\phi : X \to \mathrm{Rep_n}(G)$, where X is a variety, is called *algebraic* (or a morphism) if the image of X in $\mathcal{O}(G) \otimes \mathrm{M_n}(\mathbb{C})$ is contained in a finite dimensional subvector space U such that $\phi : X \to U$ is a morphism of varieties. It is easy to see that the G-vector bundle structures on the trivial bundle $\mathbb{C}^n \times X$ are in one-to-one correspondence with the algebraic maps $X \to \mathrm{Rep_n}(G)$.

The action of $\mathrm{GL_n}(\mathbb{C})$ on $\mathrm{M_n}(\mathbb{C})$ by conjugation induces an action on $\mathrm{Rep_n}(G)$ whose orbits are the equivalence classes of n-dimensional representations of G. For every orbit $O \subset \mathrm{Rep_n}(G)$ its closure $\overline{O}$ in $\mathcal{O}(G) \otimes \mathrm{M_n}(\mathbb{C})$ is contained in $\mathrm{Rep_n}(G)$, and $O = \bar{O}$ if and only if the corresponding representation is completely reducible (cf. [**Kr, II.2.7 Satz 3**]).

From now on we assume that G is reductive. Then every orbit of $\mathrm{Rep_n}(G)$ is closed in $\mathcal{O}(G)\otimes\mathrm{M_n}(\mathbb{C})$, and the number of orbits in $\mathrm{Rep_n}(G)$ is *countable*, and even finite provided the character group of G is finite (e.g. if G semisimple). It follows that for every algebraic map $\phi : X \to \mathrm{Rep_n}(G)$, X connected, the image of X is contained in a single $\mathrm{GL_n}(\mathbb{C})$-orbit O. In addition, the stabilizer of a representation is a Levi-subgroup of $\mathrm{GL_n}(\mathbb{C})$, hence a product of GL_r's, and therefore the orbit map $\mathrm{GL_n}(\mathbb{C}) \to O$ has a local section (with respect to the Zariski-topology). From this one easily obtains the following result; we will give a different proof suggested by G. SCHWARZ.

PROPOSITION 1. *Let G be a reductive group and let $\mathcal{V}$ be a G-vector bundle on X, where G acts trivially on X.*

1. *$\mathcal{V}$ is locally trivial in the Zariski-topology. In particular, the representations $\mathcal{V}_x$ are all equivalent in case X is connected.*
2. *There is an isomorphism of G-vector bundles*
$$\mathcal{V} \xrightarrow{\sim} \bigoplus_{\omega} M_\omega \otimes \mathcal{V}_\omega,$$
where M_ω is a simple G-module and $\mathcal{V}_\omega$ a vector bundle on X (with trivial G-action).

PROOF. We may assume that X is connected. Let M be a simple G-module. It follows from the next lemma that
$$\mathcal{V}_M := (M^* \otimes \mathcal{V})^G$$
is a vector bundle over X. Furthermore, the canonical homomorphism $M \otimes \mathcal{V}_M \to \mathcal{V}$ is an injective G-homomorphism whose image is the M-isotypic component of $\mathcal{V}$. As a consequence we see that $\mathcal{V}$ is isomorphic to the G-vector bundle $\bigoplus_M M \otimes \mathcal{V}_M$, where M runs through the simple G-modules occurring in a fibre of $\mathcal{V}$. This proves the proposition. □

LEMMA 1. *The fixed point set $\mathcal{V}^G$ is a vector bundle over X.*

PROOF. We may assume that X is irreducible and that $\mathcal{V}$ is isomorphic to $X \times \mathbb{C}^n$ as a vector bundle. Then the projection $p : \mathcal{V} \to X$ is the quotient under the obvious action of $G \times \mathbb{C}^*$, and it follows from LUNA's slice theorem that the fibres of $p : \mathcal{V}^G \to X$ are vector subspaces of $\mathbb{C}^n$ of constant dimension d. Hence $\mathcal{V}^G$ is a subvector bundle of $\mathcal{V}$. (In fact, $p : \mathcal{V}^G \to X$ corresponds to a morphism from X to $\mathrm{Grass}_d(n)$, the Grassmanian, and the canonical bundle on $\mathrm{Grass}_d(n)$ is locally trivial in the Zariski-topology.) □

