THE PICARD GROUP OF A G-VARIETY

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Table of Contents

Introduction

Let G be a reductive algebraic group and X an algebraic G-variety which admits a quotient $\pi : X \to X/\!\!/G$. In this article we describe several results concerning the Picard group $\mathrm{Pic}(X/\!\!/G)$ of the quotient and the group $\mathrm{Pic}_G(X)$ of G-line bundles on X. For some further development of the subject we refer to the survey articles [Kr89a], [Kr89b].

We also give the proofs of some results which have been used in our first article *"Local Properties of Algebraic Group Actions"* in this volume; it will be quoted by [LP].

§ 1 Two Results of ROSENLICHT

We first describe two results about the group of invertible functions on an irreducible algebraic variety which are due to ROSENLICHT [Ro61, Theorems 1, 2, and 3]. They hold for an algebraically closed field k of arbitrary characteristic.

1.1. Let X be an irreducible algebraic variety. We denote by $\mathcal{O}(X)^*$ the group of *invertible regular functions* on X, i.e., morphisms $X \to k^*$.

Proposition (cf. [FI74, Lemma 2.1]. *Let X, Y be two irreducible algebraic varieties. Then the canonical map*

$$\mathcal{O}(X)^* \times \mathcal{O}(Y)^* \longrightarrow \mathcal{O}(X \times Y)^*$$

is surjective.

PROOF: We choose normal points $x_0 \in X$ and $y_0 \in Y$. Let $f \in \mathcal{O}(X \times Y)^*$, and consider the function

$$F : X \times Y \longrightarrow k^*, \quad F(x,y) := f(x_0,y_0)^{-1} f(x,y_0) f(x_0,y).$$

We have to show that $F = f$. For this it is sufficient to prove that these two functions coincide in a neighborhood of (x_0, y_0) of the form $U \times V$, where $U \subset X$, $V \subset Y$ are open subsets. Hence we can assume that X and Y are both affine and normal.

Let $\overline{X}$, $\overline{Y}$ be normal projective varieties which contain X and Y as open subsets, and consider f and F as rational functions on $\overline{X} \times \overline{Y}$. By construction, the divisor $\left(\frac{f}{F}\right)$ of the rational function $\frac{f}{F} \in k(\overline{X} \times \overline{Y})$ has a support contained in $((\overline{X} \smallsetminus X) \times \overline{Y}) \cup (\overline{X} \times (\overline{Y} \smallsetminus Y))$. Hence, it is a sum of divisors of the form $D \times \overline{Y}$ and $\overline{X} \times E$ where D and E are irreducible components of $\overline{X} \smallsetminus X$ and $\overline{Y} \smallsetminus Y$ (of codimension 1), respectively. If $\frac{f}{F}$ has a zero of order $d > 0$ along $D \times \overline{Y}$, then $\frac{f}{F}$ is regular on an open set U which meets $D \times \{y_0\}$, and vanishes on $U \cap (D \times \{y_0\})$. But $f(x, y_0) = F(x, y_0)$ for all $x \in X$ which leads to a contradiction. Similarly, we see that $\frac{f}{F}$ cannot have poles on $D \times \overline{Y}$. Interchanging the roles of $\overline{X}$ and $\overline{Y}$ it follows that the divisor of $\frac{f}{F}$ is zero, i.e., $\frac{f}{F} = 1$. □

1.2. Proposition (cf. [FI74, Corollary 2.2]). *Let G be a connected algebraic group. Then every regular function $f : G \to k^*$ whose value at the unit element $e \in G$ is 1, is a character.*

PROOF: It follows from the proposition above that there exist functions $r_1, r_2 \in \mathcal{O}(G)^*$ such that $f(g_1 g_2) = r_1(g_1) r_2(g_2)$ for all $g_1, g_2 \in G$. Multiplying r_1 and r_2 by a scalar we may assume that $r_1(e) = r_2(e) = 1$. But this implies that $f = r_i$ $(i = 1, 2)$ and the claim follows. □

