APPENDIX C

Fiber Bundles, Slice Theorem and Applications

Contents

1. Introduction: Local Cross Sections and Slices

Luna's famous Slice Theorem gives a "local description" of an action of a reductive group G on an affine variety. It is modeled on the case of compact transformation groups, but one has to take into account the existence of non-closed orbits. Also one has to modify the concept of "local" and of "open neighborhoods" which make the whole story much more complicated. We first describe the situation of a compact group acting continuously on a nice topological space.

1.1. Free actions and cross sections. Let K be a compact group and let X be a K-*space*, i.e. a Hausdorff topological space with a continuous action of G. Then the orbit space X/G is again Hausdorff and the quotient map $\pi\colon X \to X/G$ is open, closed and proper.

Assume that the point $x \in X$ has a trivial stabilizer. Then one might expect that in a suitable neighborhood of the orbit Kx the action is free and X looks like $K \times U$ where K acts by left multiplication on K. This is indeed the case under very mild assumptions, e.g. if K is a compact Lie group and X is locally compact.

A *cross section* is a continuous map $\sigma\colon X/G \to X$ such that $\pi \circ \sigma$ is the identity on X/G. A *local cross section* defined on $U \subseteq X/G$ is a cross section of $\pi^{-1}(U) \to U$. A first result for compact transformation groups in this setting is the following, see [**Bre72**, Chap. II, Theorem 5.4].

1.1.1. Proposition. *Assume that K is a compact Lie group and that X is locally compact. If $x \in X$ has a trivial stabilizer, $K_x = \{e\}$, then there is a local cross section σ in a neighborhood U of $\pi(x)$ such that $\pi^{-1}(U) \simeq K \times U$. Thus a free action of K on X looks locally like $K \times U$.*

1.1.2. Example. Let us look at an algebraic example. Take the finite group $G = \mathbb{Z}/2$ acting on $X := \mathbb{C}$ by $\pm\,\mathrm{id}$. Then the orbit space X/G can be identified with $\mathbb{C}$ where the quotient map $\pi\colon X \to \mathbb{C}$ is given by $\pi(z) := z^2$. Removing the origin $\{0\} \in X$, the action is free and the quotient $\pi\colon \dot{X} := X \setminus \{0\} \to \dot{\mathbb{C}} := \mathbb{C} \setminus \{0\}$ is a 2-fold covering. This is clearly locally trivial in the $\mathbb{C}$-topology, but not locally trivial in the Zariski-topology. However, looking at the two fiber products

$$\begin{array}{ccc} F = \mathbb{C} \cup \mathbb{C} & \longrightarrow & X \\ \big\downarrow{\scriptstyle\tilde{\pi}} & & \big\downarrow{\scriptstyle\pi} \\ \mathbb{C} & \xrightarrow{z \mapsto z^2} & \mathbb{C} \end{array} \qquad\qquad \begin{array}{ccc} \dot{F} = \dot{\mathbb{C}} \cup \dot{\mathbb{C}} & \longrightarrow & \dot{X} \\ \big\downarrow{\scriptstyle\tilde{\pi}} & & \big\downarrow{\scriptstyle\pi} \\ \dot{\mathbb{C}} & \xrightarrow{z \mapsto z^2} & \dot{\mathbb{C}} \end{array}$$

we find that $F \simeq \mathcal{V}(xy) \subseteq \mathbb{C}^2$, the union of two lines intersecting in the origin, and that $\dot{F}$ is the disjoint union of two copies of $\dot{\mathbb{C}}$, interchanged by G and each one mapped isomorphically to $\dot{\mathbb{C}}$ under $\tilde{\pi}$. Thus the quotient π can be trivialized, not with an open covering of $\dot{\mathbb{C}}$, but with the "étale" surjective map $\dot{\mathbb{C}} \to \dot{\mathbb{C}}, z \mapsto z^2$.

1.2. Associated bundles and slices. Assume again that K is a compact group and X a K-space. What can we say if the action is not free? More precisely, how does X look like in a neighborhood of an orbit $O \simeq K/H$? In order to explain this we make the following construction. Consider an H-space Y and define

$$X := K \times^H Y := (K \times Y)/H$$

where H acts freely on the product $K \times Y$ by $h(g, y) := (gh^{-1}, hy)$. We will denote the orbit of (g, y) by $[g, y] \in K \times^H Y$. This space is called *twisted product* or *associated bundle*. It has a number of remarkable properties. First of all, we have an action of K on $K \times^H Y$

induced by the left multiplication on K: $g'[g,y] := [g'g,y]$. Then, there is a natural closed embedding $Y \hookrightarrow K \times^H Y$, $y \mapsto [e,y]$.

1.2.1. PROPOSITION. (1) *There is a canonical bijection between the K-orbits in $K \times^H Y$ and the H-orbits in Y given by $O = K[g,y] \mapsto Hy = O \cap Y$. This map induces a homeomorphism of orbit spaces $(G \times^H Y)/G \xrightarrow{\sim} Y/H$, the inverse map is given by $Hy \mapsto G[e,y]$.*

(2) *The projection $K \times Y \to K$ induces a K-equivariant map $p\colon K \times^H Y \to K/H$ which is a locally trivial bundle with fiber Y: $p^{-1}(gH) = gY$.*

Except for the last statement, the proofs are easy exercises and are left to the reader. For the last statement, one has to use the fact that the projection $K \to K/H$ admits local cross sections.

1.2.2. EXAMPLE. Let us give again an algebraic example. Take $G := \mathbb{C}^*$ and $H := \{\pm 1\} \subseteq \mathbb{C}^*$, and consider the action of H on $Y := \mathbb{C}$ by $\pm\,\mathrm{id}$ as in the example above. Then the associated bundle $G \times^H Y$ has the following description:

$$\mathbb{C}^* \times^H \mathbb{C} \xrightarrow{\sim} \mathbb{C}^* \times \mathbb{C}, \ [t,z] \mapsto (t^2, tz),$$

the $\mathbb{C}^*$-action on $\mathbb{C}^* \times \mathbb{C}$ is given by $t(s,x) = (t^2 s, tx)$, and the closed embedding $\mathbb{C} \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ is $z \mapsto (1,z)$. Thus $\mathbb{C}^*(s,x) \cap \mathbb{C} = \{\pm x\}$ and $(\mathbb{C}^* \times \mathbb{C})/\mathbb{C}^* \simeq \mathbb{C}$ where the quotient map $\pi\colon \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$ is given by $(s,x) \mapsto x^2$. Finally, $p\colon \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^*/H \simeq \mathbb{C}^*$ is the projection $\mathrm{pr}_{\mathbb{C}^*}$ and so p is a trivial bundle with fiber $\mathbb{C}$.

1.2.3. REMARK. There is an easy criterion to show that a given K-space X is an associated bundle. Assume that there is a K-equivariant map $p\colon X \to K/H$ with some closed subgroup $H \subseteq K$. Then $Y := p^{-1}(eH)$ is an H-space, and we have a canonical homeomorphism

$$\varphi\colon K \times^H Y \xrightarrow{\sim} X, \ [g,y] \mapsto gy.$$

In fact, φ is continuous and bijective, and the inverse map is given by $x \mapsto [p(x), \widetilde{p(x)}^{-1} x]$ where $\widetilde{p(x)} \in K$ is a representative of $p(x)$. We use here again the fact that $K \to K/H$ has local cross sections.

Now we can formulate the local structure theorem for actions of compact groups, see [**Bre72**, Chap. II, Theorem 5.4].

1.2.4. THEOREM. *Let K be a compact Lie groups and X a locally compact K-space. For any $x \in X$ there is a locally closed and K_x-stable subset $S \subseteq X$ containing x such that*

(1) *KS is an open neighborhood of Kx,*
(2) *$K \times^{K_x} Y \to KY$, $[g,y] \mapsto gy$, is a homeomorphism.*

Such an $S \subseteq X$ is called a *slice in x*, and KS is called a *tube about Kx*. The theorem together with Proposition 1.2.1 above tells us that the action of K in a neighborhood of an orbit $O = Kx$ is completely determined by the action of H_x on a slice in x.

2. Flat and Étale Morphisms

In this section we discuss the concept of "local" in algebraic geometry. Since there are no "small" open neighborhoods in the Zariski-topology we will replace them by so-called "étale neighborhoods". For this we have to define étale morphisms and to describe their basic properties. In the smooth case, a morphism is étale in a point if and only if its differential is an isomorphism. In general, one has to ask in addition that the morphism is flat.

In this section, we will use some results from the literature, and we refer to [**Har77**, III.9], [**Mat89**, 3.7 and 8], and [**Eis95**, Section 6] for more details and proofs. Our approach is based on "standard étale morphisms" (Example 2.1.13)

2.1. Unramified and étale morphisms. Let $\varphi\colon X \to Y$ be a morphism, let $x \in X$ and set $y := \varphi(x) \in Y$. Then the morphism φ induces a homomorphism $\varphi^*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ of local rings, i.e. $\varphi^*(\mathfrak{m}_y) \subseteq \mathfrak{m}_x$.

2.1.1. Definition. The morphism φ is *unramified in* $x \in X$ if $\mathfrak{m}_x = \varphi^*(\mathfrak{m}_y)\mathcal{O}_{X,x}$. More geometrically, this means that x is an isolated point of the fiber $F := \varphi^{-1}(y)$ and F is reduced in x.

Recall that the differential $d\varphi_x\colon T_xX \to T_yY$ vanishes on $T_xF \subseteq T_xX$, and that $T_xF = \ker d\varphi_x$ in case the fiber is reduced in x. It follows that φ is unramified in x if and only if the differential $d\varphi_x$ is injective. A immediate consequence is that an unramified morphism $\varphi\colon X \to Y$ has finite reduced fibers.

2.1.2. Exercise. Show that the subset $\{x \in X \mid \varphi \text{ is unramified in } x\} \subseteq X$ is open. (Hint:)

Another important concept is flatness. It will play a central rôle in all what follows. Unfortunately, there is no easy "geometric meaning" of flatness; it is a purely algebraic concept.

2.1.3. Definition. If R is a ring, then an R-module M is called *flat* if the functor $N \mapsto N \otimes_R M$, N an R-module, is left exact. A morphism $\varphi\colon X \to Y$ is called *flat in* $x \in X$ if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,\varphi(x)}$-module (with respect to $\varphi^*\colon \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$).

We have the following "Local Criterion for Flatness", see [**Eis95**, Theorem 6.8].

2.1.4. Lemma. *Let $\varphi\colon X \to Y$ be a morphism, let $x \in X$ and set $y := \varphi(x) \in Y$. Then φ is flat in x if and only if the map $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is injective.*

2.1.5. Exercise. Show that the projection $\mathrm{pr}_Y\colon X \times Y \to Y$ is flat.

2.1.6. Exercise. If $\varphi\colon X \to Y$ is flat in $x \in X$, then $\varphi^*\colon \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is injective. (Hint: For $h \in \mathfrak{m}_{\varphi(x)}$ denote by $\mathfrak{a}_h \subseteq \mathcal{O}_{Y,\varphi(x)}$ the kernel of $\mu_h : f \mapsto hf$. Then we get an exact sequence $0 \to \mathfrak{a}_h \otimes_{\mathcal{O}_{Y,\varphi(x)}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \xrightarrow{\mu_h} \mathcal{O}_{X,x}$. Hence $\mu_h|_{\mathcal{O}_{X,x}} = 0$ if and only if $h = 0$.)

Finally, we define étale morphisms which will be the algebraic-geometric replacement for local isomorphisms.

2.1.7. Definition. The morphism $\varphi\colon X \to Y$ is *étale* in $x \in X$ if φ is unramified and flat in x. Equivalently, φ^* induces an isomorphism $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \xrightarrow{\sim} \mathfrak{m}_x$ where $y := \varphi(x)$.

2.1.8. Examples.
(1) An open immersion $X \hookrightarrow Y$ is étale.
(This is clear since $\mathcal{O}_{X,x} = \mathcal{O}_{Y,x}$ for all $x \in X$.)
(2) If $\varphi\colon X \to Y$ is étale in $x \in X$, then the differential $d\varphi_x\colon T_xX \to T_{\varphi(x)}Y$ is an isomorphism.
(Since $\mathfrak{m}_y^n \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \xrightarrow{\sim} \mathfrak{m}_x^n$ for all n (see the following exercise) it follows that $\mathfrak{m}_y/\mathfrak{m}_y^2 \xrightarrow{\sim} \mathfrak{m}_x/\mathfrak{m}_x^2$ is an isomorphism.)
(3) If $\varphi\colon X \to Y$ is étale in $x \in X$ and $y := \varphi(x)$, then X is smooth in x if and only if Y is smooth in y.
(The following exercise implies that the canonical maps $\mathfrak{m}_y^n/\mathfrak{m}_y^{n+1} \to \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ are isomorphisms for all $n \geq 0$. Hence $\mathrm{gr}_{\mathfrak{m}_y} \mathcal{O}_{Y,y} \simeq \mathrm{gr}_{\mathfrak{m}_x} \mathcal{O}_{X,x}$, and the claim follows from Theorem A.4.10.1.)

2.1.9. Exercise. If $\varphi\colon X \to Y$ is étale in $x \in X$, then the maps $\mathfrak{m}_y^n \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathfrak{m}_x^n$ are isomorphisms for all $n \geq 0$.

In case X and Y are smooth, there is simple criterion for φ to be étale, see [**Har77**, III. Proposition 10.4 and Exercise 10.3].

2.1.10. Proposition. *Assume that X is smooth in x and Y is smooth in y. Then φ is étale in x if and only if the differential $d\varphi_x\colon T_xX \to T_yY$ is an isomorphism.*

Using the implicit function theorem it follows that an étale morphism between smooth varieties is a local homeomorphism in the $\mathbb{C}$-topology. We will see that this holds in general for any étale morphism, as a consequence of Proposition 2.1.14.

Let us recall some basic properties of flat and étale morphisms. We refer to [**Har77**, III.9], [**Mat89**, 3.7 and 8], and [**Eis95**, Section 6] for more details and proofs.