COROLLARY. *Every G-vector bundle on X is trivial in case every vector bundle on X is trivial.*

2.2. Homogeneous bundles. Now assume that X is a homogeneous G-variety, i.e. $X = G/H$, where H is a closed algebraic subgroup of G. It is well known that every G-vector bundle on G/H is of the form
$$G \star^H N \to G/H.$$
Here N is an H-module and $G \star^H N$ is the orbit space $(G \times N)/H$, where the (right-)H-action is given by $(g, n) \cdot h := (gh, h^{-1} \cdot n)$. These bundles are usually called *homogeneous vector bundles.*

A G-vector bundle $G \star^H N$ is trivial if and only if the representation of H on N extends to a representation of G. This follows immediately from the remark below. As an example we see that every SL_2-vector bundle on $\mathbb{C}^2 \setminus \{0\}$ (obvious SL_2-action on $\mathbb{C}^2 \setminus \{0\}$) is trivial. In fact $\mathbb{C}^2 \setminus \{0\} \xleftarrow{\sim} \mathrm{SL}_2 / U$, where
$$U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \right\} \xrightarrow{\sim} \mathbb{C}^+,$$
and every representation of U extends to a representation of SL_2 (cf. [**Kr, III.3.9 Lemma**]).

Remark 2. For every quasi-affine H-variety Y the orbit space $G\star^H Y$ exists as an algebraic G-variety and is a fibre bundle over G/H with fibre Y. (Fibre bundle means that it is *locally trivial in the étale topology.*) This is clear if the H-action on Y extends to an action of G, because the isomorphism $G\times Y \to G\times Y$, $(g,y)\mapsto (g, g\cdot y)$, induces a G-isomorphism

$$G\star^H Y \xrightarrow{\sim} G/H \times Y,$$

and the diagram

$$\begin{tikzcd} G\times Y \arrow[r] \arrow[d] & G\star^H Y \arrow[d] \\ G \arrow[r] & G/H \end{tikzcd}$$

is a pullback. In general, every quasi-affine H-variety Y is a locally closed H-stable subvariety of a G-variety Z. Hence

$$G\star^H Y \hookrightarrow G\star^H Z \xrightarrow{\sim} G/H\times Z,$$

and the claim follows.

2.3. Principal bundles. Let us consider a principal G-bundle $\pi : X \to Y$. This means that X is a G-variety with all orbits isomorphic to G, that the fibres of π are G-orbits and that π is locally trivial in the étale topology, i.e. there is a surjective étale morphism $Y'\to Y$ and a G-isomorphism

$$\begin{tikzcd} G\times Y' \arrow[r, "\sim"] \arrow[d] & X\times_Y Y' \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ Y' \arrow[r, equal] & Y' \arrow[r] & Y \end{tikzcd}$$

over Y'.

Proposition 2. *Let $\pi : X\to Y$ be a principal G-bundle. The pullback $\mathcal{W}\mapsto \pi^*\mathcal{W}$ defines an equivalence*

$$\pi^* : \mathrm{Vec}(Y) \xrightarrow{\sim} \mathrm{Vec}_G(X).$$

Again, this is well known; it can be proved in several ways. In the next section we will consider a more general situation from which the proposition follows as a special case (see Proposition 3 in the next paragraph).

§ 3. Algebraic Quotients and Pullbacks

Let G be reductive and X a G-variety.

Definition. A morphism $\pi : X\to Y$ is an (algebraic) *quotient* if π is constant on G-orbits and satisfies the following condition:

(Q) For every affine open subset $U\subset Y$ the inverse image $\pi^{-1}(U)$ is affine and $\pi^* : \mathcal{O}_Y(U) \xrightarrow{\sim} \mathcal{O}_X(\pi^{-1}(U))^G$ is an isomorphism.

If a quotient exists then it is unique; we will denote it by $\pi : X \to X/\!\!/G$. The quotient map π has a number of remarkable properties (see [**Kr, II.3.2**] or [**MF, Chap. 1, § 2**]): It is *G-closed*, i.e. the image of a closed G-stable subset is closed, and *G-separating*, i.e. disjoint closed G-stable subsets have disjoint images. It follows from this that $X/\!\!/G$ carries the quotient (Zariski-)topology and that every fibre of π contains a unique closed orbit. (Here we use the fact that every G-variety contains closed orbits.)