1.3. For an irreducible variety X we denote by $E(X)$ the quotient $\mathcal{O}(X)^*/k^*$.

Proposition. (i) *The group $E(X)$ is free abelian and finitely generated.*

(ii) *If X is a G-variety where G is connected linear algebraic group, then the canonical action of G on $E(X)$ is trivial.*

PROOF: (i) If X' is a nonempty quasi-projective open subset of X consisting of normal points, then $\mathcal{O}(X)^*$ is a subgroup of $\mathcal{O}(X')^*$. Hence we may assume that X is normal and quasi-projective: $X \subset \mathbf{P}^n$. Let $\overline{X}$ denote the normalisation of the closure of X in $\mathbf{P}^n$.

If $D \subset \overline{X}$ is a closed irreducible subvariety of codimension 1, i.e., an irreducible divisor, then the local ring $\mathcal{O}_{D,\overline{X}}$ is the valuation ring of a discrete normalized valuation ν_D of the field $k(\overline{X})$ of rational functions on $\overline{X}$. (Here we use the normality of $\overline{X}$, [BAC7].) Denote by $D_1, D_2, \ldots, D_m$ the irreducible components of $\overline{X} \setminus X$ which are of codimension 1 in $\overline{X}$.

Let $f \in \mathcal{O}(X)^*$. If $\nu_{D_i}(f) \geq 0$ for $i = 1, 2, \ldots, m$, then $\nu_D(f) \geq 0$ for every irreducible divisor D, and so f is a regular function on $\overline{X}$, hence a constant. This shows that the homomorphism

$$\operatorname{div} : \mathcal{O}(X)^* \longrightarrow \bigoplus_{i=1}^{m} \mathbf{Z}\, D_i, \quad f \mapsto \textstyle\sum_i \nu_{D_i}(f) D_i,$$

induces an injection of $E(X)$ into a finitely generated free abelian group. This implies the first claim.

(ii) Let $f \in \mathcal{O}(X)^*$ and consider the morphism

$$G \times X \longrightarrow k^*, \quad (g, x) \mapsto f(g^{-1}x).$$

By Proposition 1.1 there exist $p \in \mathcal{O}(G)^*$ and $q \in \mathcal{O}(X)^*$ such that

$$\begin{aligned} f(g^{-1}x) &= p(g)q(x), \quad (g \in G, x \in X) \\ p(e) &= 1. \end{aligned}$$

Putting $g = e$ we obtain $f = q$. Hence, f is invariant modulo the constant functions. □

§ 2 G-Linearization of the Trivial Bundle

From now on we assume that the base field k is algebraically closed and of characteristic zero. Nevertheless, the results of the next two paragraphs hold in arbitrary characteristic and most of the proofs can be carried over to the general case.

2.1. Let G be a linear algebraic group and X an irreducible G-variety. Recall that a G-linearization of a line bundle L on X is a lifting of the G-action to L

which is linear on the fibers; see our first article *"Local Properties of Algebraic Group Actions"* [LP]. A line bundle L on X together with a G-linearization is called a *G-line bundle*; we denote by $\mathrm{Pic}_G(X)$ *the set of isomorphism classes of G-line bundles on X*. It has a group structure given by the tensor product. Forgetting the G-linearization we obtain a canonical homomorphism

$$\nu : \mathrm{Pic}_G(X) \to \mathrm{Pic}(X)$$

whose kernel consists of the G-linearizations of the trivial bundle on X (up to isomorphism). Such an action is given by a morphism

$$c : G \times X \longrightarrow k^* = \mathrm{GL}(1, k)$$

satisfying

$$c(gh, x) = c(g, hx)\, c(h, x), \quad g, h \in G, x \in X. \tag{1}$$

Define

$$\gamma : G \to \mathcal{O}(X)^* \quad \text{by} \quad \gamma(g)(x) := c(g^{-1}, x).$$