2.1.11. Lemma.
(1) *Let $\psi\colon X \xrightarrow{\eta} Y \xrightarrow{\varphi} Z$ be a composition. If η and φ are flat (resp. étale), then ψ is flat (resp. étale). If ψ and η are flat (resp. étale) and η is surjective, then φ is flat (resp. étale).*
(2) *If $\varphi\colon X \to Y$ is flat in $x \in X$, then $\varphi^*\colon \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is injective, and for every ideal $\mathfrak{a} \subseteq \mathcal{O}_{Y,\varphi(x)}$ we have $\mathfrak{a}\mathcal{O}_{X,x} \cap \mathcal{O}_{Y,\varphi(x)} = \mathfrak{a}$. In particular, $\mathcal{O}_{X,x}/\mathfrak{a}\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,\varphi(x)}/\mathfrak{a}$ and $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,\varphi(x)}$.*
(3) *For an arbitrary morphism $\varphi\colon X \to Y$ the set of points $x \in X$ where φ is flat (resp. étale) is open in X.*
(4) *A flat morphism $\varphi\colon X \to Y$ is open and equidimensional, i.e., if φ is flat in $x \in X$, then $\dim_x X = \dim_{\varphi(x)} Y + \dim_x \varphi^{-1}(\varphi(x))$.*

Proof. (1) This is an easy exercise which we leave to the reader.

(2) This follows immediately from the definition, see [**Mat89**, Theorem 7.5].

(3) For flatness this is [**Mat89**, Theorem 24.3]. For the étaleness one remarks that the set of points $x \in X$ where the differential $d\varphi_x$ is injective is open, see Exercise 2.1.2.)

(4) See [**Har77**, Chap. III, Exercise 9.1 and Proposition 9.5] or [**Mat89**, Theorem 15.1]. □

A morphism $\varphi\colon X \to Y$ is called *faithfully flat* if it is flat and surjective. If X and Y are affine this is equivalent to the following condition: *A homomorphism $N \to M$ of $\mathcal{O}(Y)$-modules is injective if and only if $\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} N \to \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} M$ is injective.* Here is a useful application, on the level of rings.

2.1.12. Lemma. *Let A be a ring, and let R be an A-algebra. Let B/A be faithfully flat and assume that $B \otimes_A R$ is a finitely generated B-algebra. Then R is finitely generated over A.*

This behavior is usually expressed in the following way. *If an A-algebra R becomes finitely generated under a faithfully flat base change, then R is finitely generated.* We might ask here which other properties of an A-algebra behave in a similar way. E.g. being an integral domain or being reduced are such properties.

Proof. The ring R is the union of finitely generated A-subalgebras R_ν. Since the tensor product commutes with direct limits, $\varinjlim(B \otimes_A R_\nu) \xrightarrow{\sim} B \otimes_A \varinjlim R_\nu = B \otimes_A R$, there is a ν such that $B \otimes_A R_\nu \xrightarrow{\sim} B \otimes_A R$. Since B/A is faithfully flat, this implies that $R_\nu = R$. □

The following example gives a general construction of an étale morphism reflecting what we usually have in mind. Unfortunately, the proof is not easy and needs some work.

2.1.13. Example (standard étale morphism). Let U be an affine variety, and let $F \in \mathcal{O}(U)[t]$ be a monic polynomial. Then the projection onto U induces a morphism $\eta\colon \mathcal{V}_{U\times\mathbb{C}}(F) \to U$, and the following holds:

(1) *The morphism η is étale in any $(u,a) \in \mathcal{V}_{U\times\mathbb{C}}(F)$ such that $F'(u,a) \neq 0$, where $F' := \frac{dF}{dt} \in \mathcal{O}(U)[t]$.*

(2) *Define $Z := \mathcal{V}_{U\times\mathbb{C}}(F)_{F'}$. Then $\mathcal{O}(U)[t]_{F'}/(F) \xrightarrow{\sim} \mathcal{O}(Z)$ is an isomorphism, i.e. the ideal $(F) \subseteq \mathcal{O}(U)[t]_{F'}$ is perfect.*

Proof. (a) The morphism $p\colon \mathcal{V}_{U\times\mathbb{C}}(F) \to U$ is finite and surjective, and $Z \subseteq \mathcal{V}_{U\times\mathbb{C}}(F)$ is open. Set $R := \mathcal{O}(U)[t]_{F'}/(F)$ so that $\mathcal{O}(Z) = R/\sqrt{(0)}$. For any $u \in p(Z)$ we get $R/\mathfrak{m}_u R = \mathbb{C}[t]/(F(u,t)$ and this is a product of copies of $\mathbb{C}$. It follows that $R/\mathfrak{m}_u R = \mathcal{O}(Z)/\mathfrak{m}_u\mathcal{O}(Z)$. Hence $p\colon Z \to U$ has discrete and reduced fibers, and so $p\colon Z \to U$ is unramified. Clearly, R is flat over $\mathcal{O}(U)$. So if we show that R is reduced, then (2) follows, and $\eta\colon Z \to U$ is flat, hence (1).

(b) From (b) we see that $\tilde{\mathfrak{m}}_z := \mathfrak{m}_z R \subseteq R$ is a maximal ideal, and that we get surjective homomorphisms

$$\mathfrak{m}_{p(z)}/\mathfrak{m}_{p(z)}^2 \twoheadrightarrow \mathfrak{m}_z/\mathfrak{m}_z^2 \twoheadrightarrow \tilde{\mathfrak{m}}_z/\tilde{\mathfrak{m}}_z^2.$$

This implies that $R_{\tilde{\mathfrak{m}}_z}$ is a regular local ring in case $p(z) \in U$ is a smooth point. Hence $R_{\tilde{\mathfrak{m}}_z} = \mathcal{O}_{Z,z}$, because a regular local ring is an integral domain.

(c) Now we look at the canonical map $\varphi\colon R \to \prod_{z\in Z'} R_{\tilde{\mathfrak{m}}_z}$ where $Z' := \{z \in Z \mid p(z)$ smooth in $U\}$. We want to show that φ is injective which implies that R is reduced. If $r \in \ker\varphi$, then, for every $z \in Z'$, there is an $s_z \notin \tilde{\mathfrak{m}}_z$ such that $s_z r = 0$. This implies that $\operatorname{Ann}(r) \nsubseteq \tilde{\mathfrak{m}}_z$ for all $z \in Z'$. If $r \neq 0$, then $\operatorname{Ann}(r)$ is contained in an associated prime of R. Since every irreducible component of Z contains smooth points, it follows that every minimal prime of R is contained in $\tilde{\mathfrak{m}}_z$ for some $z \in Z'$. So it remains to see that R has no embedded primes, i.e. every zero divisor is contained in a minimal prime.

(d) It suffices to prove this for the algebra $A := \mathcal{O}(U)[t]/(F)$. Let $\mathfrak{p} \subseteq A$ be an associated prime which is not minimal, and let $\mathfrak{p}' \subseteq \mathfrak{p}$ be a minimal prime. Then $\mathfrak{p}' \cap \mathcal{O}(U) \subsetneqq \mathfrak{p} \cap \mathcal{O}(U)$. If $a \in \mathfrak{p} \cap \mathcal{O}(U) \setminus \mathfrak{p}' \cap \mathcal{O}(U)$, then multiplication with a is injective on $\mathcal{O}(U)$, but has a kernel on A. This contradicts the fact that A is flat over $\mathcal{O}(U)$. $\square$

A morphism of the form $\eta\colon Z \to U$ as above is called *a standard étale morphism.* These morphisms have many nice properties, e.g. *a standard étale morphism is a local homeomorphism in the* $\mathbb{C}$*-topology.* In fact, this is obvious for $U = \mathbb{C}^n$ by the implicit function theorem, and using a closed embedding $U \hookrightarrow \mathbb{C}^n$ one gets a fiber product of the form

$$\begin{array}{ccc}
\mathcal{V}_{U\times\mathbb{C}}(F)_{F'} & \xrightarrow{\subseteq} & \mathcal{V}_{\mathbb{C}^n\times\mathbb{C}}(\tilde{F})_{\tilde{F}'} \\
\big\downarrow & & \big\downarrow \\
U & \xrightarrow{\subseteq} & \mathbb{C}^n
\end{array}$$

where $\tilde{F} \in \mathcal{O}(\mathbb{C}^n)[t]$ is a lift of $F \in \mathcal{O}(U)[t]$. Another point is the following. If $F \in \mathcal{O}(U)[t]$ has degree d as a polynomial in t, then the standard étale morphism $\eta\colon \mathcal{V}_{U\times\mathbb{C}}(F)_{F'} \to U$ has also degree d. In particular, if η is injective, then $d = 1$, hence F is linear, and so η is an open immersion. We will see below that this holds for every étale morphism.

The next result shows that every étale morphism is "locally standard".

2.1.14. Proposition. *Let $\varphi\colon X \to Y$ be a morphism, and assume that φ is étale in $x_0 \in X$. Then there is an affine open neighborhood U of $\varphi(x_0)$, a standard étale morphism*

$\eta\colon Z \to U$ and an open immersion of a neighborhood V of x_0 into Z such that $\varphi(V) \subseteq U$ and $\varphi|_V = \eta|_V\colon V \to U$:

$$\begin{tikzcd} X & V \arrow[l, "\supseteq"', "open"] \arrow[r, "\subseteq", "open"'] \arrow[d, "\eta|_V", "\varphi|_V ="'] & Z \arrow[d, "\eta"] \\ Y & U \arrow[l, "\supseteq"', "open"] \arrow[r, equal] & U \\ X \arrow[from=1-1, to=2-1, "\varphi"] \end{tikzcd}$$

Let us first recall the following "Local Criterion for Flatness", see [**Eis95**, Theorem 6.8].

2.1.15. LEMMA. *Let $\varphi\colon X \to Y$ be a morphism, let $x \in X$ and set $y := \varphi(x) \in Y$. Then φ is flat in x if and only if the map $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is injective.*

As a consequence we get the next lemma which will be used in the final part of the proof of Proposition 2.1.14.

2.1.16. LEMMA. *Let $\mu\colon Z_1 \to Z_2$ and $\eta_2\colon Z_2 \to U$ be morphisms. Assume that $\eta_1 := \eta_2 \circ \mu$ is flat in $z_1 \in Z_1$ and that η_2 is étale in $z_2 := \mu(z_1)$. Then μ is flat in z_1.*

$$\begin{tikzcd} Z_1 \arrow[r, "\mu"] \arrow[dr, "\eta_1"'] & Z_2 \arrow[d, "\eta_2"] \\ & U \end{tikzcd}$$

If η_1 is étale in z_1 (and η_2 étale in z_2), then μ is étale in z_1.

PROOF. Since η_2 is étale in z_2, we get $\mathfrak{m}_u \mathcal{O}_{Z_2,z_2} = \mathfrak{m}_{z_2}$ where $u := \eta_2(z_2)$. It follows that the first map in the composition

$$\mathfrak{m}_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{Z_1,z_1} \to \mathfrak{m}_{z_2} \otimes_{\mathcal{O}_{Z_2,z_2}} \mathcal{O}_{Z_1,x_1} \to \mathcal{O}_{Z_1,z_1}$$

is surjective. Since η_1 is flat in z_1 the composition is injective, hence the second map is injective, and this implies, by the lemma above, that μ is flat in z_1. The second claim follows, because μ is unramified in z_1 in case η_1 is unramified in z_1. $\square$

2.1.17. REMARK. Lemma 2.1.15 has the following generalization, see [**Mat89**, Theorem 22.3]. *Let $\varphi\colon X \to Y$ be a morphism, let $x \in X$ and set $y := \varphi(x) \in Y$, and let $I \subseteq \mathcal{O}_{Y,y}$ be an ideal. Then φ is flat in x if and only if the following holds: (i) $\mathcal{O}_{X,x}/I\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}/I$, and (ii) the map $I \otimes \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is injective.*
This has the following nice application, generalizing Lemma 2.1.16.

2.1.18. PROPOSITION. *Consider the diagram*

$$\begin{tikzcd} X_1 \arrow[r, "\mu"] \arrow[dr, "\varphi_1"'] & X_2 \arrow[d, "\varphi_2"] \\ & Y \end{tikzcd}$$

where φ_1 is flat. Assume that for every $y \in Y$ the induced morphism of the (schematic) fibers $\varphi_1^{-1}(y) \to \varphi_2^{-1}(y)$ is flat. Then μ is flat.

PROOF. Choose $x_1 \in X_1$, and put $x_2 := \eta(x_1)$ and $y := \varphi_1(x_1) = \varphi_2(x_2)$. Set $I := \mathfrak{m}_y\mathcal{O}_{X_2,x_2} \subseteq \mathcal{O}_{X_2,x_2}$. Then the local ring of the schematic fiber $\varphi_2^{-1}(y)$ in x_1 is $\mathcal{O}_{X_1,x_1}/I\mathcal{O}_{X_1,x_1}$ which is flat over the local ring $\mathcal{O}_{X_2,x_2}/I$ of the schematic fiber $\varphi_2^{-1}(y)$ in x_2, by assumption. Moreover, $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X_1,x_1} \to I \otimes_{\mathcal{O}_{X_2,x_2}} \mathcal{O}_{X_1,x_1}$ is surjective, and the composition with $\iota\colon I \otimes_{\mathcal{O}_{X_2,x_2}} \mathcal{O}_{X_1,x_1} \to \mathcal{O}_{X_1,x_1}$ is injective, because φ_1 is flat in x_1. Thus ι is injective, and the claim follows from the remark above. $\square$

Proof of Proposition 2.1.14. We can assume that every irreducible component of X contains x_0.

(a) There is an open embedding $X \hookrightarrow \tilde{X}$ and a finite morphism $\tilde{\varphi}\colon \tilde{X} \to Y$ such that $\tilde{\varphi}|_X = \varphi$. Thus we can assume that φ is finite and surjective.

(b) There exists an affine open neighborhood $U \subseteq Y$ of $y_0 := \varphi(x_0)$ and a closed embedding $\rho\colon V := \varphi^{-1}(U) \hookrightarrow U \times \mathbb{C}$ of the form $x \mapsto (\varphi(x), h(x))$ where $h(x_0) = 1$.

(c) There is an $F \in \mathcal{O}(U)[t]$ with the following properties: (i) F vanishes on the image of Y; (ii) $F'(y_0, 1) \neq 0$; (iii) the leading term of F does not vanish in y_0. Localizing U at the leading term of F we can assume that F is monic.

Now we can finish the proof. By (c) we have a closed immersion $V \hookrightarrow \mathcal{V}_{U\times\mathbb{C}}(F)$. Since $F'(y_0, 1) \neq 0$ we can replace V by the open set $V' = V \cap \mathcal{V}_{U\times\mathbb{C}}(F)_{F'}$ containing x_0, and we get a closed immersion $V' \hookrightarrow \mathcal{V}_{U\times\mathbb{C}}(F)_{F'}$. Moreover, the induced morphism $\mathcal{V}_{U\times\mathbb{C}}(F)_{F'} \to U$ is a standard étale map. Thus we are in the situation of Lemma 2.1.16 which implies that the image of V' is open in $\mathcal{V}_{U\times\mathbb{C}}(F)_{F'}$. □

Let us draw some important consequences.