Examples. (a) If X is an affine G-variety then the quotient exists. It is given by $X/\!\!/G := \operatorname{Spec} \mathcal{O}(X)^G$, the maximal spectrum of the invariant ring $\mathcal{O}(X)^G$, which is a finitely generated $\mathbb{C}$-algebra by Hilbert's finiteness theorem (cf. [**Kr, II.3.2**])

(b) The canonical morphism $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the quotient under the scalar $\mathbb{C}^*$-action.

(c) Every principal G-bundle $\pi : X \to Y$ (2.3) is a quotient morphism.

(d) Given any quotient $\pi : X \to X/\!\!/G$ and a locally closed subvariety $Y \subset X/\!\!/G$, then $\pi : \pi^{-1}(Y) \to Y$ is also a quotient.

Let us assume now that X is a G-variety which admits a quotient $\pi : X \to X/\!\!/G$. If $\mathcal{W}$ is a vector bundle on $X/\!\!/G$ then the pullback $\pi^*\mathcal{W}$ is clearly a G-vector bundle on X, and we obtain the following diagram:

$$\begin{array}{ccc} \pi^*\mathcal{W} & \longrightarrow & X \\ \Big\downarrow{\scriptstyle\tilde{\pi}} & & \Big\downarrow{\scriptstyle\pi} \\ \mathcal{W} & \longrightarrow & X/\!\!/G \end{array}$$

It follows that $\tilde{\pi}$ is a quotient, too. In fact, $\pi^*\mathcal{W}/\!\!/G$ has a natural vector bundle structure and $\mathcal{W} \simeq \pi^*\mathcal{W}/\!\!/G$ as a vector bundle over $X/\!\!/G$. The following proposition characterizes the G-vector bundles which are obtained in this way.

Proposition 3. *A G-vector bundle $\mathcal{V}$ on X is isomorphic to a pullback $\pi^*\mathcal{W}$ if and only if it satisfies the following condition:*

(PB) *For every $x \in X$ such that the orbit Gx is closed, the isotropy group G_x acts trivially on the fibre $\mathcal{V}_x$.*

In this case $\mathcal{V}/\!\!/G$ is a vector bundle over $X/\!\!/G$ isomorphic to $\mathcal{W}$.

(The following proof is due to F. Knop.)

Proof. It is clear that a pullback bundle satisfies the condition (PB). For the other implication one easily reduces to the case where $X/\!\!/G$ (and hence X) is affine. Then the global sections $P := \mathcal{V}(X)$ form a projective G-$\mathcal{O}(X)$-module (see Remark 1 in § 1). Suppose that the fixed elements P^G form a projective $\mathcal{O}(X)^G$-module. (Recall that $\mathcal{O}(X)^G = \mathcal{O}(X/\!\!/G)$.) Then $P^G \otimes_{\mathcal{O}(X/\!\!/G)} \mathcal{O}(X) \xrightarrow{\sim} P$, where the isomorphism is given by $p \otimes f \mapsto fp$, and the proposition follows. Now the condition (PB) implies that for every closed orbit Gx the restriction $\mathcal{V}|_{Gx}$ is trivial:

$$\mathcal{V}|_{Gx} \xrightarrow[G]{\sim} \mathbb{C}^r \times Gx,$$

where G acts trivially on $\mathbb{C}^r$. This means that $P \otimes_{\mathcal{O}(X)} \mathcal{O}(Gx) \xrightarrow{\sim} \mathcal{O}(Gx)^r$. Using the canonical surjective homomorphism $P \twoheadrightarrow P \otimes_{\mathcal{O}(X)} \mathcal{O}(Gx)$ and taking invariants we obtain a surjective homomorphism

$$P^G \longrightarrow\!\!\!\!\!\rightarrow P^G \otimes_{\mathcal{O}(X/\!\!/G)} \mathcal{O}(\pi(x)) \simeq \mathbb{C}^r.$$

Now one shows that the kernel of this map is $\mathbf{m}_y\, P^G$, where $\mathbf{m}_y$ denotes the maximal ideal of the point $y = \pi(x)$, hence $P^G/\mathbf{m}_y\, P^G \simeq \mathbb{C}^r$ for all $y \in X/\!\!/G$. Since P^G is a finitely generated $\mathcal{O}(X/\!\!/G)$-module this implies the claim. □

REMARK 3. This proves Proposition 2 of the previous paragraph.