Then the equality (1) becomes the usual cocycle condition:

$$\gamma(gh) = ({}^g\gamma(h))\gamma(g), \quad g, h \in G, \tag{2}$$

where the action of G on the functions $\mathcal{O}(X)$ is defined by $({}^g u)(x) := u(g^{-1}x)$ ($g \in G, u \in \mathcal{O}(X), x \in X$). Changing the trivialization by an isomorphism

$$X \times k \xrightarrow{\sim} X \times k, \quad (x, \lambda) \mapsto (x, u(x)\lambda)$$

where $u \in \mathcal{O}(X)^*$, transforms $c(g, x)$ into $c(g, x)u(gx)u(x)^{-1}$. Hence, γ is transformed into an equivalent cocycle $g \mapsto c(g)\, {}^g u\, u^{-1}$ in the usual sense. It follows that $\ker \nu$ is given by the group $\mathrm{H}^1_{\mathrm{alg}}(G, \mathcal{O}(X)^*)$ *of classes of algebraic cocycles.* (By definition, a map $c : G \to \mathcal{O}(X)^*$ is algebraic, if the corresponding map $G \times X \to k^*$, $(g, x) \mapsto c(g)(x)$ is a morphism.)

2.2. Lemma. *There is an exact sequence*

$$0 \to \mathrm{H}^1_{\mathrm{alg}}(G, \mathcal{O}(X)^*) \to \mathrm{Pic}_G(X) \xrightarrow{\nu} \mathrm{Pic}(X).$$

If X is normal it has an extension by the homomorphism $\mathrm{Pic}(X) \xrightarrow{\rho} \mathrm{Pic}(G)$, *defined by*

$$\rho(L) := (\varphi^*(L) \otimes p_X^*(L)^{-1})|_{G \times \{x_0\}}$$

where $\varphi : G \times X \to X$ is the G-action, $p_X : G \times X \to X$ the projection and $x_0 \in X$ an arbitrary point.

PROOF: The first part is clear from what we said above. Also, by definition, the image of the map ν are the G-linearizable line bundles on X. If X is normal it follows from [LP, Lemma 2.3 and Lemma 4.1] that the kernel of ρ is the subgroup of G-linearizable bundles, too. ◻

2.3. Proposition. *There is a long exact sequence*

$$\begin{aligned} 1 \to k^* &\to (\mathcal{O}(X)^*)^G \to E(X)^G \to \mathcal{X}(G) \to \\ &\to \mathrm{H}^1_{\mathrm{alg}}(G, \mathcal{O}(X)^*) \to \mathrm{H}^1(G/G^0, E(X)) \end{aligned} \tag{3}$$

where $\mathcal{X}(G)$ is the character group of G and G^0 the connected component of the unit element of G.

PROOF: We start with the short exact sequence

$$1 \to k^* \to \mathcal{O}(X)^* \to E(X) \to 1 \tag{4}$$

of G-modules, and show that (3) is the "algebraic part" of the long exact cohomology sequence of (4). Consider first the homomorphism

$$\delta : E(X)^G \to \mathrm{H}^1(G, k^*)$$

which associates to $\bar{p} \in E(X)^G$ the class of the cocycle $\delta(\bar{p}) : g \mapsto {}^g p\, p^{-1}$, where $p \in \mathcal{O}(X)^*$ is a representative of $\bar{p}$. It is clear that $\delta(\bar{p})$ belongs to $\mathcal{O}(G)^*$. Hence $\delta(\bar{p})$ is a character of G, because G acts trivially on k^*. It is also obvious that the inclusion $k^* \hookrightarrow \mathcal{O}(X)^*$ induces a homomorphism $\mathcal{X}(G) \to \mathrm{H}^1_{\mathrm{alg}}(G, \mathcal{O}(X))$.