2.1.19. Proposition. (1) *Consider the following fiber product.*

$$\begin{tikzcd}
U \times_Y X \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{\varphi}"] & X \arrow[d, "\varphi"] \\
U \arrow[r, "\eta"] & Y
\end{tikzcd}$$

If η is étale, then the fiber product is reduced and $\tilde{\eta}$ is étale.

(2) *An injective étale morphism is an open immersion.*

Proof. (1) We can assume that X, Y and U are affine. If $X \to Y$ is a standard étale morphism, $\mathcal{O}(X) = \mathcal{O}(Y)[t]_{F'}/(F)$ where $F \in \mathcal{O}(Y)[t]$ is a monic polynomial, then $\mathcal{O}(U) \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \simeq \mathcal{O}(U)[t]_{G'}/(G)$ where $G = \eta^*(F)$, hence $U \times_Y X$ is reduced and $U \times_Y X \to U$ is also a standard étale morphism. Now the claim follows from Proposition 2.1.14 above.

(2) We have seen above that an injective standard étale morphism is an open immersion. Hence the claim follows from Proposition 2.1.14 □

2.2. Étale base change. The situation of the first statement of Proposition 2.1.19 above is a special case of the following setup. Let S be a variety, let $p\colon X \to S$ and $q\colon Y \to S$ two S-varieties, and let $\varphi\colon X \to Y$ be an S-morphism, i.e. $q \circ \varphi = p$. If $\eta\colon S' \to S$ is a morphism we obtain S'-varieties $X' := S' \times_S X$ and $Y' := S' \times_S Y$ and an induced S'-morphism $\varphi'\colon X' \to Y'$, as shown in the following diagram:

(∗)

$$\begin{tikzcd}
X' \arrow[rr, "\eta_X"] \arrow[dr, "\varphi'"] \arrow[ddr, "p'"'] & & X \arrow[dr, "\varphi"] \arrow[ddr, "p"'] & \\
& Y' \arrow[rr, "\eta_Y"] \arrow[d, "q'"] & & Y \arrow[d, "q"] \\
& S' \arrow[rr, "\eta"] & & S
\end{tikzcd}$$

This is usually expressed by saying that $\varphi'\colon X' \to Y'$ *is obtained from* $\varphi\colon X \to Y$ *by the base change* $\eta\colon S' \to S$. A basic question is what happens in case of a flat or étale base change. E.g. the first statement of Proposition 2.1.19 above says that for a an étale base change $\eta\colon S' \to S$, the fiber products X' and Y' are reduced and η_X, η_Y are again étale.

We will have more statements of this form later, but let us first prove the following useful result.

2.2.1. LEMMA. *Let $\varphi\colon X \to Y$ be an abstract map between varieties. If $\eta\colon X' \to X$ is an étale and surjective morphism such that the composition $\varphi \circ \eta$ is a morphism, then φ is a morphism.*

$$\begin{tikzcd} X' \arrow[r, "\eta"] \arrow[dr, "\eta\circ\varphi"'] & X \arrow[d, "\varphi"] \\ & Y \end{tikzcd}$$

PROOF. Denote by $\Gamma_\varphi \subseteq X \times Y$ the graph of the map φ. We have to show that Γ_φ is closed and that the induced map $p\colon \Gamma_\varphi \to X$ is an isomorphism. By assumption, the composition $\psi := \eta \circ \varphi$ is a morphism, and we get the following commutative diagram:

$$(12)\qquad \begin{tikzcd} \Gamma_\psi \arrow[r, "\subseteq"] \arrow[d, "\gamma:=", "(\eta\times\mathrm{id}_Y)|_{\Gamma_\psi}"'] & X' \times Y \arrow[r, "\mathrm{pr}"] \arrow[d, "\eta\times\mathrm{id}_Y"] & X' \arrow[d, "\eta"] \\ \Gamma_\varphi \arrow[r, "\subseteq"] & X \times Y \arrow[r, "\mathrm{pr}"] & X \end{tikzcd}$$

Since η is surjective we see that $(\eta \times \mathrm{id}_Y)^{-1}(\Gamma_{\varphi)} = \Gamma_\psi$. It follows that $X \times Y \setminus \Gamma_\varphi$ is the image of the open set $X' \times Y \setminus \Gamma_\psi$ which is open, because $\eta \times \mathrm{id}_Y$ is flat. Hence Γ_φ is closed. Now the outer diagram of (12) is a fiber product, hence γ is étale and surjective, and the induced horizontal map $\Gamma_\psi \to X'$ is an isomorphism. Therefore, $\Gamma_\varphi \to X$ is a bijective étale morphism, by Lemma 2.1.11(1), and thus an isomorphism, by Proposition 2.1.19(2). $\square$

2.2.2. EXAMPLES.
(1) If an S-variety X becomes smooth under an étale surjective base change $S' \to S$, then X is also smooth (see Example 2.1.8(3)).
(2) If an S-morphism $\varphi\colon X \to Y$ becomes an isomorphism under an étale surjective base change $S' \to S$, then φ is an isomorphism. (This follows from the lemma above applied to the map φ^{-1}.)

The next example is a very special case of the Slice Theorem for finite groups.

2.2.3. EXAMPLE. Let G be a finite group acting on the affine variety X, and denote by $\pi\colon X \to X/G$ the quotient. Define $X' := \{x \in X \mid G_x = \{e\}\}$. Then

(1) *X' is open in X and $\pi(X')$ is open in X/G.*
(2) *The map $(g, x) \mapsto (x, gx)\colon G \times X' \to X' \times_{\pi(X')} X'$ is a G-equivariant isomorphism:*

$$\begin{tikzcd} G \times X' \arrow[r, "\simeq", "(g,x)\mapsto(x,gx)"'] \arrow[d, "\mathrm{pr}_{X'}"] & X' \times_{\pi(X')} X' \arrow[r] \arrow[d, "p"] & X' \arrow[d, "\pi|_{X'}"] \\ X' \arrow[r, equal] & X' \arrow[r, "\pi|_{X'}"] & X/G \end{tikzcd}$$

(3) *The induce morphism $\pi|_{X'}\colon X' \to X/G$ is étale.*

PROOF. (1) The first statement is clear since $X \setminus X' = \bigcup_{g\neq e} X^g$.

(2) For any $g \in G$ the morphism $\iota_g\colon X' \to X' \times_{\pi(X')} X'$, $x \mapsto (x, gx)$, is a closed immersion, because $p \circ \iota_g = \mathrm{id}_{X'}$. Hence the fiber product $X' \times_{\pi(X')} X'$ is the disjoint union of copies of X', proving (2).

(3) For the last statement we can embed X as a closed G-stable subset into a representation V of G and thus assume that $X = V$. The following argument was indicated to

us by G.W. SCHWARZ. We claim that for any $x \in V'$ there exist $n := \dim V$ invariant functions $p_1, \dots, p_n$ vanishing in x such that the differentials $(dp_1)_x, \dots, (dp_n)_x$ form a basis of the cotangent space $(T_x V)^* = \mathfrak{m}_x/\mathfrak{m}_x^2$. In fact, the following Exercise 2.2.5 shows that for a given cotangent vector $\xi \in (T_x V)^*$ there is an $f \in \mathcal{O}(V)$ vanishing on Gx such that $df_{gx} = g\xi$ for all $g \in G$. It is easy to that gf has the same property for all $g \in G$ and so the invariant $p := 1/|G| \sum_g gf$ vanishes at x and satisfies $dp_x = \xi$.

It follows that the G-invariant morphism $p := (p_1, \dots, p_n) \colon V \to \mathbb{C}^n$ is unramified in gx for all $g \in G$. Replacing V by a suitable G-stable affine open neighborhood U of x we can assume that the fiber $p^{-1}(p(x))$ is equal to Gx and is therefore reduced. This means that the ideal $I(Gx) \subseteq \mathcal{O}(U)$ is generated by the invariants $p_1, \dots, p_n$. But then, the maximal idea $\mathfrak{m}_{\pi(x)} = I(Gx) \cap \mathcal{O}(U)^G$ is also generated by $p_1, \dots, p_n$, showing that $\pi(x)$ is a smooth point of U/G and that $d\pi_x \colon T_x U \to T_{\pi(x)} U/G$ is an isomorphism. Now the claim follows from Proposition 2.1.10. $\square$

2.2.4. EXERCISE. (1) Let $\mathfrak{a} \subseteq \mathcal{O}(X)$ be an ideal. For any $x \notin \mathcal{V}_X(\mathfrak{a})$ we have $\mathfrak{m}_x \cap \mathfrak{a} + \mathfrak{m}_x^2 = \mathfrak{m}_x$, i.e. the map $\mathfrak{m}_x \cap \mathfrak{a} \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is surjective.

(2) Let $x_1, \dots, x_n \in X$ be n different points. Then the canonical map
$$\mathfrak{m}_{x_1} \cap \mathfrak{m}_{x_2} \cap \dots \cap \mathfrak{m}_{x_n} \to \bigoplus_i \mathfrak{m}_{x_i}/\mathfrak{m}_{x_i}^2$$
is surjective.
(Hint: Use (1) with $\mathfrak{a} := \mathfrak{m}_{x_2}^2 \cap \dots \cap \mathfrak{m}_{x_n}^2$ to show that the image of this map contains $\mathfrak{m}_{x_1}/\mathfrak{m}_{x_1}^2 \oplus (0) \cdots \oplus (0)$.)

2.2.5. EXERCISE. Use the previous exercise to show that for a finite set of points $x_1, \dots, x_n \in X$ and cotangent vectors $\xi_i \in (T_{x_i} X)^*$ there is an $f \in \mathcal{O}(X)$ such that $df_{x_i} = \xi_i$ for all $i = 1, \dots n$.

3. Fiber Bundles and Principal Bundles

Fiber bundles with fiber F are morphisms $\varphi \colon B \to X$ which look locally like $U \times F$. In order to get a useful concept, one has to replace the ZARISKI-open neighborhoods of a point $x \in X$ by *étale neighborhoods* which are defined as étale morphisms $\eta \colon U \to X$ such that $x \in \eta(U)$. One can define intersections of étale neighborhoods by taking the fiber product, and one can even introduce an *étale topology*.

3.1. Additional structures, s-varieties.

In many applications we are dealing with varieties with an additional structure, shortly s-varieties. E.g. a vector space, a quadratic space (i.e. a vector space with a nondegenerate quadratic form), an affine space, a G-variety (i.e. a variety with an action of an algebraic group G), or a G-module. We will not give a formal definition, but we will need the fact that it is always clear what an isomorphism between two such s-varieties is. In particular, for every s-variety F the automorphism group $\operatorname{Aut}(F)$ is a well-defined subgroup of $\operatorname{Aut}(|F|)$ where $|F|$ denotes the underlying variety.

In the examples above, we see that $\operatorname{Aut}(F) \subseteq \operatorname{Aut}(|F|)$ is a closed subgroup in case $|F|$ is affine. E.g., for a vector space V we have $\operatorname{Aut}(V) = \operatorname{GL}(V)$, for a quadratic space (Q, q) we have $\operatorname{Aut}(Q, q) = O(Q, q)$, and for an affine space A we have get $\operatorname{Aut}(A) = \operatorname{Aff}(A)$, the group of affine transformations. For an affine G-variety X we have $\operatorname{Aut}(X) = \operatorname{Aut}_G(|X|) = \operatorname{Aut}(|X|)^G$, the group of G-equivariant automorphisms of X, and for a G-module M we get $\operatorname{Aut}(M) = \operatorname{GL}(M)^G$.

3.1.1. REMARK. In many cases, the s-variety F is determined by the pair $(|F|, \operatorname{Aut}(F))$. This means the following: F is isomorphic to E if and only if there is an isomorphism $\varphi \colon |F| \xrightarrow{\sim} |E|$ which defines an isomorphism $\operatorname{Aut}(F) \xrightarrow{\sim} \operatorname{Aut}(E)$ by $g \mapsto \varphi \circ g \circ \varphi^{-1}$. A necessary and sufficient condition for this is that $\mathrm{Aut}(F) \subseteq \mathrm{Aut}(|F|)$ is self-normalizing. As an exercise, the interested reader might check that the following subgroups of $\mathrm{Aut}(\mathbb{C}^n)$ are self-normalizing: $\mathrm{GL}_n, \mathrm{O}_n, \mathbb{C}^*$.

3.1.2. Lemma. *Let G be a reductive group acting on an affine variety X. If there are no nonconstant invariants, then* $\mathrm{End}_G(X)$ *and* $\mathrm{Dom}_G(X)$ *are affine algebraic semi-groups, and* $\mathrm{Aut}_G(X)$ *is an affine algebraic group. Moreover,* $\mathrm{Aut}_G(X)$ *is closed in* $\mathrm{Dom}_G(X)$ *and* $\mathrm{Dom}_G(X)$ *is open in* $\mathrm{End}_G(X)$.

(Here $\mathrm{Dom}(X)$ denotes the semigroups of dominant endomorphisms.)

Proof. First it is clear that $\mathrm{End}_G(X) \subseteq \mathrm{End}(X)$ and $\mathrm{Aut}_G(X) \subseteq \mathrm{Aut}(X)$ are both closed. Since there are no invariants the isotypic components of $\mathcal{O}(X)$ are finite dimensional. This implies that we can find a finite direct sum $W \subseteq \mathcal{O}(X)$ of isotypic components of $\mathcal{O}(X)$ which generates $\mathcal{O}(X)$. Thus, we get an injective morphism $\iota\colon \mathrm{End}_G(X) \hookrightarrow \mathrm{End}_G(W)$, and a commutative diagram

$$\begin{array}{ccccc} \mathrm{End}_G(X) & \xrightarrow{\iota} & \mathrm{End}_G(W) & \xrightarrow{\subseteq} & \mathrm{Hom}_G(W, \mathcal{O}(X)) \\ \downarrow{\subseteq} & & \downarrow{\subseteq} & & \downarrow{\subseteq} \\ \mathrm{End}(X) & \xrightarrow{\hookrightarrow} & \mathrm{Hom}(W, \mathcal{O}(X)) & = & \mathrm{Hom}(W, \mathcal{O}(X)) \end{array}$$

which shows that ι is a closed immersion. Hence $\mathrm{End}_G(X)$ an algebraic semigroup. Similarly, we see that $\mathrm{Dom}_G(X)$ is an algebraic semigroup and that $\mathrm{Aut}_G(X)$ is an algebraic group. For the remaining claims we use [**FK16**, Proposition 3.2.1] which shows that, for any affine variety X, $\mathrm{Aut}(X)$ is closed in $\mathrm{Dom}(X)$ and $\mathrm{Dom}(X)$ is open in $\mathrm{End}(X)$. $\square$

3.2. Fiber bundles. Let F be an affine s-variety.

3.2.1. Definition. A *fiber bundle over Y with fiber F* is a morphism $p\colon B \to Y$ with the following properties:

(1) Every fiber $p^{-1}(y)$ is an s-variety isomorphic to F;
(2) For every point $y \in Y$ there is an étale neighborhood $\eta\colon U \to Y$ such that $U \times_Y B$ is U-isomorphic to $U \times F$, i.e. there is an isomorphism $\varphi_U\colon U \times F \xrightarrow{\sim} U \times_X B$ such that the induced morphisms $F \xrightarrow{\sim} \{u\} \times F \xrightarrow{\varphi_U} p^{-1}(\eta(u)$ are isomorphisms of s-varieties for all $u \in U$.