COROLLARY. *Assume that the isotropy groups G_x generate G. Then a vector bundle $\mathcal{W}$ on $X/\!\!/G$ is trivial if and only if $\pi^*\mathcal{W}$ is a trivial G-vector bundle.*

PROOF. If $\pi^*\mathcal{W} \simeq M \times X$, then every isotropy group G_x acts trivially on M. Therefore it follows from the assumption that M is the trivial G-module, and so $\mathcal{W} \simeq \pi^*\mathcal{W}/\!\!/G \simeq M \times X/\!\!/G$. □

§ 4. The Linearization Problem

4.1. Linearizable actions. An action of G on the affine space $\mathbb{A}^n$ is called *linearizable* if it becomes linear after a polynomial change of coordinates, i.e. there is a G-equivariant algebraic isomorphism $\mathbb{A}^n \xrightarrow{\sim} V$, where V is a representation of G. In the last years a lot of work has been invested in solving the following interesting problem (a positive response has been conjectured by KAMBAYASHI [**Ka79, Conjecture 3.1**]):

LINEARIZATION PROBLEM. *Is every action of a reductive group G on affine space $\mathbb{A}^n$ linearizable?*

Very recently SCHWARZ gave the first examples of non-linearizable reductive group actions on affine space [**Sch89**] (see 4.3). Before that, a number of "small" cases have been handled positively, using various methods (see [**KS89**]). We refer to [**Kr89**] for a report about the Linearization Problem.

4.2. Equivariant Serre-Problem. The construction of the counterexamples uses the following result which relates the Linearization Problem with the question whether every G-vector bundle on a representation space is trivial or not (see [**BH87**]).

PROPOSITION 4. *Assume that the linearization problem has a positive answer, and let V be a representation of the reductive group G. Then*

(a) *Every G-vector bundle on V is trivial.*
(b) *Every vector bundle on $V/\!\!/G$ is trivial.*

PROOF. Let $p : \mathcal{V} \to V$ be a G-vector bundle on V. As a variety $\mathcal{V}$ is an affine space, because every vector bundle on $\mathbb{A}^n$ is trivial (Theorem of QUILLEN and SUSLIN, see [**Qu76**]). On $\mathcal{V}$ we have an action of $\hat{G} := \mathbb{C}^* \times G$, where $\mathbb{C}^*$ acts by scalar multiplication on the fibres of $\mathcal{V}$. It follows that in the diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{p} & V \\ & & \Big\downarrow{\scriptstyle \pi} \\ & & V/\!\!/G \end{array}$$

the map p is the quotient by $\mathbb{C}^*$. Hence the composition $\pi \circ p$ is the quotient by $\hat{G}$. By assumption we can linearize the action of $\hat{G}$ on $\mathcal{V}$ and obtain a $\hat{G}$-isomorphism

$$\varphi : \mathcal{V} \xrightarrow{\sim} W,$$

where W is a representation of $\hat{G}$. Consider the $\hat{G}$-stable decomposition $W = W^{\mathbb{C}^*} \oplus F$. It follows from LUNA's slice theorem [**Lu73**] that $\mathbb{C}^*$ acts on F by scalar multiplication. Hence the linear projection $pr : W \to W^{\mathbb{C}^*}$ is the quotient by $\mathbb{C}^*$, and W is the trivial G-vector bundle $F \times W^{\mathbb{C}^*}$ on $W^{\mathbb{C}^*}$. This shows that φ induces the following isomorphisms:

$$\begin{array}{ccc}
\mathcal{V} & \xrightarrow[\sim]{\varphi} & W \\
\Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\
V & \xrightarrow[\sim]{\bar{\varphi}} & W^{\mathbb{C}^*} \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\
V/\!\!/G & \xrightarrow{\sim} & W^{\mathbb{C}^*}/\!\!/G = W/\!\!/\hat{G}
\end{array}$$

In particular, we obtain $\mathbb{C}^*$-equivariant isomorphisms

$$\varphi : \mathcal{V}_x \xrightarrow{\sim} \{\bar{\varphi}(x)\} \times F.$$

Since the $\mathbb{C}^*$-action is by scalar multiplication on both sides this means that φ is a linear isomorphism. Therefore we obtain a G-vector bundle isomorphism

$$\mathcal{V} \xrightarrow{\sim} \bar{\varphi}^* W = F \times V.$$

This proves (a), and (b) is a consequence of (a) by the corollary of Proposition 3 in § 3. □

4.3. Counterexamples. Let V be a representation of the reductive group G. It is not hard to see that every G-line bundle on V is trivial (see Corollary 1 of Proposition 7). For G-vector bundles of higher rank this is not true, due to the counterexamples of SCHWARZ. Let V_i denote the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension $i+1$ and consider the adjoint representation $V = V_2$.