Finally, let $\gamma : G \to \mathcal{O}(X)^*$ be an algebraic cocycle, and denote by $\bar{\gamma} : G \to E(X)$ the composition of γ with the projection $\mathcal{O}(X)^* \to E(X)$. By Proposition 1.1 there exist $\varphi \in \mathcal{O}(G^0)^*$ and $u \in \mathcal{O}(X)^*$ such that $\gamma(g) = \varphi(g)u$ for all $g \in G^0$. It is easy to see from the cocycle condition (2) for γ that u is constant, i.e., $\bar{\gamma}|_{G^0} = 1$. In addition, since G^0 acts trivially on $E(X)$ (Proposition 1.3ii), the cocycle condition $\bar{\gamma}(gh) = {}^g\bar{\gamma}(h)\,\bar{\gamma}(g)$ implies $\bar{c}(G^0 h) = \bar{c}(h)$. Hence, the projection $\mathcal{O}(X)^* \to E(X)$ induces a homomorphism $\mathrm{H}^1_{\mathrm{alg}}(G, \mathcal{O}(X)^*) \to \mathrm{H}^1(G/G^0, E(X))$.

Thus we have established the existence of the sequence (3); it remains to prove the exactness. Since the image of δ is contained in $\mathcal{X}(G)$ it is clearly exact at $\mathcal{X}(G)$. For the exactness at $\mathrm{H}^1_{\mathrm{alg}}$ we remark that every $d \in \mathrm{H}^1(G, k^*)$ whose image is in $\mathrm{H}^1_{\mathrm{alg}}$, belongs to $\mathcal{X}(G)$. □

§ 3 Picard Group of Homogeneous Spaces

3.1. In this paragraph the group G is assumed to be connected. Let H be a closed subgroup of G. Recall that we have a canonical homomorphism

$$\mathcal{E} : \mathcal{X}(H) \longrightarrow \operatorname{Pic} G/H, \quad \chi \mapsto E_\chi$$

whose image is the set of G-linearizable bundles on G/H (see [LP, Example 2.1]). In fact, $E_\chi := (G \times k)/H$ is a G-line bundle on G/H and $\mathcal{X}(H) \xrightarrow{\sim} \operatorname{Pic}_G(G/H)$.

3.2. Proposition. (i) *The sequence*

$$\mathcal{X}(G) \xrightarrow{\mathrm{res}} \mathcal{X}(H) \xrightarrow{\mathcal{E}} \operatorname{Pic}(G/H) \xrightarrow{\pi^*} \operatorname{Pic}(G)$$

is exact.

(ii) *If H is solvable and connected then π^* is surjective.*

PROOF: (i) We show the exactness at $\mathrm{Pic}(G/H)$ and leave the rest to the reader (cf. [Po74, p. 316], [FI74, Proposition 3.1]). Let $L \in \mathrm{Pic}(G/H)$ and denote by $\varphi : G \times G/H \to G/H$ the action of G on G/H. By [LP, Lemma 4.2] we know that

$$\varphi^*(L) \simeq p_G^*(M) \otimes p_{G/H}^*(N)$$

where

$$\begin{aligned} M &:= \varphi^*(L)|_{G\times\{eH\}} \simeq \pi^*(L) \\ N &:= \varphi^*(L)|_{\{e\}\times G/H} \simeq L. \end{aligned}$$

Hence

$$\begin{aligned} \pi^*(L) = 0 &\iff \varphi^*(L) \simeq p_{G/H}^*(L) \\ &\iff L \text{ is } G\text{-linearizable} \\ &\iff L \text{ belongs to the image of } \mathcal{E} \end{aligned}$$

by what we said above.

(ii) If H is connected and solvable then $\pi : G \to G/H$ is locally trivial in the Zariski-topology ([Se58, 4.4 proposition 14]): There is an affine open subset U of G/H such that $\pi^{-1}(U) \simeq U \times H$. Let us denote by $D_1, \ldots, D_n$ the irreducible components of the complement of U in G/H. Then we obtain the following commutative diagram with exact rows (see [Ha77, Chap. II, Proposition 6.5]):

$$\begin{array}{ccccccc} \bigoplus \mathbf{Z}\, D_i & \to & \mathrm{Cl}(G/H) & \to & \mathrm{Cl}(U) & \to & 0 \\ \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\pi^*} & & \downarrow{\scriptstyle\pi_U^*} & & \\ \bigoplus \mathbf{Z}\, \pi^{-1}(D_i) & \to & \mathrm{Cl}(G) & \to & \mathrm{Cl}(\pi^{-1}(U)) & \to & 0 \end{array}$$