The set of isomorphism classes of fiber bundles over X with fiber F is denoted by $H^1(X, F)$.

In our definition, every fiber of p has given the structure of F, by (1), and condition (2) makes sure that this structure is locally trivial in the étale topology. Clearly, a stronger condition would be that a fiber bundle is locally trivial in the Zariski-topology. We denote by $H^1_{Zar}(X, F) \subseteq H^1(X, F)$ the subset of isomorphism classes of those fiber bundles which are locally trivial in the Zariski-topology. Note also that (2) implies that the fibers of p are reduced.

3.2.2. Remark. It is clear from the definition that for $b \in B$ and $x := p(b) \in X$ the tangent map $dp_b\colon T_bB \to T_xX$ is surjective with kernel $\ker dp_b = T_b p^{-1}(x)$. In particular, B is smooth in b if and only if the fiber $p^{-1}(x) \simeq F$ is smooth in b and X is smooth in x.

3.2.3. Example. Let F be an s-variety such that $\mathrm{Aut}(F)$ is trivial. Then every fiber bundle $\varphi\colon B \to X$ with fiber F is trivial. In fact, for every fiber $p^{-1}(x)$ there is a unique isomorphism $\psi_x\colon p^{-1}(x) \xrightarrow{\sim} F$, and this collection $(\psi_x)_{x \in X}$ defines a map $\psi\colon B \to F$. We claim that ψ is a morphism and that $(\psi, \varphi)\colon B \to F \times X$ is an isomorphism. If the bundle

$B \to X$ is trivial over the étale neighborhood $\eta\colon U \to X$, we get the following commutative diagram:

$$\begin{tikzcd} F \arrow[r, equal] & F \arrow[r, equal] & F \\ U \times F \arrow[u, "\mathrm{pr}_F"] \arrow[r, "\simeq"] \arrow[d, "\mathrm{pr}_U"'] & U \times_X B \arrow[u, "\psi\circ\eta_B"] \arrow[r, "\eta_B"] \arrow[d] & B \arrow[u, "\psi"] \arrow[d, "\varphi"] \\ U \arrow[r, equal] & U \arrow[r, "\eta"] & X \end{tikzcd}$$

Thus, $\psi \circ \eta_B$ is a morphism, and so $\psi|_{\varphi^{-1}(U)}$ is a morphism, by Lemma 2.2.1. The claim follows.

3.2.4. Example. Take $F = \mathbb{C}^n$ together with the action of the affine group Aff_n. Then $\mathrm{Aut}(F) = \mathrm{Aut}_{\mathrm{Aff}_n}(\mathbb{C}^n)$ is trivial, because a regular automorphism of $\mathbb{C}^n$ commuting with all affine transformations is trivial.
(To see this, one first shows that a regular automorphism of $\mathbb{C}^n$ commuting with the scalar multiplications is linear. From that the claim follows immediately.)
As a consequence, *every fiber bundle with fiber the* Aff_n*-variety* $\mathbb{C}^n$ *is trivial.*

3.2.5. Example. If F is a vector space V, then one can show (see section ???) that every fiber bundle over X with fiber V is locally trivial in the Zariski-topology, and these bundles are usually called *vector bundles* over X. The same is true if $F = \mathbb{A}^n$ considered as affine n-space. If X is affine, then every affine space bundle over X has the structure of a vector bundle, but this does not hold in general. E.g., define $B := \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ where $\Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal, and let $p\colon B \to \mathbb{P}^1$ be the morphism induced by the projection onto the first factor. Then B is an affine line bundle, trivial over $\mathbb{P}^1 \setminus \{0\}$ and $\mathbb{P}^1 \setminus \{\infty\}$, but it cannot be a line bundle, because B is affine and so p has no sections.

A similar, but weaker concept is that of a *fibration with fiber* F by what we mean a flat surjective morphism $p\colon B \to X$ with the condition that every fiber is (reduced and) isomorphic to F. A famous unsolved problem here is whether every affine fibration with fiber $\mathbb{C}^n$ is a fiber bundle, see [**KR14**, Section 5]. This is not the case if the base X is not normal. It is known to be true for $n = 1$ and X normal, and for $n = 2$ and X a smooth curve. In these cases, the bundle is even locally trivial in the Zariski-topology.

3.2.6. Remark. Assume that the fiber F is a G-variety. *Then one has a canonical* G*-action on the total space* B *of every fiber bundle* $B \to X$ *with fiber* F*.*
In fact, there is an action of G on every fiber, and therefore a well-defined "abstract" action of G on B which becomes a regular action under an étale base change, by condition (2). Hence, the claim follows from the next lemma.

3.2.7. Lemma. *Let* Z *be a variety with an "abstract" action of an algebraic group* G*. Assume that there is* G*-variety* $\tilde{Z}$ *and a surjective étale and* G*-equivariant morphism* $\xi\colon \tilde{Z} \to Z$*. Then the action of* G *on* Z *is regular.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} \tilde{Y} := G \times \tilde{Z} \arrow[r, "\tilde{\varphi}", "\simeq"'] \arrow[d, "\mathrm{id}_G \times \xi"'] & G \times \tilde{Z} \arrow[d, "\mathrm{id}_G \times \xi"] \\ Y := G \times Z \arrow[r, "\varphi", "bijective"'] & G \times Z \end{tikzcd}$$

where $\varphi(g, z) := (g, gz)$ and $\tilde{\varphi}(g, \tilde{z}) := (g, g\tilde{z})$. Then $\mathrm{id}_G \times \xi$ is étale and surjective, and the composition $\varphi \circ (\mathrm{id}_G \times \xi)$ is a morphism, and so the claim follows from Lemma 2.2.1. $\square$

3.3. Principal bundles. An important special case is the following. Take $F := G$, an algebraic group considered as a G-variety where G acts by *right multiplication*. A bundle with fiber G is called a *principal G-bundle*. The usual definition is the following which is equivalent, by Lemma 3.2.7 above.

3.3.1. Definition. Let G be an algebraic group. A *principal G-bundle over X* is a variety P together with a right action by G and a G-invariant morphism $\rho\colon P \to X$ with the following property: For every $x \in X$ there is an étale neighborhood $\eta\colon U \to X$ such that the fiber product $U \times_X P$ is G-isomorphic to $U \times G$ over U.

We denote by $H^1(X, G)$ the set of isomorphism classes of principal G-bundles over X and by $H^1_{Zar}(X, G) \subseteq H^1(X, G)$ the subset of those which are locally trivial in the Zariski-topology.

3.3.2. Example. A typical example is the following. Let H be an algebraic group and let $G \subseteq H$ be a closed subgroup. It is known that the left cosets $H/G := \{hG \mid h \in H\}$ form a smooth quasi-projective variety with the usual universal properties, see [**Bor91**, Chap. II, Theorem 6.8]. It follows that the projection $\pi\colon H \to H/G$ is a principal G-bundle. In fact, we have

$$\begin{array}{ccccc}
H \times G & \xrightarrow[(h,g)\mapsto[h,hg]]{\simeq} & H \times_{H/G} H & \xrightarrow[[h,h']\mapsto h']{\tilde{\pi}} & H \\
\Big\downarrow{\scriptstyle \mathrm{pr}_H} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\
H & = & H & \xrightarrow{\pi} & H/G
\end{array}$$

i.e., the fiber product $H \times_{H/G} H$ is G-isomorphic to $H \times G$, hence a trivial principal G-bundle over H. Since the differential $d\pi_h$ is surjective for all $h \in H$, the next lemma shows that for every $h \in H$ there is a locally closed smooth subvariety $S \subseteq H$ such that $p|_S\colon S \to H/G$ is étale. Clearly, $S \times_{H/G} H \simeq S \times G$, and the claim follows.

3.3.3. Lemma. *Let $\varphi\colon X \to Y$ be a morphism of smooth varieties. Assume that $d\varphi_x\colon T_xX \to T_{\varphi(x)}Y$ is surjective for some $x \in X$. Then there is a closed subvariety $S \subseteq X$, containing x and smooth in x, such that $\varphi|_S\colon S \to Y$ is étale in x.*

Proof. We can assume that X and Y are both affine. By assumption, φ^* induces an injection $\mathfrak{m}_y/\mathfrak{m}_y^2 \hookrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$. Thus we can find a subspace $W \subseteq \mathfrak{m}_x$ of dimension $r = \dim X - \dim Y$ such that $\mathfrak{m}_x = W \oplus \varphi^*(\mathfrak{m}_y) \oplus \mathfrak{m}_x^2$. Define $S := \mathcal{V}_X(W) \subseteq X$. Then $\dim S \geq \dim X - r = \dim Y$, by Krull's Theorem, and $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_{x,S}/\mathfrak{m}_{x,S}^2$ is surjective, because $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_{x,S}/\mathfrak{m}_{x,S}^2$ is surjective and W is in the kernel. Since $\dim \mathfrak{m}_y/\mathfrak{m}_y^2 = \dim Y$, it follows that $\dim S = \dim Y$, that S is smooth in x and that $\mathfrak{m}_y/\mathfrak{m}_y^2 \xrightarrow{\sim} \mathfrak{m}_{x,S}/\mathfrak{m}_{x,S}^2$ is an isomorphism. This shows that $\varphi|_S\colon S \to Y$ is étale in x, by Proposition 2.1.10. □

3.3.4. Remark. We will see later in section 3.7 that for $G = \mathrm{GL}_n$, SL_n or Sp_n every principal G-bundle is locally trivial in the Zariski-topology. The same holds for every connected solvable group.

3.4. Associated bundles. The following construction of an *associated bundle to a principal bundle* is essential for the rest of this section. Let H be an algebraic group, and let $p\colon P \to X$ be a principal H-bundle. If Y is an affine H-variety, then we can form the orbit space

$$P \times^H Y := (P \times Y)/H$$

where H acts by $h(a, x) := (ah^{-1}, hx)$. This is clearly a free action, and we have a canonical map $q\colon P \times^H Y \to X$, $[a, x] \mapsto p(a)$, whose fibers are isomorphic to Y. Here $[a, x]$ denotes the H-orbit of (a, x).

3.4.1. Lemma. *The orbit space $P\times^H Y$ has the structure of a variety with the following properties:*

(1) *The canonical map $P \times Y \to P \times^H Y$ is a principal H-bundle;*
(2) *The map $q\colon P \times^H Y \to X$ is a fiber bundle with fiber Y.*

Proof. We can assume that X is affine, and so P is affine, too. In fact, if $X = \bigcup X_i$ is an affine covering, then $P_i := p^{-1}(X_i) \to X_i$ is a principal H-bundle, and $P \times^H Y = \bigcup_i P_i \times^H Y$. Now it is clear that if the $P_i \times^H Y$ satisfy the properties of lemma, then so does $P \times^H Y$.

There is an obvious candidate for $(P \times Y)/H$, namely the "affine variety" with coordinate ring $\mathcal{O}(P \times Y)^H$, but we have to show that this algebra is finitely generated. If $\eta\colon U \to Y$ is a surjective étale morphism trivializing the principal bundle $P \to X$ with U affine, we obtain the following commutative diagram of affine schemes (with obvious morphisms):

$$\begin{tikzcd}
U \times H \times Y \arrow[r, "\tilde\eta\times\mathrm{id}_Y"] \arrow[d, "\mathrm{pr}"] & P \times Y \arrow[d, "q"] \\
U \times Y \arrow[r] \arrow[d, "\mathrm{pr}"] & \operatorname{Spec}\mathcal{O}(P\times Y)^H \arrow[d, "p"] \\
U \arrow[r, "\eta"] & X
\end{tikzcd}$$

It follows that the outer diagram is a fiber product. Now we claim that

$$(*) \qquad \mathcal{O}(U) \otimes_{\mathcal{O}(X)} \mathcal{O}(P \times Y)^H = (\mathcal{O}(U) \otimes_{\mathcal{O}(X)} \mathcal{O}(P \times Y))^H = \mathcal{O}(U) \otimes \mathcal{O}(Y).$$

This implies that the invariant ring $\mathcal{O}(P \times X)^H$ is finitely generated (see Lemma 2.1.12), hence $Q := \operatorname{Spec}\mathcal{O}(P \times Y)^H$ is an affine variety, and then that all diagrams are fiber products, hence Q is the orbit space $(P \times Y)/H$ and $p\colon Q \to X$ is a fiber bundle with fiber X.

In order to show the first equality in $(*)$ we look at the exact sequence of $\mathcal{O}(X)$-modules

$$0 \longrightarrow \mathcal{O}(P\times Y)^h \longrightarrow \mathcal{O}(P\times Y) \xrightarrow{r\mapsto r-hr} \mathcal{O}(P\times Y)$$

where $h \in H$, and use that $\mathcal{O}(U)$ is flat over $\mathcal{O}(X)$. The second equality in $(*)$ is clear, because $\mathcal{O}(U) \otimes_{\mathcal{O}(X)} \mathcal{O}(P \times Y) = \mathcal{O}(U) \otimes \mathcal{O}(H) \otimes \mathcal{O}(Y)$. □

3.5. Functorial properties. Let $\varphi\colon G \to H$ be a homomorphism of algebraic groups, and let $P \to X$ be a principal G-bundle. Then we define a principal H-bundle $\varphi_*(P)$ as the associated bundle (3.4)

$$\varphi_*(P) := P \times^G H := (P \times H)/G$$

where G acts in the following way: $g(p, h) := (pg^{-1}, \varphi(g)h)$. In fact, for the trivial bundle $P = X \times G$ we get

$$\varphi_*(P) = P \times^G H = (X \times G \times H)/G \xrightarrow{\sim} X \times H$$

where the isomorphism is given by $[x, g, h] \mapsto (x, \varphi(g)h)$. It follows that the associated bundle $P \times^G H$ is a principal H-bundle. Note that $\varphi_*(P) = P/N$ in case $\varphi\colon G \to H$ is surjective with kernel $N \subseteq G$.