PROPOSITION 5 ([**Sch89**]). *For every $i \geq 3$ there are infinitely many non-isomorphic* SL_2*-vector bundles $\mathcal{V}$ on V whose zero fibre $\mathcal{V}_0$ is* SL_2*-isomorphic to V_i.*

By Proposition 4 above this implies that there are *non-linearizable* $\mathbb{C}^* \times \mathrm{SL}_2$*-actions on $\mathbb{A}^n$ for every $n \geq 7$.* SCHWARZ showed in addition that the underlying SL_2-actions are also non-linearizable ($n \neq 8$).

REMARK 4. It can be shown that all these examples are *holomorphically trivial* G-vector bundles (see [**Sch89**]). In particular, the corresponding SL_2-actions can be linearized if we allow a holomorphic change of coordinates. Clearly, every G-vector bundle on V is trivial in the differentiable setting, since V has an obvious G-retraction to the origin.

§ 5. Some Results

In this paragraph we assume that G is reductive.

5.1. Stability. Let X be a G-variety. A G-vector bundle $\mathcal{V}$ on X is called *stably trivial* if there is a trivial G-vector bundle $\mathcal{V}_0 = M \times X$ such that $\mathcal{V} \oplus \mathcal{V}_0$ is trivial. The following result is due to BASS-HABOUSH and THOMASON, see [**BH87**].

THEOREM 1. *Every G-vector bundle on a representation space V is stably trivial.*

In fact their results are more general and are expressed in terms of algebraic K-Theory. In view of Proposition 4 we might ask the following question:

PROBLEM. *Is it true that every vector bundle on $V/\!\!/G$ is stably trivial or even trivial?*

5.2. Small quotients. Let V be a representation of G. If every G-invariant function on V is a constant, i.e. $V/\!\!/G = \{*\}$, then every G-vector bundle on V is trivial. This is an easy consequence of LUNA's slice theorem; a more general result will be given in Theorem 3. For a one-dimensional quotient we have the following result.

THEOREM 2. *Let V be a representation of G with* $\dim V/\!\!/G = 1$. *Assume that the generic orbit is closed and has trivial stabilizer. Then every G-vector bundle on V is trivial.*

This can be proven using methods developed in joint work with SCHWARZ [**KS89**], where we study the Linearization Problem for actions with a one-dimensional quotient.

5.3. Fix-pointed actions. An action of G on X is called *fix-pointed* if every closed orbit in X is a fixed point. Such actions have been studied by BASS and HABOUSH. They obtained the following result ([**BH85, 10.2**], cf. [**Kr89, § 5.5**]).

THEOREM 3. *Let X be a fix-pointed G-variety which admits a quotient $\pi : X \to X/\!\!/G$. Then π induces an isomorphism $X^G \xrightarrow{\sim} X/\!\!/G$, and we have an equivalence*

$$\mathrm{Vec}_G(X) \xrightarrow{\sim} \mathrm{Vec}_G(X^G)$$

given by restriction, i.e. every G-vector bundle on X is of the form $\bigoplus M_\omega \otimes \pi^*\mathcal{V}_\omega$, where the M_ω are simple G-modules and the $\mathcal{V}_\omega$ are vector bundles on $X/\!\!/G$.

In particular, we obtain the following corollary:

COROLLARY. *Let X be as above and assume that every vector bundle on $X/\!\!/G$ (or on X) is trivial. Then every G-vector bundle on X is trivial.*

Typical examples for the corollary are:

1. Fix-pointed actions on $\mathbb{A}^n$;
2. Actions on normal affine varieties with a fixed point and only constant invariants.