Since H, as a variety, is isomorphic to $k^{*p} \times k^q$ (see [LP, 4.1]) it follows from [Ha77, Chap. II, Proposition 6.6] that π_U^* is an isomorphism, and so π^* is surjective. □

3.3. A connected algebraic group G is called *simply connected* if every finite covering of G is trivial. It follows that every line bundle on G/H is linearizable where H is any closed subgroup of G.

Corollary. *Let G be semisimple and simply connected and let $B \subset G$ be a Borel subgroup. Then $\mathcal{E} : \mathcal{X}(B) \xrightarrow{\sim} \mathrm{Pic}(G/B)$ is an isomorphism.*

PROOF: In fact, if G is semisimple then $\mathcal{X}(G) = 1$, and G simply connected implies that $\mathrm{Pic}\, G = 0$ by [LP, Proposition 4.6]. Now the claim follows. □

§ 4 Picard Group of Quotients

4.1. Let G be a *reductive* algebraic group and X an irreducible G-variety. Recall that a morphism $\pi : X \to Y$ is a *quotient of* X *by* G if

(a) π is constant on the G-orbits and

(b) for every affine open $U \subset Y$ the inverse image $\pi^{-1}(U)$ is affine and $\mathcal{O}_Y(U) \xrightarrow{\sim} \mathcal{O}_X(\pi^{-1}(U))^G$.

Whenever such a quotient exists it is unique and will be denoted by

$$\pi : X \to X/\!\!/G.$$

Let $M \in \operatorname{Pic}(X/\!\!/G)$; then $\pi^* M$ is obviously a G-line bundle on X. The following proposition describes the G-line bundles which arise in this way (cf. [Ma80, Proposition 5]).

4.2. Proposition. *The homomorphism*

$$\pi^* : \operatorname{Pic}(X/\!\!/G) \to \operatorname{Pic}_G(X)$$

is injective, and for every $M \in \operatorname{Pic}(X/\!\!/G)$ we have $\pi^* M/\!\!/G \simeq M$ in a canonical way. The image of π^* consists of those G-line bundles L on X which satisfy the following condition:

(PB) *For every $x \in X$ whose G-orbit is closed, the isotropy group G_x acts trivially on the fiber L_x of L.*

((PB) stands for "pull back".)

4.3. Remark. It follows from the description of Pic_G of a homogeneous space in 3.1 that the condition (PB) is equivalent to

(PB′) *For every closed orbit O of G in X the restriction of L to O is a trivial G-line bundle.*

Proof of Proposition 4.2: Obviously we have $\pi^* M/\!\!/G \simeq M$; in particular, π^* is injective. It is also clear that the condition (PB) holds for every G-line bundle L in the image of π^*.

Conversely, let $L \in \operatorname{Pic}_G(X)$ and assume that L satisfies the condition (PB). We will show that for an affine variety X the bundle L is a pull-back of a line bundle on $X/\!\!/G$. Then the general case follows immediately from the properties of π and the injectivity of π^*.

To simplify notations, let $A := \mathcal{O}(X)$ and denote by P the A-module of global section of L and by $P(x) := P/\boldsymbol{m}_x P$ the fiber of L, where $\boldsymbol{m}_x$ is the maximal ideal of $x \in X$. Clearly, P is a projective G-A-module of finite type.