3.5.1. Proposition. *Let $N \subseteq G$ be a normal subgroup. Then the corresponding sequence*

$$H^0(X,N) \to H^0(X,G) \to H^0(X,G/N) \xrightarrow{\delta} H^1(X,N) \to H^1(X,G) \to H^1(X,G/N)$$

is an exact sequence of pointed sets.

Here $H^0(X, G) := G(X) := \mathrm{Mor}(X, G)$, and the boundary map δ is defined in the following way: Consider $\sigma \in H^0(X, G)$ as a section of the trivial bundle $X \times G/N \to X$, and let $p\colon G \to G/N$ be the projection. Then $Q := (\mathrm{id}_X \times p)^{-1}(\sigma(X)) \subseteq X \times G$ is a principal N-bundle, and we set $\delta(\sigma) := [Q]$.

PROOF. We will always confuse elements from $H^0(X, H)$ with sections of the trivial bundle $X \times H \to X$.

(a) *Exactness at $H^0(X, G/N)$:* Assume that $\delta(\sigma)$ is trivial. This means that $Q := (\mathrm{id}_X \times p)^{-1}(\sigma(X)) \subseteq X \times G$ has a section τ, and it follows that σ is the image of τ under the map $G \to G/N$. The other inclusion is clear.

(b) *Exactness at $H^1(X, N)$:* Let $Q \to X$ be a principal N-bundle and assume that $Q \times^N G$ has a section σ. Then, by construction, $\delta(\sigma) = [Q]$. Again, the other inclusion is clear.

(c) *Exactness at $H^1(X, G)$:* Let $P \to X$ be a principal G-bundle and assume that the image $P \times^G G/N = P/N$ has a section σ. If $q\colon P \to P/N$ is the projection, then $Q := q^{-1}(\sigma(X)) \subseteq P$ is a principal N-bundle, and $P \simeq Q \times^N G$. The other inclusion is clear. $\square$

3.6. The correspondence between fiber bundles and principal bundles. In this section we prove the following theorem. We will use some elementary facts about *ind-varieties* (see [**FK16**]).

3.6.1. THEOREM. *Let F be an s-variety such that that $\mathrm{Aut}(F)$ is an algebraic group. Then, for any variety X, there is a canonical bijective correspondence between fiber bundles over X with fiber F and principal $\mathrm{Aut}(F)$-bundles over X.*

PROOF. (a) Let $P \to X$ be a a principal $\mathrm{Aut}(F)$-bundle. Then, by Lemma 3.4.1, the associated bundle $P \times^{\mathrm{Aut}(F)} F$ is a fiber bundle with fiber F.

(b) For the other direction, starting with a fiber bundle $q\colon B \to X$ with fiber F, we define

$$P := \{(x, \varphi) \in X \times \mathrm{Mor}(F, B) \mid \varphi\colon F \xrightarrow{\sim} q^{-1}(x)\} \subseteq X \times \mathrm{Mor}(F, B).$$

We have to show that this is a variety and that the map $p\colon P \to X$, $(x, \varphi) \mapsto x$, is a principal $\mathrm{Aut}(F)$-bundle. It is clear that $\mathrm{Aut}(F)$ acts on P from the right, $(x, \varphi)\sigma := (x, \varphi \circ \sigma)$, and that this action is simply transitiv on the fibers $p^{-1}(x) = \mathrm{Iso}(F, q^{-1}(x))$. Define

$$\tilde{P} := \{(x, \varphi) \in X \times \mathrm{Mor}(F, B) \mid \varphi(F) \subseteq q^{-1}(x)\} \subseteq X \times \mathrm{Mor}(F, B).$$

It is easy to see that this is a closed ind-subvariety of $X \times \mathrm{Mor}(F, B)$.

There is an affine variety X' and a surjective étale morphism $\eta\colon X' \to X$ such that $B' := X' \times_X B$ is trivial. Define $P' \to X'$ and $\tilde{P}' \to X'$ as above:

$$P' := \{(x, \varphi) \in X' \times \mathrm{Mor}(F, B') \mid \varphi\colon F \xrightarrow{\sim} {q'}^{-1}(x)\} \subseteq X' \times \mathrm{Mor}(F, B'),$$
$$\tilde{P}' := \{(x, \varphi) \in X' \times \mathrm{Mor}(F, B') \mid \varphi(F) \subseteq {q'}^{-1}(x)\} \subseteq X' \times \mathrm{Mor}(F, B').$$

Since $B' = X' \times F$ we get an isomorphism of ind-varieties $\tilde{\psi}\colon X' \times \mathrm{End}(F) \xrightarrow{\sim} \tilde{P}'$ given in the following way: $(x, \sigma) \mapsto (x, \iota_x \circ \sigma)$ where $\iota_x\colon F \to B'$ is the closed immersion $z \mapsto (x, z) \in X' \times F = B'$. Clearly, $\tilde{\psi}$ induces an isomorphism $\psi\colon X' \times \mathrm{Aut}(F) \xrightarrow{\sim} P'$ which implies that P' is a variety.

Finally, we have a surjective map $\tilde{\eta}\colon \tilde{P}' \to \tilde{P}$, $(x', \varphi') \mapsto (\eta(x'), \eta_B \circ \varphi')$ where $\eta_B\colon B' \to B$ is the canonical morphism. It is clear from the definition that $\tilde{\eta}$ is a morphism of ind-varieties and that $\tilde{\eta}(P') = P$. This gives the following commutative diagram:

$$\begin{tikzcd}
X' \times \mathrm{Aut}(F) \arrow[r, "\tilde{\Psi}", "\simeq"'] \arrow[d, "\subseteq"] & P' \arrow[r] \arrow[d, "\subseteq"] & P \arrow[d, "\subseteq"] \\
X' \times \mathrm{End}(F) \arrow[r, "\Psi", "\simeq"'] \arrow[d] & \tilde{P}' \arrow[r, "\tilde{\eta}"] \arrow[d, "p'"] & \tilde{P} \arrow[d, "p"] \\
X' \arrow[r, equal] & X' \arrow[r, "\eta"] & X
\end{tikzcd}$$

It is clear from the construction that the diagrams are (set-theoretic) fiber-products. This implies that $\tilde{\eta}$ is étale which means that for any algebraic subset $A \subseteq \tilde{P}$ the inverse image $A' := \tilde{\eta}^{-1}(A) \subseteq \tilde{P}'$ is also algebraic and $\tilde{\eta}\colon A' \to A$ is étale. Now we use [**FK16**, Proposition 3.2.1] which shows that, for any affine variety F, $\mathrm{Aut}(F)$ is open in $\mathrm{Dom}(F)$ and $\mathrm{Dom}(F)$ is closed in $\mathrm{End}(F)$ where $\mathrm{Dom}(F)$ denotes the semigroup of dominant endomorphisms. Since images of open sets under étale maps are open this implies first that the image of $X' \times \mathrm{Dom}(F)$ is open in $\tilde{P}$, and then that $P = \tilde{\eta}^{-1}(\tilde{\eta}(P))$ is closed in this image, hence algebraic. The rest is easy and is left to the reader. □

3.7. Special groups. In this section we collect results about special groups. The references are [**Ser58**] and some unpublished notes of DOMINGO LUNA. Let us first recall the definition from which the importance of this notion is clear.

3.7.1. DEFINITION. A linear algebraic group G is called *special* if every principal G-bundle is locally trivial in the Zariski-topology.

Clearly, special groups are connected. The following result can be found in [**Ser58**]. We will deduced it from Lemma 3.7.4 below.

3.7.2. PROPOSITION.
(1) *If* $1 \to G' \xrightarrow{\iota} G \xrightarrow{\pi} G'' \to 1$ *is an exact sequence of algebraic groups where* G' *and* G'' *are special, then* G *is special. In particular, products of special groups are special.*
(2) *The groups* GL_n, SL_n *and* Sp_n *are special.*
(3) *Every connected solvable group is special.*

3.7.3. REMARK. GROTHENDIECK [**Gro58**] has shown that SL_n and Sp_n are the only simple groups which are special.

The poof is based on the following unpublished result due to LUNA.

3.7.4. LEMMA. *Let G be a reductive group and W a G-module. Assume that W contains a G-orbit Gw_0 isomorphic to G and that there is a G-equivariant retraction $\rho\colon U \to Gv_0$ where U is an open G-stable neighborhood of Gw_0. Then G is special.*

PROOF. Let $\pi\colon P \to Y$ be a principal G-bundle and consider a fiber $F := \pi^{-1}(y_0)$. We may assume that Y and P are affine. Choose a G-equivariant isomorphism $\alpha\colon F \xrightarrow{\sim} Gw_0 \subseteq W$. Then α extends to a G-equivariant morphism $\tilde{\alpha}\colon P \to W$, because W is a vector space and G is reductive. It follows that $\tilde{U} := \tilde{\alpha}^{-1}(U)$ is a G-stable open neighborhood of F and that $\tilde{\rho} := \alpha^{-1} \circ \rho \circ \tilde{\alpha}\colon \tilde{U} \to F$ is an G-equivariant retraction. If S is a fiber of $\tilde{\rho}$, then the map $G \times S \to \tilde{U}$, $(g, s) \mapsto gs$, is an G-equivariant isomorphism. Hence the bundle P is trivial over the neighborhood $\pi(\tilde{U})$ of y_0. □

Proof of Proposition 3.7.2. (1) Let $P \to X$ be a principal G-bundle. Choosing a suitable open covering of X we can assume that $\pi_*(P)$ is trivial. Then $P = \iota_*(P')$ for a suitable G'-principal bundle P', by Proposition 3.5.1. Since P' is locally trivial in the Zariski-topology the same holds for P.

(2) Since $\mathrm{GL}_n \subseteq M_n$ is an open orbit, the claim follows from Lemma 3.7.4. For SL_n we take the module $M := \mathrm{M}_n$ where SL_n acts by left-multiplication. Then $U := \mathrm{GL}_n \subseteq \mathrm{M}_n$ retracts equivariantly to SL_n by $A \mapsto A \begin{bmatrix} \det A^{-1} & \\ & \ddots \end{bmatrix}$. Finally, for Sp_n

(3) For a connected solvable group G the unipotent elements form a closed normal unipotent subgroup U, and the quotient G/U is a torus (see ???). Since $\mathbb{C}^* = \mathrm{GL}_1$ is special by (2) it follows from (1) that every torus is special. Moreover, we will see below in section 3.9 (Proposition 3.9.4) that every principal U-bundle over an affine variety is trivial. Now the claim follows from (1). □

3.8. Locally trivial group schemes and torsors. Let G be an algebraic group and X a variety. A fiber bundle $\mathfrak{G} \to X$ with fiber G is a *smooth group scheme over* X which means that the multiplication $\mathfrak{G} \times_X \mathfrak{G} \to \mathfrak{G}$ and the inverse $\mathfrak{G} \to \mathfrak{G}$ are morphism over X satisfying the usual properties. In particular, the sections $\mathfrak{G}(X) := \mathrm{Mor}_X(X, \mathfrak{G})$ form a group. We will call $\mathfrak{G}$ a *locally trivial group scheme (over X) with fiber G*. It is called *trivial* if $\mathfrak{G}$ is isomorphic to $X \times G$ (over X). It is rather obvious how to define *homomorphisms of group schemes, closed subgroup schemes, normal subgroup schemes, the center of a group scheme*, etc. The concept of a *quotient* $\mathfrak{G}/\mathfrak{H}$ is more delicate. It is uniquely defined by the universal property of such a quotient, but it is not clear whether it exists. We will need this only in a very special situation.

If U is a commutative unipotent group, then U has a canonical structure of a vector space, given by the exponential map $\exp\colon \mathrm{Lie}\, U \xrightarrow{\sim} U$. In particular, $\mathrm{Aut}(U) = \mathrm{GL}(U)$. This implies the following result.

3.8.1. Lemma. *Let $\mathfrak{U} \to X$ be a locally trivial group scheme with fiber a commutative unipotent group U. Then $\mathfrak{U}$ has a canonical structure of a vector bundle over X.*

For any group scheme, one has the notion of a *torsor* generalizing the principal "principal bundles" defined above. We will need this only for a locally trivial group scheme $\mathfrak{G}$ over X with fiber G.

3.8.2. Definition. A $\mathfrak{G}$*-torsor* is a morphism $\mathcal{P} \to X$ with an action $\mathcal{P} \times_X \mathfrak{G} \to \mathcal{P}$ of $\mathfrak{G}$ from the right such that $\mathcal{P}$ is locally isomorphic to $\mathfrak{G}$ with the $\mathfrak{G}$-action by right multiplication. The $\mathfrak{G}$-torsor $\mathcal{P}$ is called *trivial* if it is isomorphic to $\mathfrak{G}$ where $\mathfrak{G}$ acts by right multiplication.

One has to be careful in this setting. Although every $\mathfrak{G}$-torsor $\mathcal{P}$ is locally isomorphic to $U \times G$ where G is acting by right multiplication, there is in general no global G-action on $\mathcal{P}$, except if $\mathfrak{G}$ is trivial, i.e. $\mathfrak{G} \simeq X \times G$. In that case, a $\mathfrak{G}$-torsor is the same as a principal G-bundle.

3.9. Unipotent group schemes. The main result here is the following.

3.9.1. Proposition. *Let $\mathfrak{U} \to X$ be a locally trivial group scheme with fiber a commutative unipotent group U. If X is affine, then every $\mathfrak{U}$-torsor is trivial.*

Proof. Let $\mathcal{P} \to X$ be a $\mathfrak{U}$-torsor, and let $S := \{s \in \mathcal{O}(X) \mid \mathcal{P}_s \text{ is trivial over } X_s\}$. We will show that $1 \in S$. First of all there is a finite covering $X = \bigcup X_{s_i}$ such that $\mathfrak{U}_{s_i} \xrightarrow{\sim} X_{s_i} \times U$ is trivial. Thus $\mathcal{P}_{s_i}$ is trivial, because every principal U-bundle over an affine variety is trivial. It follows that S generates the unit ideal $(1) = \mathcal{O}(X)$.