5.4. Tori. The last result in this section deals with tori. We refer to [**Kr89, § 7**] for the notation used below.

THEOREM 4. *Let V be a representation of a torus T. Assume that the principal stratum $(V/\!\!/G)_{\mathrm{pr}}$ has a complement of codimension ≥ 2 and that the quotient $V/\!\!/G$ is (locally) factorial. Then $V/\!\!/G$ is smooth, $\pi : V \to V/\!\!/G$ has a section and every T-vector bundle on V is trivial.*

OUTLINE OF PROOF. One first shows that the fibre bundle $\pi : V_{\mathrm{pr}} \to (V/\!\!/G)_{\mathrm{pr}}$ is trivial. We will see in the next section (6.2) that any G-vector bundle $\mathcal{V}$ on V is locally trivial over the quotient $V/\!\!/G$. The group functor of automorphisms of $\mathcal{V}$ is represented by an algebraic group scheme $\mathfrak{A}$ over $V/\!\!/G$. The triviality of the fibre bundle over $(V/\!\!/G)_{\mathrm{pr}}$ implies that $\mathfrak{A}|_{(V/\!\!/G)_{\mathrm{pr}}}$ is "constant" with fibre $\mathfrak{A}_0$ isomorphic to a product of $\mathrm{GL_n}$'s. Hence $\mathcal{V}|_{(V/\!\!/G)_{\mathrm{pr}}}$ can be understood as a principal bundle with structure group $\mathfrak{A}_0$. Now the codimension 2 condition can be used to show that this bundle has an extension to all of $V/\!\!/G$, hence is trivial. □

§ 6. Some Methods

In this section we assume again that G is reductive.

6.1. Equivariant Nakayama Lemma. The following result can be found in [**BH85**].

LEMMA 2. *Let X be an affine G-variety, $Y \subset X$ a closed G-stable subvariety and $\mathcal{V}$, $\mathcal{W}$ two G-vector bundles on X.*

1. *Every G-homomorphism $\varphi : \mathcal{V}|_Y \to \mathcal{W}|_Y$ extends to a G-homomorphism $\tilde{\varphi} : \mathcal{V} \to \mathcal{W}$.*
2. *Assume that all closed orbits are contained in Y. Then the extension $\tilde{\varphi}$ of φ is unique.*

It is easy to see that this lemma implies Theorem 3 of the previous paragraph and its corollary.

6.2. Local triviality. In general we cannot expect that G-vector bundles are *locally trivial*, i.e. that there is a covering $X = \bigcup U_i$ by G-stable Zariski-open sets U_i such that the restrictions to the U_i's are trivial. In fact, we have seen in 2.2 that a G-vector bundle $G \star^H N$ on the homogeneous space G/H is trivial if and only if the representation of H on N extends to a representation of G.

Nevertheless, the situation is different when the base X is a representation space:

PROPOSITION 6. *Every G-vector bundle on a representation space V of G is locally trivial.*

PROOF. Consider a closed orbit Gx in V. Then $\mathcal{V}|_{Gx} \xrightarrow{\sim} G \star^H W$, where $H := G_x$ and $W := \mathcal{V}_x$. Now V^H is a linear subspace, and the H-vector bundle $\mathcal{V}|_{V^H}$ is trivial by the corollary of 2.1 Proposition 1: $\mathcal{V}|_{V^H} \xrightarrow{\sim} W \times V^H$. Since $0 \in V^H$, the representation of H on W extends to a representation of G, and therefore $\mathcal{V}|_{Gx} \xrightarrow{\sim} G/H \times W$ is trivial. By Lemma 2(a) above this isomorphism extends to a G-homomorphism $\mathcal{V} \to W \times V$ which is an isomorphism in a neighbourhood of Gx. □

Remark 5. In the proof above we only needed the following two facts about the underlying affine variety X:

1. $X^G \neq \emptyset$;
2. For every isotropy group H of a point of a closed orbit the fixed point set X^H is connected.