We first show that P^G generates the A-module P. Let $x \in X$ be a point whose orbit Gx is closed, and denote by $I = I(x)$ the ideal of Gx. By assumption, we have $P/IP \simeq A/I$. Consider the commutative diagram

$$\begin{array}{ccccccc}
P & \twoheadrightarrow & P/IP & \simeq & A/I & \twoheadrightarrow & P(x) \\
\cup & & \cup & & & & \\
P^G & \twoheadrightarrow & (P/IP)^G & \simeq & (A/I)^G & &
\end{array}$$

which consists of surjective homomorphisms (since G is reductive); it is obtained by factorizing the canonical homomorphism $\varepsilon : P \to P(x)$. It follows that $\varepsilon(P^G) = P(x)$, i.e.,

$$P = Q + \boldsymbol{m}_x P$$

where $Q := AP^G$. Hence $P_{\boldsymbol{m}_x} = Q_{\boldsymbol{m}_x}$ by Nakayama's Lemma. On the other hand, the support S of the A-module P/Q is a G-stable closed subset of X. But we have just seen that S does not contain any closed orbit. Hence S is empty and therefore $P = Q$ as required.

Next we observe that multiplication with elements of A induces a surjective G-equivariant homomorphism

$$I \otimes_k P^G \to IP.$$

Since G is reductive this implies that $(IP)^G = I^G P^G = \boldsymbol{m}_{\pi(x)} P^G$ for every $x \in X$ with a closed orbit. It follows that $P^G / \boldsymbol{m}_{\pi(x)} P^G$ is isomorphic to $(P/IP)^G$. But $(P/IP)^G \simeq k$ by assumption, and so P^G is a projective A^G-module of rank 1. Now it is easy to see that $A \otimes_{A^G} P^G \xrightarrow{\sim} P$, and the claim follows. □

4.4. Remark. The above proposition holds for G-vector bundles (with the same proof, see [Kr89b, §3]).

4.5. Remark. We denote by $S_1, S_2, \ldots, S_r \subseteq X/\!\!/G$ the closed (connected) Luna strata ([Lu73]; see [Kr89a, §7]). We claim that condition (PB) is equivalent to

(PB″) *For $i = 1, 2, \ldots, r$ there exists $x_i \in X$ such that the orbit Gx_i is closed, $\pi(x_i) \in S_i$ and such that the isotropy group G_{x_i} acts trivially on the fiber L_{x_i}.*

In order to prove this assertion, we denote by U the image in $X/\!\!/G$ of the set of all elements $x \in X$ such that Gx is closed and the isotropy group G_x acts trivially on L_x; condition (PB) is equivalent to $U = X/\!\!/G$. It follows from the proof of Proposition 4.2 that U is open in $X/\!\!/G$. This implies that U meets every stratum of the Luna stratification. Therefore, it remains to show that U contains the whole stratum whenever it contains one point of it. Hence, we are reduced to the case where $X/\!\!/G$ is one stratum and every orbit in X is closed. Choose a point $x \in X$ and denote by Y the fixed point set of G_x in X. Then the restriction of L to Y is a (connected) "algebraic family" of representations of the reductive group G_x, and, if one member is trivial, then all others are trivial, too (see [Kr89b, 2.1]). Since $\pi(Y) = X/\!\!/G$ and $U \neq \varnothing$ the claim follows.

§ 5 Résumé and Applications

5.1. Collecting our results from the previous paragraphs we obtain the following proposition:

Proposition. *Let G be a reductive algebraic group and X an irreducible G-variety which admits a quotient $\pi : X \to X/\!\!/G$. Then we have the following commutative diagram with an exact line and exact columns:*

$$
\begin{tikzcd}
 & 1 \arrow[d] & & \\
 & E(X/\!\!/G) \arrow[d] & & \\
 & E(X)^G \arrow[d] & 1 \arrow[d] & \\
 & \mathcal{X}(G) \arrow[d] & \operatorname{Pic} X/\!\!/G \arrow[d] & \\
1 \arrow[r] & \mathrm{H}^1_{\mathrm{alg}}(G, \mathcal{O}(X)^*) \arrow[r] \arrow[d] & \operatorname{Pic}_G X \arrow[r] \arrow[d, "\delta"] & \operatorname{Pic} X \\
 & \mathrm{H}^1(G/G^0, E(X)) & \prod_{x \in \mathcal{C}} \mathcal{X}(G_x) &
\end{tikzcd}
$$

In the product $\prod_{x\in\mathcal{C}} \mathcal{X}(G_x)$ the index set $\mathcal{C} \subset X$ is a set of representatives of the closed orbits in X, and the map δ is obtained by associating to a line bundle L the characters of the isotropy groups G_x on the fibers L_x, $x \in \mathcal{C}$. More precisely, by Remark 4.5, one can take for $\mathcal{C}$ the finite set of points x_i of condition (PB$''$).