Next we claim that S is an ideal. For this we have only to show that for $s_1, s_2 \in S$ we have $s_1 + s_2 \in S$. If $Y := X_{s_1+s_2}$, then $Y = Y_{s_1} \cup Y_{s_2}$, and $Y_{s_i} = X_{(s_1+s_2)s_i}$. Thus the claim follows from the next lemma. $\square$

3.9.2. Lemma. *Let $\mathfrak{U} \to Y$ be a locally trivial group scheme with fiber a commutative unipotent group U, and let $\mathcal{P} \to Y$ be a $\mathfrak{U}$-torsor. Assume that Y is affine and that $Y = Y_{s_1} \cup Y_{s_2}$ for some $s_1, s_2 \in \mathcal{O}(Y)$. If the torsors $\mathcal{P}_{s_i} \to Y_{s_i}$ are both trivial, then so is $\mathcal{P}$.*

Proof. (a) If $O \subseteq Y$ is an open set, then the sections $\sigma \in \Gamma(O, \mathfrak{U})$ correspond to automorphisms $\tilde{\sigma}$ of $\mathfrak{U}|_O$ as a $\mathfrak{U}$-torsor: $\tilde{\sigma}(u) = u + \sigma(p(u))$, and $\sigma(x) = \tilde{\sigma}(0_x)$ where $0_x \in \mathfrak{U}_x$ is the neutral element. Thus, we can identify the group $\mathrm{Aut}_{\mathfrak{U}}(\mathfrak{U}|_O)$ with the additive group $\Gamma(O, \mathfrak{U})$, i.e. $(\sigma + \tau)\tilde{} = \tilde{\sigma} \circ \tilde{\tau}$.

(b) Now let $\varphi_i \colon \mathfrak{U}_{s_i} \xrightarrow{\sim} \mathcal{P}_{s_i}$ be isomorphisms of $\mathfrak{U}$-torsors. Then $\psi := (\varphi_2|_{\mathfrak{U}_{s_1 s_2}})^{-1} \circ \varphi_1|_{\mathfrak{U}_{s_1 s_2}}$ is an automorphism of $\mathfrak{U}_{s_1 s_2}$ as a $\mathfrak{U}$-torsor. We claim that ψ can be written in the form $\psi = \psi_2 \circ \psi_1^{-1}$ where $\psi_i \in \mathrm{Aut}_{\mathfrak{U}}(\mathfrak{U}_{s_i})$. Then $\mathrm{id}_{\mathfrak{U}_{s_1 s_2}} = \psi_2^{-1} \circ \psi \circ \psi_1$, and so, replacing φ_i by $\varphi_i' := \varphi_i \circ \psi_i$, we see that $\varphi_1'|_{\mathfrak{U}_{s_1 s_2}} = \varphi_2'|_{\mathfrak{U}_{s_1 s_2}}$. It follows that φ_1', φ_2' define an isomorphism $\mathfrak{U} \xrightarrow{\sim} \mathcal{P}$ as $\mathfrak{U}$-torsors.

(c) It remains to prove the claim in (b). In terms of sections, as explained in (a), the claim is equivalent to the condition $\Gamma(Y_{s_1}, \mathfrak{U}) + \Gamma(Y_{s_2}, \mathfrak{U}) = \Gamma(Y_{s_1 s_2}, \mathfrak{U})$. Let $\sigma \in \Gamma(Y_{s_1 s_2}, \mathfrak{U}) = \Gamma(Y, \mathfrak{U})_{s_1 s_2}$. Then there is is an $m > 0$ such that $f := (s_1 s_2)^m \sigma \in \Gamma(Y, \mathfrak{U})$, and we can find $h_1, h_2 \in \mathcal{O}(Y)$ such that $1 = h_1 s_1^m + h_2 s_2^m$. Hence

$$\sigma = h_1 s_1^m \sigma + h_2 s_2^m \sigma = h_1 \frac{f}{s_2^m} + h_2 \frac{f}{s_1^m} \in \Gamma(Y, \mathfrak{U})_{s_2} + \Gamma(Y, \mathfrak{U})_{s_1} = \Gamma(Y_{s_2}, \mathfrak{U}) + \Gamma(Y_{s_1}, \mathfrak{U}),$$

and the claim follows. $\square$

3.9.3. Lemma. *Let $\mathfrak{U}$ be a locally trivial group scheme over X with fiber a unipotent group U. Then the center $Z(\mathfrak{U}) \subseteq \mathfrak{U}$ is a locally trivial group scheme with fiber $Z(U)$, and the quotient $\mathfrak{U}/Z(\mathfrak{U})$ exists and is also a locally trivial group scheme, with fiber $U/Z(U)$.*

Proof. We will assume that $\mathfrak{U}$ is locally trivial in the Zariski-topology. The first part is clear, because $Z(X \times U) = X \times Z(U)$, and in this case we get $X \times \mathfrak{U}/X \times Z(\mathfrak{U}) = X \times U/Z(U)$. On the other hand, $\mathfrak{U}/Z(\mathfrak{U}) \to X$ is well-defined as a family of groups over X: $(\mathfrak{U}/Z(\mathfrak{U}))_x := \mathfrak{U}_x/Z(\mathfrak{U})_x$ for all $x \in X$. We have to show that this space carries the structure of a variety such that $\mathfrak{U}/Z(\mathfrak{U}) \to X$ is a locally trivial group scheme and that the projection $p \colon \mathfrak{U} \to \mathfrak{U}/Z(\mathfrak{U})$ has the universal property of a quotient.

Start with a finite covering $X = \bigcup_i X_i$ such that $\mathfrak{U}|_{X_i}$ is trivial, and fix isomorphisms $\varphi_i \colon X_i \times U \xrightarrow{\sim} \mathfrak{U}|_{X_i}$. Then φ_i induces an isomorphism $X_i \times Z(U) \xrightarrow{\sim} Z(\mathfrak{U})|_{X_i}$ and a bijection $\bar{\varphi}_i \colon X_i \times U/Z(U) \xrightarrow{\sim} (\mathfrak{U}/Z(\mathfrak{U}))|_{X_i}$ which is fiberwise an isomorphism of groups. We endow $(\mathfrak{U}/Z(\mathfrak{U}))|_{X_i}$ with the structure given by the bijection $\bar{\varphi}_i$. If $x \in X$ belongs to several X_j, then it is easy to see that the structure of $\mathfrak{U}/Z(\mathfrak{U})$ in a neighborhood of $(\mathfrak{U}/Z(\mathfrak{U}))_x$ is well defined, i.e. does not depend on the choice of the X_j containing x. Thus the structure of $\mathfrak{U}/Z(\mathfrak{U})$ as a variety is well defined, and with this structure $\mathfrak{U}/Z(\mathfrak{U})$ is a locally trivial group scheme with fiber $U/Z(U)$. Now it is not difficult to see that $p \colon \mathfrak{U} \to \mathfrak{U}/Z(\mathfrak{U})$ is a homomorphism of group schemes with the universal property of a quotient. $\square$

3.9.4. Proposition. *Let $\mathfrak{U}$ be a locally trivial unipotent group scheme over an affine variety X. Then every $\mathfrak{U}$-torsor is trivial. In particular, for a unipotent algebraic group U every principal U-bundle over an affine variety is trivial.*

PROOF. Let $P \to X$ be a $\mathfrak{U}$-torsor. Then one can form the quotient $\bar{P} := P/Z(\mathfrak{U})$ which is a $\mathfrak{U}/Z(\mathfrak{U})$-torsor. (This is clear in case $\mathfrak{U}$ is locally trivial in the ZARISKI-topology, and needs some work in general.) By induction on $\dim U$ we can assume that $\bar{P}$ is trivial, i.e. that we have a section $\sigma\colon X \to \bar{P}$. If we take the inverse image of $\sigma(X) \subseteq \bar{P}$ under $\pi\colon P \to \bar{P}$ we obtain a $Z(\mathfrak{U})$-torsor $\pi^{-1}(\sigma(X)) \subseteq P$ which is trivial by Proposition 3.9.1. Hence, there is a section $\tau\colon X \to \pi^{-1}(\sigma(X))$, and therefore P is trivial as well. □

3.9.5. THEOREM. *Let G be an algebraic group, and let $U \subseteq G$ be the unipotent radical. For any affine variety X the canonical map $H^1(X, G) \to H^1(X, G/U)$ is injective.*

PROOF. We have the exact sequence $H^1(X, U) \to H^1(X, G) \xrightarrow{\rho} H^1(X, G/U)$ of pointed sets (Proposition 3.5.1). Let $P \to X$ be a principal G-bundle and $[P] \in H^1(X, G)$ its isomorphism class. In order to describe the fiber $\rho^{-1}(\rho([P])) \subseteq H^1(X, G)$ we use the so-called twist construction. Define

$$\mathfrak{G} := P \times^G G := (P \times G)/\!\!/G$$

where G acts in the following way: $g(p, h) := (pg^{-1}, ghg^{-1})$. This is a free action and so the quotient $P \times G \to P \times^G G$ is a principal G-bundle.

We define a multiplication on $\mathfrak{G}$ by $[p, h_1] \cdot [p, h_2] := [p, h_1h_2]$ where we use the fact that G acts transitively on the fiber P_x. It is easy to see that this does not depend on the choice of $p \in P_x$ and that it defines a morphism $\mathfrak{G} \times_X \mathfrak{G} \to \mathfrak{G}$. We claim that $\mathfrak{G}$ is a locally trivial group scheme over X with fiber G. In fact, if $P = X \times G$ is trivial, then

$$P \times^G G = (X \times G \times G)/\!\!/G \xrightarrow{\sim} X \times G,\ (x, g, h) \mapsto (x, ghg^{-1}),$$

and this map is an isomorphism of group schemes over X. Since $U \subseteq G$ is normal, we see that $\mathfrak{U} := P \times^G U = (P \times U)/\!\!/G \subseteq \mathfrak{G}$ is a closed subgroup scheme, and it is locally trivial with fiber U. Now the twist construction says that there is a canonical bijection between the image of $H^1(X, \mathfrak{U})$ in $H^1(X, \mathfrak{G})$ and the fiber $p^{-1}(p([P]))$. Thus the claim follows from Proposition 3.9.4. □

3.10. Some applications.

3.10.1. COROLLARY. *Let X be an affine variety. Then every affine space bundle $A \to X$, i.e. fiber bundle with fiber $\mathbb{A}^n$ considered as an affine space, has the structure of a vector bundle.*

PROOF. If $A \to X$ is an affine space bundle, then $A \simeq P \times^{\mathrm{Aff}_n} \mathbb{A}^n$ where $P \to X$ is a principal Aff_n-bundle, see Theorem 3.6.1. It follows from Theorem 3.9.5 that the inclusion $\mathrm{GL}_n \hookrightarrow \mathrm{Aff}_n$ induces a bijection $H^1(X, \mathrm{GL}_n) \xrightarrow{\sim} H^1(X, \mathrm{Aff}_n)$. Hence $P \simeq P' \times^{\mathrm{GL}_n} \mathrm{Aff}_n$ where P' is a principal GL_n-bundle, and so

$$A \simeq P \times^{\mathrm{Aff}_n} \mathbb{A}^n \simeq P' \times^{\mathrm{GL}_n} \mathrm{Aff}_n \times^{\mathrm{Aff}_n} \mathbb{A}^n \simeq P' \times^{\mathrm{GL}_n} \mathbb{A}^n,$$

showing that A has the structure of a vector bundle over X. □

3.10.2. COROLLARY. *Let G be a reductive group, and let V be a G-module without invariants. Then every fiber bundle with fiber V is a G-vector bundle.*

PROOF. If V has no invariants, then $\mathrm{Aut}_G(V)$ is an algebraic group, by Lemma 3.1.2. Since $0 \in V$ is the only closed orbit, it follows that $\mathrm{Aut}_G(V)$ fixes 0. Therefore, we get a homomorphism $\mathrm{Aut}_G(V) \to \mathrm{GL}(T_0V) = \mathrm{GL}(V)$ whose image is $\mathrm{GL}(V)^G$. It is clear that the kernel is a unipotent group since it stabilizes the flag $\mathfrak{m}_0 \supseteq \mathfrak{m}_0^2 \supseteq \mathfrak{m}_0^3 \supseteq \cdots$. As in the previous corollary, we get from Theorem 3.9.5 that every fiber bundle with fiber V and automorphism group $\mathrm{Aut}_G(V)$ has the structure of G-vector bundle, i.e. of a fiber bundle with fiber V and automorphism group $\mathrm{GL}(V)^G$. □

The following corollary is due to Bass-Haboush [**BH85**]. Recall that an action of an algebraic group on a variety is called *fix-pointed* if every closed orbit is a fixed point.

3.10.3. Corollary. *Let X be an affine G-variety where G is reductive. Assume that X is smooth and that the action is fix-pointed. Then the quotient $X \to X/\!\!/G$ has the structure of a G-vector bundle.*

Proof. We will see later, as a consequence of the Slice Theorem, that the quotient is a fiber bundle with fiber a G-module V without invariants (Theorem 5.3.1). Now the claim follows from the previous corollary. □

4. The Slice Theorem

Now we have all the tools to formulate the important Slice Theorem due to Domingo Luna [**Lun73**]. The proof is based on the so-called Fundamental Lemma which was given in the later work [**Lun75**]. We first define the associated bundles in the case of reductive groups and describe its properties which take into account that in general orbits are not closed. An important ingredient is Matsushima's Theorem which states that the stabilizer of a closed orbit under a reductive group is again a reductive group. We will not prove this here but refer to the literature.

4.1. Associated bundles for reductive groups. Let G be an algebraic group, $H \subseteq G$ a closed subgroup, and let Y be an affine H-variety. Assume that H is reductive. Then we can form the quotient

$$G \times^H Y := (G \times Y)/\!\!/H$$

where H acts on the product $G \times Y$ by $h(g, y) := (gh, h^{-1}y)$. We will denote by $[g, y] \in G \times^H Y$ the image of (g, y). The group G acts on $G \times^H Y$ by $g[g', y] := [gg', y]$, and the projection $\mathrm{pr}\colon G \times Y \to G$ induces a G-equivariant morphism $p\colon G \times^H Y \to G/H$.

4.1.1. Lemma. *The quotient map $\pi\colon G \times Y \to G \times^H Y$ is a principal H-bundle, and $p\colon G \times^H Y \to G/H$ is a fiber bundle with fiber Y. In particular, $G \times^H Y$ is smooth in $[g, y]$ if and only if Y is smooth in y.*

Proof. By Lemma 4.1.2 below there exists a G-module V and a closed H-equivariant embedding $\iota\colon Y \hookrightarrow V$. From this we get the commutative diagram

$$\begin{tikzcd}
G \times Y \arrow[r, "\mathrm{id}_G \times \iota", "\subseteq"'] \arrow[d, "\pi_{G\times Y}"] & G \times V \arrow[r, "(g,x)\mapsto(g,gx)", "\simeq"'] \arrow[d, "\pi_{G\times V}"] & G \times V \arrow[d, "p\times \mathrm{id}_V"] \\
G \times^H Y \arrow[r, "\tau", "\subseteq"'] & G \times^H V \arrow[r, "\simeq"'] & G/H \times V
\end{tikzcd}$$

where τ is a closed immersion and $G \times Y = \pi_{G\times X}^{-1}(G \times^H Y)$. Now the first claim follows since $p \times \mathrm{id}_V\colon G \times V \to G/H \times V$ is a principal H-bundle.