6.3. Cohomology. As a consequence of the proposition above one shows that the isomorphism classes of G-vector bundles $\mathcal{V}$ on a representation space V whose fibre $\mathcal{V}_0$ over the origin is isomorphic to a given G-module M, can be described by the *non-abelian cohomology set*

$$\mathrm{H}^1(V/\!\!/G, \mathfrak{A}_M),$$

where $\mathfrak{A}_M$ is the *automorphism group scheme* of the trivial G-vector bundle $M \times V$, representing the functor which associates to every $U \subset V/\!\!/G$ the automorphism group of the G-vector bundle $\pi^{-1}(U) \times M$. It turns out that $\mathfrak{A}_M$ is an affine group scheme over $V/\!\!/G$ whose general fibre is a semidirect product of a unipotent group by a product of $\mathrm{GL_n}$'s.

Assume, for example, that M is the trivial G-module, or more generally that G acts on M via multiplication by a character. Then $\mathfrak{A}_M$ is isomorphic to the "constant" group scheme $\mathrm{GL_n} \times V/\!\!/G$. This implies that every G-vector bundle $\mathcal{V}$ on V whose zero fibre $\mathcal{V}_0$ is G-isomorphic to M is trivial, provided that every vector bundle on $V/\!\!/G$ is trivial.

§ 7. G-Line Bundles

In the last paragraph we consider the special case of *G-line bundles* on G-varieties, where G is an arbitrary algebraic group. This case has been studied by different authors, and most of the following material can be found in the literature (e.g. [**Mag80**], [**Pop74**]). For a more detailed exposition of these results we refer to the forthcoming Seminar Notes [**DMV**].

Let X be an irreducible G-variety with quotient $\pi : X \to X/\!\!/G$. We denote by $\mathcal{O}(X)^*$ the group of invertible global functions on X and define $E(X) := \mathcal{O}(X)^*/\mathbb{C}^*$. By a result of Rosenlicht [**Ro56**], this group is always finitely generated. In addition, the connected component G° of the unit element $e \in G$ acts trivially on $E(X)$.

The *character group* $\mathcal{X}(G)$ of G is the subgroup of $\mathcal{O}(G)^*$ of all algebraic group homomorphisms $G \to \mathbb{C}^*$. Again it has been shown by Rosenlicht [**Ro56**] that for a connected group G, every invertible function $f \in \mathcal{O}(X)^*$ with $f(e) = 1$ is automatically a character. In particular, $\mathcal{O}(G)^* = \mathbb{C}^* \cdot \mathcal{X}(G)$, and we have a canonical isomorphism $\mathcal{X}(G) \xrightarrow{\sim} E(G)$.

Proposition 7. *In the following diagram the row and the column sequences are exact:*

$$\begin{tikzcd}
 & 1 \arrow[d] & & \\
 & E(X/\!\!/G) \arrow[d] & & \\
 & E(X)^G \arrow[d] & 1 \arrow[d] & \\
 & \mathcal{X}(G) \arrow[d] & \operatorname{Pic} X/\!\!/G \arrow[d, "\pi^*"] & \\
1 \arrow[r] & \mathrm{H}^1(G, \mathcal{O}(X)^*) \arrow[r, "\mu"] \arrow[d] & \operatorname{Pic}_G X \arrow[r] \arrow[d, "\rho"] & (\operatorname{Pic} X)^G \\
 & \mathrm{H}^1(G/G^\circ, E(X)) & \prod_{x \in \mathcal{C}} \mathcal{X}(G_x) &
\end{tikzcd}$$

In $\prod_{x\in\mathcal{C}} \mathcal{X}(G_x)$ the set $\mathcal{C} \subset X$ is a set of representatives of the closed orbits in X, and the map ρ is obtained by associating to a line bundle $\mathcal{L}$ the characters of the isotropy groups G_x on the fibres $\mathcal{L}_x$, $x \in \mathcal{C}$. As usual $\mathrm{H}^1(G, \cdot)$ denotes the group of *algebraic cocycles* modulo *algebraic coboundaries*, and the map μ is defined as follows: Every cocycle $\gamma : G \to \mathcal{O}(X)^*$ determines a G-structure on the trivial line bundle $\mathbb{C} \times X$, and one obtains isomorphic G-line bundles if and only if the corresponding cocycles are equivalent.

Proof of Proposition 7. The first column is obtained from the exact sequence

$$1 \to \mathbb{C}^* \to \mathcal{O}(X)^* \to E(X) \to 1$$

by applying the G-fixed point functor and its derived functors $\mathrm{H}^1(G, \cdot)$. The second column is a reformulation of Proposition 3 (§ 3) for G-line bundles. The exactness of the row is clear from the definition of the map μ. □

Remark 6. If X is normal and G connected, then the cokernel of the canonical map $\operatorname{Pic}_G X \to (\operatorname{Pic} X)^G$ is a torsion group. This follows from a result of Sumihiro stating that, under our hypotheses, a suitable power of every line bundle admits a G-linearization ([**Sum74**], [**MF**]).