5.2. Example. Let M be a connected linear algebraic group and $G \subset M$ a closed subgroup. We assume that G is reductive (although the following is true in general). We know that $E(M) \simeq \mathcal{X}(M)$ (Proposition 1.2) and that the action of M (and hence also of G) on $E(M)$ is trivial (Proposition 1.3ii). From the diagram of Proposition 5.1 we obtain the following exact sequence (cf. [Po74])

$$
\begin{aligned}
1 \to E(M/G) \;&\longrightarrow\; \mathcal{X}(M) \longrightarrow \mathcal{X}(G) \longrightarrow \\
&\longrightarrow\; \operatorname{Pic}(M/G) \longrightarrow \operatorname{Pic}(M) \longrightarrow \operatorname{Pic}(G)
\end{aligned}
$$

where the last map is the restriction from M to G (cf. Lemma 2.2). In particular, if $\operatorname{Pic} M = 0$ then $\operatorname{Pic}(M/G) \simeq \mathcal{X}(G)/\operatorname{Im}(\mathcal{X}(M))$.

Proof of exactness at $\operatorname{Pic}(M)$: Let $L \in \operatorname{Pic}(M)$ such that $L|_G$ is trivial. Recall that $L^* := L \setminus \{\text{zero section}\}$ has the structure of a linear algebraic group such that the projection $p : L^* \to G$ is a group homomorphism with central kernel, isomorphic to k^* (see [LP, Lemma 4.3]). Our assumption implies that the group $p^{-1}(G) \subseteq L^*$ is isomorphic to $G \times k^*$, i.e., G can be considered as a subgroup of L^*. It follows that L admits a G-linearization. But clearly, every G-linearizable bundle on M is a pullback of a line bundle on M/G (Proposition 4.2). □

5.3. Corollary. *Assume that $\mathcal{O}(X)^* = k^*$ and that $X^G \neq \varnothing$. Then, in the following diagram*

$$\begin{tikzcd}
 & & 1 \arrow[d] & \\
 & & \mathrm{Pic}\, X/\!\!/G \arrow[d, "\gamma"] & \\
1 \arrow[r] & \mathcal{X}(G) \arrow[r, "\alpha"] & \mathrm{Pic}_G X \arrow[r, "\beta"] \arrow[d, "\delta"] & \mathrm{Pic}\, X \\
 & & \prod_{x\in C} \mathcal{X}(G_x) &
\end{tikzcd}$$

the row and the column are both exact, and the two compositions $\delta\circ\alpha$ and $\beta\circ\gamma$ are injective. In particular, if $\mathrm{Pic}\, X = 0$, *then* $\mathrm{Pic}(X/\!\!/G) = 0$ *and* $\mathcal{X}(G) \xrightarrow{\sim} \mathrm{Pic}_G X$.

(It clearly suffices to assume that the isotropy groups G_x generate G.)

PROOF: In view of the proposition it remains to prove the assertions concerning the compositions $\delta' := \delta\circ\alpha$ and $\beta' := \beta\circ\gamma$. But δ' is the product of the restrictions of the characters to the subgroups G_x, and the claim follows since there are fixed points by assumption. Now the injectivity of β' follows immediately. □

5.4. Example. For every G-action on an affine space k^n with a fixed point (e.g., for a representation of G) we have $\mathrm{Pic}(k^n/\!\!/G) = 0$ and $\mathrm{Pic}_G(k^n) \simeq \mathcal{X}(G)$.

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