As for the second, we have the fiber product

$$\begin{tikzcd}
G \times Y \arrow[r, "\pi"] \arrow[d] & G \times^H Y \arrow[d, "p"] \\
G \arrow[r] & G/H
\end{tikzcd}$$

and the same argument as in Example 3.3.2, using Lemma 3.3.3, shows that p is a fiber bundle. The last claim follows from Remark 3.2.2. □

4.1.2. Lemma. *Let G be an algebraic groups and $H \subseteq G$ a closed subgroup. If H is reductive, then every affine H-variety Y is H-isomorphic to a closed H-stable subvariety of a G-module V.*

Proof. It suffices to show that every simple H-module W is isomorphic to an H-stable submodule of a G-module V. Since $H \subseteq G$ is H-stable with respect to left multiplication, the projection $p\colon \mathcal{O}(G) \to \mathcal{O}(H)$ is H-equivariant. We know that $\mathcal{O}(H)$ contains every simple H-module W. Then $p^{-1}(W)$ is H-stable and we can find a H-equivariant embedding $W \hookrightarrow p^{-1}(W)$, because H is reductive. Now the claim follows. □

Lemma 4.1.1 shows that $G \times^H Y$ is, as a set, equal to the orbit space $(G \times Y)/H$ discussed in the first section for compact groups. We will now show that all the properties formulated in Proposition 1.2.1 carry over to the algebraic setting where H is reductive. Note that we have a closed H-equivariant embedding $Y \hookrightarrow G \times^H Y$, $y \mapsto [e, y]$; we will often identify Y with its image in $G \times^H Y$.

4.1.3. Proposition.
(1) *There is a bijection between the H-orbits in Y and the G-orbits in $G \times^H Y$, given by $Hy \mapsto G(e, y)$.*
(2) *The embedding $Y \hookrightarrow G \times^H Y$ induces an isomorphism $Y /\!\!/ H \xrightarrow{\sim} (G \times^H Y)/\!\!/ G$.*
(3) *If $Y_0 \subseteq Y$ is closed and H-stable, then the canonical map $G \times^H Y_0 \to G \times^H Y$ is a closed embedding with image GY_0. Moreover, $GY_0 \cap Y = Y_0$.*
(4) *A orbit $G[e, y']$ is contained in the closure $\overline{G[e, y]}$ if and only if Hy' is contained in the closure $\overline{Hy}$.*

Proof. (1) We have already seen this in the first section.

(2) The quotient $(G \times^H Y)/\!\!/ G$ can be obtained from $G \times Y$ by first taking the quotient by G and then the quotient by H. Clearly, $(G \times Y)/\!\!/ G = Y$, and so $(G \times^H Y)/\!\!/ G = (G \times Y)/\!\!/(G \times H) = ((G \times Y)/\!\!/ G)/\!\!/ H = Y /\!\!/ H$.

(3) The first part is clear since $G \times Y_0 \subseteq G \times Y$ is closed and H-stable, and the last statement follows from the fact that $p\colon G \times^H Y \to G/H$ and $p\colon G \times^H Y_0 \to G/H$ are fiber bundles with fibers Y and Y_0, respectively, by Lemma 4.1.1.

(4) This is an easy consequence from the previous statement (3). □

4.1.4. Remarks.
(1) If Y is smooth, then $G \times^H Y$ is also smooth. If Y is normal, then $G \times^H Y$ is also normal.
(2) For any $[g, y] \in G \times^H Y$ we have $\dim_{[g,y]} G \times^H Y = \dim_y Y + \dim G = \dim H$.
(3) For any $y = [e, y] \in Y \subseteq G \times^H Y$ we have $T_y Gy \cap T_y Y = T_y Hy$.
(4) If $y_0 \in Y \subseteq G \times^H Y$ is a fixed point under H, then $T_{y_0}(G \times^H Y) = T_{y_0} G y_0 \oplus T_{y_0} Y$.

All this is not difficult to see, and the proofs are left to the reader.

4.2. The construction of slices. Let us start with the following situation. Let G be an algebraic group, let V be a G-module, and let $O = Gx \subseteq V$ be an orbit such that the stabilizer G_x is reductive. Then the tangent space to the orbit is a G_x-stable subspace $T_x Gx \subseteq T_x V = V$, hence admits a G_x-stable complement W:

$$V = T_x V = T_x Gx \oplus W.$$

We have $\dim W = \dim V - \dim Gx = \dim V - \dim G + \dim G_x$. Define the G_x-stable affine subspace $S := x + W \subseteq V$, and consider the morphism $\Phi\colon G \times S \to V$, $(g, s) \mapsto gs$. Since $\Phi(gh, h^{-1}s) = \Phi(g, s)$ for all $h \in G_x$ we obtain a G-equivariant morphism

$$\varphi\colon G \times^{G_x} S \to V,\ [g, s] \mapsto gs.$$

4.2.1. Lemma. *The morphism $\varphi\colon G \times^{G_x} S \to V$ is étale in a G-stable open neighborhood of the point $[e, x] \in G \times^{G_x} S$.*

PROOF. The composition $\Phi\colon G \times S \to G \times^{G_x} S \to V$ is given by $(g, s) \mapsto gs$ and thus $d\Phi_{(e,x)}\colon \operatorname{Lie} G \oplus W \to V$ has the form $(A, w) \mapsto Ax + w$, hence is surjective. Since $G \times^{G_x} S$ and V are both smooth (Remark 4.1.4(1)), and

$$\dim G \times^{G_x} S = \dim G/G_x + \dim W = \dim T_x Gx + \dim W = \dim V$$

it follows that $d\varphi_{[e,x]}$ is an isomorphism and thus φ is étale in $[e, x]$, by Proposition 2.1.10. Since φ is G-equivariant the set of points where φ is étale is G-stable and open. □

We know that the G-stable open set has the form $G \times^{G_x} S'$ where $x \in S' \subseteq S$ is open and H-stable, but we cannot expect that S' contains with every orbit its closure or is even saturated, i.e., $S' = \pi_S^{-1}(\pi_S(S'))$, as we have seen in the compact case.

If G is also reductive, the situation is much better. First of all, we have the following fundamental theorem, due to MATSUSHIMA.

4.2.2. THEOREM (MATSUSHIMA [**Mat60**]). *If the orbit Gx is closed, then the stabilizer G_x is reductive.*

Thus we can construct the slices above for every closed orbit $Gx = \overline{Gx}$ and obtain commutative diagram

$$\begin{array}{ccc} G \times^{G_x} S & \xrightarrow{\varphi} & V \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_V} \\ S/\!\!/G_x & \xrightarrow{\bar\varphi} & V/\!\!/G \end{array}$$

where φ is étale in an open neighborhood of $[e, x]$, i.e. in an open set of the form $U = G \times^{G_x} S'$. But still we do not know if U and its image $\varphi(U)$ are saturated. We will see in the next section that LUNA's Slice Theorem will imply this, but is even much stronger.

4.3. Excellent morphisms and the Slice Theorem. The basic definition is the following, see [**Lun75**].

4.3.1. DEFINITION. Let X, Y be affine G-varieties. A G-equivariant morphism $\varphi\colon X \to Y$ is called *excellent* if the following holds:

(i) The induced morphism $\varphi/\!\!/G\colon X/\!\!/G \to Y/\!\!/G$ is étale;
(ii) The morphism $(\pi_X, \varphi)\colon X \to X/\!\!/G \times_{Y/\!\!/G} Y$ is an isomorphism.

In particular, $\varphi\colon X \to Y$ is also étale.

$$\begin{array}{ccccc} X & \xrightarrow[\simeq]{(\pi_X,\varphi)} & X/\!\!/G \times_{Y/\!\!/G} Y & \longrightarrow & Y \\ \big\downarrow{\scriptstyle \pi_X} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_Y} \\ X/\!\!/G & = & X/\!\!/G & \xrightarrow{\varphi/\!\!/G} & Y/\!\!/G \end{array}$$

Now we can formulate the main result of this appendix.

4.3.2. THEOREM (LUNA'S SLICE THEOREM). *Let X be an affine G-variety where G is reductive, and let $O = Gx$ be a closed orbit. Then there exists a locally closed affine G_x-stable subset $S \subseteq X$ containing x such that the morphism $\varphi\colon G \times^{G_x} S \to X$, $[g, s] \mapsto gs$, is excellent and the image $GS \subseteq X$ is open and affine, i.e. the diagram*

$$\begin{array}{ccc} G \times^{G_x} S & \xrightarrow[{[g,s]\mapsto gs}]{\varphi} & X \\ \big\downarrow & & \big\downarrow \\ S/\!\!/G_x & \xrightarrow{\varphi/\!\!/G} & X/\!\!/G \end{array}$$

is a fiber product and both maps φ and $\varphi/\!\!/G$ are étale.

A morphism $G \times^{G_x} S \to X$ as in the theorem, or simply the locally closed subset $S \subseteq X$ with the properties of the theorem is called *an étale slice in x*. In general, the structure of the slice S might be rather complicated. But if X is smooth in x we can say more. In this case we have again a G_x-stable decomposition $T_xX = T_xGx \oplus W$ where W is G_x isomorphic to the normal space $N_x := T_xX/T_xGx$ to the orbit in x.

4.3.3. THEOREM. *In addition to the assumptions of the Slice Theorem above assume that X is smooth in x. Then there exists an étale slice $S \subseteq X$ and an excellent G_x-equivariant morphism $\mu\colon S \to N_x$ with affine image. In particular, both diagrams*

$$\begin{array}{ccccc} G \times^{G_x} N_x & \xleftarrow{\mu} & G \times^{G_x} S & \xrightarrow{\varphi} & X \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi_X} \\ N_x/\!\!/G_x & \xleftarrow{\bar\mu} & S/\!\!/G_x & \xrightarrow{\bar\varphi} & X/\!\!/G \end{array}$$

are fiber products and all horizontal maps are étale.

The proofs are based on the following lemma due to LUNA, called "Lemme fondamentale" in [**Lun75**].

4.3.4. FUNDAMENTAL LEMMA. *Let X, Y be affine G-varieties, let $\varphi\colon X \to Y$ be a G-equivariant morphism, and let $O \subseteq X$ be a closed orbit. Assume that*

(1) *φ is étale in O,*
(2) *$\varphi(O) \subseteq Y$ is closed,*
(3) *$O \xrightarrow{\sim} \varphi(O)$ is an isomorphism.*

Then there exists an affine saturated open neighborhood U of O such that $\varphi|_U\colon U \to Y$ is excellent.

The proof of this lemma will be given in the following section 4.4. We will first show that it implies the Slice Theorem 4.3.2 and Theorem 4.3.3 above.

PROOF OF THE SLICE THEOREM 4.3.2. We can assume that X is a closed G-stable subset of a G-module V. If we choose a G_x-stable decomposition $V = T_xV = T_xGx \oplus W$ and set $S_V := x + W \subseteq V$, then Lemma 4.2.1 shows that $\varphi\colon G \times^{G_x} S_V \to V$ is étale in a neighborhood of (e, x) and maps the closed orbit $G(e, x)$ isomorphically onto Gx. Thus, the assumptions of Fundamental Lemma 4.3.4 are satisfied, so that there exists an affine saturated open neighborhood $U \subseteq G \times^{G_x} S_V$ of (e, x) such that $\varphi|_U\colon U \to V$ is excellent. In addition, we can arrange that the image of U in V is affine. Since U is G-saturated, it has the form $G \times^{G_x} S'$ where $S' \subseteq S_V$ is open and G_x-saturated, i.e.

$$\begin{array}{ccccc} G \times^{G_x} S' & = & U & \xrightarrow{\varphi' := \varphi|_U} & V \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi_U} & & \big\downarrow{\scriptstyle\pi_V} \\ S'/\!\!/G_x & \xrightarrow{\simeq} & U/\!\!/G & \xrightarrow{\varphi'/\!\!/G} & V/\!\!/G \end{array}$$

is a fiber product, and φ' and $\varphi'/\!\!/G$ are both étale. Setting $S := S' \cap X$ we finally get a fiber product

$$\begin{array}{ccc} G \times^{G_x} S & \xrightarrow{\psi} & X \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi_X} \\ S/\!\!/G_x & \xrightarrow{\psi/\!\!/G} & X/\!\!/G \end{array}$$

with étale morphisms ψ and $\psi/\!\!/G$ and affine image $\psi(G \times^{G_x} S) = GS$, proving the Slice Theorem. $\square$

Proof of Theorem 4.3.3. Let $S \subseteq X$ be an étale slice in x. Since X is smooth in x the same holds for S, and $T_xX = T_xGx \oplus T_xS$. Hence T_xS is G_x-isomorphic to N_x.

Choose a G_x-stable complement N of $\mathfrak{m}_x^2$ in $\mathfrak{m}_x \subseteq \mathcal{O}(S)$. Then $N^* \simeq N_x$, and we get a canonical G_x-equivariant morphism $\mu\colon S \to N^* \xrightarrow{\sim} N_x$ corresponding to the embedding $N \hookrightarrow \mathcal{O}(S)$. Clearly, μ is étale in x and maps the fixed point $x \in S$ to $0 \in N_x$. The Fundamental Lemma then implies that μ is excellent in an open affine G_x-saturated neighborhood of x, and the claim follows. □

4.4. Proof of the Fundamental Lemma. The following proof is due to Knop, see [Slo89, Anhang, p. 110–112]. It is based on some unpublished notes of Luna.

We have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X/\!\!/G & \xrightarrow{\varphi/\!\!/G} & Y/\!\!/G \end{array}$$

and points $x \in X$, $y := \varphi(x) \in Y$ such that the following holds:

(1) φ is étale in x;
(2) The orbits $O := Gx$ and $\varphi(O) = Gy$ are closed;
(3) $G_x = G_y$, i.e. $O \xrightarrow{\sim} \varphi(O)$ is an isomorphism.

Let $I := I(O) \subseteq \mathcal{O}(X)$ be the ideal of O, and let $\widehat{\mathcal{O}(X)}$ be the I-adic completion. Similarly, we set $J := I(\varphi(O))$ and denote by $\widehat{\mathcal{O}(Y)}$ the J-adic completion. Furthermore, $\mathfrak{m} := I \cap \mathcal{O}(X)^G$ is the maximal ideal of $\pi_X(x)$ and $\mathfrak{n} := J \cap \mathcal{O}(Y)^G$ is the maximal ideal of $\pi_Y(y) = (\varphi/\!\!/G)(\pi_X(x))$. We denote by $\widehat{\mathcal{O}(X)^G}$ resp. $\widehat{\mathcal{O}(Y)^G}$ the corresponding $\mathfrak{m}$-adic resp. $\mathfrak{n}$-adic completions. The proof of the Fundamental Lemma will follow from the next result.