We draw a number of corollaries (cf. [**FI73**], [**Mag80**], [**Pop74**]).

Corollary 1. *Assume that* $\mathcal{O}(X)^* = \mathbb{C}^*$ *and that* $X^G \neq \emptyset$*. Then, in the following diagram*

$$\begin{tikzcd}
 & & 1 \arrow[d] & \\
 & & \operatorname{Pic} X/\!\!/G \arrow[d, "\gamma"] & \\
1 \arrow[r] & \mathcal{X}(G) \arrow[r, "\alpha"] & \operatorname{Pic}_G X \arrow[r, "\beta"] \arrow[d, "\delta"] & (\operatorname{Pic} X)^G \\
 & & \prod_{x\in\mathcal{C}} \mathcal{X}(G_x) &
\end{tikzcd}$$

the row and the column are both exact, and the two compositions $\delta \circ \alpha$ *and* $\beta \circ \gamma$ *are both injective. In particular, if* $\operatorname{Pic} X = 0$*, then* $\operatorname{Pic}(X/\!\!/G) = 0$ *and* $\mathcal{X}(G) \xrightarrow{\sim} \operatorname{Pic}_G X$*.*

(It clearly suffices to assume that the stabilizers G_x generate G.)

Example. For every G-action on an affine space $\mathbb{A}^n$ with a fixed point (e.g. for every representation of G) we have $\operatorname{Pic}(\mathbb{A}^n/\!\!/G) = 0$ and $\operatorname{Pic}_G \mathbb{A}^n \xleftarrow{\sim} \mathcal{X}(G)$. If, in addition, the quotient $\mathbb{A}^n/\!\!/G$ is locally factorial (e.g. smooth), then it is factorial.

Corollary 2. *Assume that* $\pi : X \to Y$ *is a principal* G*-bundle. Then we have an isomorphism* $\operatorname{Pic} Y \xrightarrow{\sim} \operatorname{Pic}_G X$ *and an exact sequence:*

$$1 \to E(Y) \to E(X)^G \to \mathcal{X}(G) \to \operatorname{Pic}_G X \to (\operatorname{Pic} X)^G \xrightarrow{\lambda} \operatorname{Pic} G$$

Here λ is defined in the following way: For every $x \in X$ we get a line bundle $L_x \in \operatorname{Pic} G$ by pulling back L via the orbit map $g \mapsto g{\cdot}x$, and one shows that the isomorphism class of L_x is independent of $x \in X$.

We remark that λ is surjective in case the principal bundle $\pi : X \to Y$ is locally trivial in the Zariski-topology. In addition, $\operatorname{Pic} G$ is always finite [**Ro56**].

Examples. (a) (Popov [**Pop74**]) Let G be a connected algebraic group and $H \subset G$ a closed subgroup. Then $E(G) = \mathcal{X}(G)$ has trivial G-action, and we obtain an exact sequence

$$1 \to E(G/H) \to \mathcal{X}(G) \to \mathcal{X}(H) \to \operatorname{Pic} G/H \to \operatorname{Pic} G \to \operatorname{Pic} H.$$

In particular, if $\operatorname{Pic} G = 0$, we have $\operatorname{Pic} G/H \simeq \mathcal{X}(H)/\operatorname{Im}\mathcal{X}(G)$.

(b) Let $G \subset \mathrm{GL}(V)$ be a finite subgroup containing no pseudo-reflections. Then the principal stratum V_{pr} has a complement of codimension ≥ 2, and we get isomorphisms

$$\mathcal{X}(G) \xrightarrow{\sim} \operatorname{Pic}(V/\!\!/G)_{\mathrm{pr}} \xrightarrow{\sim} \operatorname{Cl} V/\!\!/G.$$

(Cl denotes the *divisor class group*.) Hence for a finite subgroup $G \subset \mathrm{SL}(V)$ the quotient $V/\!\!/G$ is factorial if and only if $\mathcal{X}(G) = 0$.

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