4.4.1. Lemma. *The map $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ induces a commutative diagram of isomorphisms*

$$\begin{array}{ccc} \widehat{\mathcal{O}(Y)^G} \otimes_{\mathcal{O}(Y)^G} \mathcal{O}(Y) & \xrightarrow{\simeq} & \widehat{\mathcal{O}(X)^G} \otimes_{\mathcal{O}(X)^G} \mathcal{O}(X) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \widehat{\mathcal{O}(Y)} & \xrightarrow{\simeq} & \widehat{\mathcal{O}(X)} \end{array}$$

Proof. Since the set of points where φ is not étale is closed and G-stable, we can replace X by a special open set X_f where f is an invariant, and thus assume that $\varphi\colon X \to Y$ is étale. Then $\varphi\colon \varphi^{-1}(Gy) \to Gy$ is étale, and so $\varphi^{-1}(Gy)$ is a finite union of closed orbits. Thus, replacing again X by a special open set we can assume that $\varphi^{-1}(Gy) = Gx$ which implies that $I = \varphi^*(J)\mathcal{O}(X)$.

(a) We first show that $\widehat{\mathcal{O}(Y)} \to \widehat{\mathcal{O}(X)}$ is an isomorphism. Since φ is étale, hence flat, we get isomorphisms $J^n \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \xrightarrow{\sim} I^n$ for all $n \geq 0$, and exact sequences

$$0 \to J^{n+1} \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \to J^n \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \to J^n/J^{n+1} \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \to 0.$$

Thus, $J^n/J^{n+1} \xrightarrow{\sim} J^n/J^{n+1} \otimes_{\mathcal{O}(Y)/J} \mathcal{O}(X)/I \xrightarrow{\sim} J^n/J^{n+1} \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \xrightarrow{\sim} I^n/I^{n+1}$, and so, by induction on n, that $\mathcal{O}(Y)/J^n \xrightarrow{\sim} \mathcal{O}(X)/I^n$, and the claim follows.

(b) It remains to show that the vertical maps in the diagram above are isomorphisms. The coordinate ring $\mathcal{O}(X)$ is a direct sum of isotypic components $\mathcal{O}(X) = \bigoplus_{\lambda\in\Lambda_G} \mathcal{O}(X)_\lambda$ where Λ_G is the set of isomorphism classes of irreducible G-modules, and each $\mathcal{O}(X)_\lambda$ is a finitely generated $\mathcal{O}(X)^G$-module. These modules carry two filtrations, namely $\{\mathfrak{m}^\nu \mathcal{O}(X)_\lambda\}_{\nu\in\mathbb{N}}$ and $\{I^\mu \cap \mathcal{O}(X)_\lambda\}_{\mu\in\mathbb{N}}$. The completion with respect to the first filtration is $\widehat{\mathcal{O}(X)}^G \otimes_{\mathcal{O}(X)^G} \mathcal{O}(X)_\lambda$. Denoting by $\widehat{\mathcal{O}(X)_\lambda}$ the completion with respect to the second one, we get $\widehat{\mathcal{O}(X)} = \bigoplus_{\lambda\in\Lambda_G} \widehat{\mathcal{O}(X)_\lambda}$. If we show that the two filtrations are equivalent, then

$$\widehat{\mathcal{O}(X)}^G \otimes_{\mathcal{O}(X)^G} \mathcal{O}(X) = \bigoplus_{\lambda\in\Lambda_G} \widehat{\mathcal{O}(X)}^G \otimes_{\mathcal{O}(X)^G} \mathcal{O}(X)_\lambda \xrightarrow{\sim} \bigoplus_{\lambda\in\Lambda_G} \widehat{\mathcal{O}(X)_\lambda} = \widehat{\mathcal{O}(X)},$$

hence the claim.

(c) In order to see that the two filtrations are equivalent, we first remark that for all $\nu \in \mathbb{N}$ we have $\mathfrak{m}^\nu \mathcal{O}(X)_\lambda \subseteq I^\nu \cap \mathcal{O}(X)_\lambda$. We have to find a $\mu \in \mathbb{N}$ such that $I^\mu \cap \mathcal{O}(X)_\lambda \subseteq \mathfrak{m}^\nu \mathcal{O}(X)_\lambda$. For this we consider the algebra $A := \bigoplus_{n=0}^\infty I^n t^n \subseteq \mathcal{O}(X)[t]$. Let $r_1 t^{m_1}, \ldots, r_k t^{m_k}$ be generators of A^G as an algebra over $\mathcal{O}(X)^G = A_0^G$ where all m_i are positive and $r_i \in \mathfrak{m}$. The isotypic component $A_\lambda = \bigoplus_{n=0}^\infty (I^n \cap \mathcal{O}(X)_\lambda) t^n$ is a finitely generated A^G-module, with generators $s_1 t^{n_1}, \ldots, s_\ell t^{n_\ell}$ where $s_j \in \mathcal{O}(X)_\lambda$. We claim that $I^{m_0\nu+n_0} \cap \mathcal{O}(X)_\lambda \subseteq \mathfrak{m}^\nu \mathcal{O}(X)_\lambda$ where $m_0 := \max_i m_i$ and $n_0 := \max_j n_j$. In fact, let $r \in I^{m_0\nu+n_0} \cap \mathcal{O}(X)_\lambda$. Then $rt^{m_0\nu+n_0} \in A_\lambda$, and so r can be written in the form

$$rt^{m_0\nu+n_0} = \sum_j p_j(r_1 t^{m_1}, \ldots, r_k t^{m_k}) s_j t^{n_j}.$$

It follows that every monomial in p_j has degree at least ν which implies that

$$r = \sum_j p_j(r_1, \ldots, r_k) s_j \in \mathfrak{m}^\nu \mathcal{O}(X)_\lambda.$$

This completes the proof of the lemma. $\square$

4.4.2. Remark. The proof above shows that the Fundamental Lemma holds for affine schemes X, Y of finite type over an algebraically closed field of characteristic zero.

Proof of the Fundamental Lemma. Taking G-invariants in Lemma 4.4.1 we see that $(\varphi/\!\!/G)^*$ induces an isomorphism $\widehat{\mathcal{O}(Y)}^G \xrightarrow{\sim} \widehat{\mathcal{O}(X)}^G$ which means that $\varphi/\!\!/G$ is étale in $\pi(x)$. Replacing X by a suitable saturated affine open neighborhood of x we can assume that $\varphi/\!\!/G\colon X/\!\!/G \to Y/\!\!/G$ is étale. We get the following diagram where the right hand square is cartesian and the vertical maps are the quotients modulo G.

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, "\pi_X"] & X/\!\!/G \times_{Y/\!\!/G} Y \arrow[r, "\varphi"] \arrow[d, "\pi"] & Y \arrow[d, "\pi_Y"] \\
X/\!\!/G \arrow[r, equal] & X/\!\!/G \arrow[r, "\varphi/\!\!/G"] & Y/\!\!/G
\end{tikzcd}$$

By Lemma 4.4.1, applied to the morphism $\psi\colon X \to Z := X/\!\!/G \times_{Y/\!\!/G} Y$ we obtain an isomorphism $\widehat{\mathcal{O}(X)}^G \otimes_{\mathcal{O}(X)^G} \mathcal{O}(Z) \xrightarrow{\sim} \widehat{\mathcal{O}(X)}^G \otimes_{\mathcal{O}(X)^G} \mathcal{O}(X)$. Since $\mathcal{O}_{X/\!\!/G,\pi(x)} \to \widehat{\mathcal{O}(X)}^G$ is faithfully flat where $\mathcal{O}_{X/\!\!/G,\pi(x)}$ is the local ring of $X/\!\!/G$ in $\pi(x)$, this induces an isomorphism $\mathcal{O}_{X/\!\!/G,\pi(x)} \otimes_{\mathcal{O}(X)^G} \mathcal{O}(Z) \xrightarrow{\sim} \mathcal{O}_{X/\!\!/G,\pi(x)} \otimes_{\mathcal{O}(X)^G} \mathcal{O}(X)$, and we can find an $f \in \mathcal{O}(X/\!\!/G)$ such that $f(\pi(x)) \neq 0$ and $\mathcal{O}(Z)_f \simeq \mathcal{O}(X)_f$. Thus, the diagram

$$\begin{tikzcd}
X_f = \pi^{-1}((X/\!\!/G)_f) \arrow[r, "\varphi"] \arrow[d, "\pi_{X_f}"] & Y \arrow[d, "\pi_Y"] \\
(X/\!\!/G)_f \arrow[r, "\varphi/\!\!/G"] & Y/\!\!/G
\end{tikzcd}$$

is cartesian, and the claim follows. $\square$

5. Applications of the Slice Theorem

In the following G is a reductive group and X an affine G-variety where the action is nontrivial.

5.1. Representations. Assume that X is without invariants, i.e. every G-invariant is a constant: $\mathcal{O}(X)^G = \mathbb{C}$. Then X contains a unique closed orbit, $Gx = \overline{Gx}$, and there is an affine G_x-variety Y with a fixed points and without G_x-invariants, and a G-equivariant isomorphism $G \times^{G_x} Y \xrightarrow{\sim} X$. In particular, X is a fiber bundle over G/G_x.

If, in addition, X is smooth in Gx, then there is a representation W of G_x without invariants and an isomorphism $G \times^{G_x} W \xrightarrow{\sim} X$. In particular, X is a G-vector bundle over G/G_x. If the closed orbit is a smooth fixed point, then X is G-isomorphic to a representation.

5.1.1. Examples. (1) Let T be torus acting faithfully on X. If T has a smooth fixed point on X and if $\dim T \geq \dim X$, then X is isomorphic to a representation of T of dimension $\dim T$. E.g. every faithful action of an n-dimensional torus T on $\mathbb{C}^n$ is T-isomorphic to a representation.
(Hint: One has to use the fact that a faithful action of torus admits orbits with trivial stabilizer, and that every torus action on $\mathbb{C}^n$ has fixed points.)

(2) A two-dimensional smooth SL_2-variety is isomorphic to SL_2/T, SL_2/N or to the standard two-dimensional representation.
(Hint: The orbits of SL_2 in affine varieties are either fixed points or of dimension $\geq$ 2. In particular, X has no invariants. If there is a fixed point, then X is isomorphic to the two-dimensional representation $\mathbb{C}^2$. The two-dimensional orbits different from SL_2/T and SL_2/N are of the form SL_2/U_n where $U_n := \{\left[\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right] \mid a^n = 1\}$, and these are not affine. Hence the closure of such an orbit contains a fixed point, and we are in the first case.)

(3) For $n \geq 4$ the only smooth n-dimensional SL_n-varieties are the standard representation $\mathbb{C}^n$ and its dual $(\mathbb{C}^n)^*$.
(Hint: If SL_n acts non-trivially on an irreducible affine variety X of dimension $\leq n$, then $\dim X = n$ and there is a dense orbit. Thus we have to show that in case $n \geq 4$ every reductive subgroup $H \subseteq \mathrm{SL}_n$ has codimension $> n$.)

5.2. Free actions and principal bundles. Here is the main result.

5.2.1. Proposition. *If the action of G on X is free, then $X \to X/\!\!/G$ is a principal G-bundle. Moreover, X is smooth if and only if $X/\!\!/G$ is smooth.*

Proof. Let $S \subseteq X$ be an étale slice in $x \in X$. Since G_x is trivial, we get $G \times^{G_x} S = G \times S$. Hence the fiber product has the form

$$\begin{array}{ccc} G \times S & \xrightarrow[(g,s)\mapsto gs]{\varphi} & X \\ \Big\downarrow{\scriptstyle \mathrm{pr}} & & \Big\downarrow{\scriptstyle \pi_X} \\ S & \xrightarrow{\bar{\varphi}} & X/\!\!/G \end{array}$$

where the horizontal maps are étale and φ is G-equivariant. The claims follows. $\square$

5.2.2. Corollary. *Let T be a torus acting faithfully on an affine variety X. Then X contains a nonempty affine T-stable open set U which is T-isomorphic to $T \times Y$ with suitable affine variety Y.*

Proof. The subset $X' := \{x \in X \mid T_x = \{e\}\} \subseteq X$ is nonempty, open and T-stable. Therefore, the ideal $I(X \setminus X')$ is nonzero and T-stable, and we can find a T-semi-invariant

f vanishing on $X \setminus X'$. It follows that X_f is an affine T-variety contained in X', with a free action of T. By the proposition above, the quotient $X_f \to X_f/\!\!/T$ is a principal T-bundle and thus locally trivial in the Zariski-topology (Proposition 3.7.2(3)). The claim follows. $\square$

What can we say if all orbits in X are closed? Consider the representation of $\mathbb{C}^*$ on $V := \mathbb{C}^2$ given by $t(x,y) := (tx, t^2y)$, and remove the x-axis $y = 0$. Then all orbits of the the action of $\mathbb{C}^*$ on $X := V_y$ are closed and isomorphic to $\mathbb{C}^*$ with trivial stabilizer except for the orbit of $(0,1)$ whose stabilizer is $\{\pm 1\}$. The general result is the following.

5.2.3. Proposition. *Let X be an irreducible affine G-variety and assume that all orbits are closed. Then there is a closed reductive subgroup $H \subseteq G$ and a dense open set $U \subseteq X/\!\!/G$ such that $\pi^{-1}(U) \to U$ is a G-fiber bundle with fiber G/H. Moreover, all stabilizers contain a subgroup of finite index which is conjugate to H in G.*

Proof. Let $O = Gx \subseteq X$ be an orbit, $O = \overline{O}$, and let $S \subseteq X$ be a slice in x which we can assume to be connected. Then we obtain a fiber product

$$\begin{array}{ccc} G \times^{G_x} S & \xrightarrow[{[g,s]\mapsto gs}]{\varphi} & X \\ \big\downarrow{\scriptstyle \mathrm{pr}} & & \big\downarrow{\scriptstyle \pi_X} \\ S & \xrightarrow{\bar\varphi} & X/\!\!/G \end{array}$$

where the horizontal maps are étale and φ is G-equivariant. In particular, S is irreducible and all G_x-orbits in S are closed. Since S contains a fixed point this implies that all G_x orbits are finite. In particular, G_x^0 acts trivially on S. Since the finite group G_x/G_x^0 contains only finitely many subgroups, there is an open dense set of S whose stabilizers in G are equal to a fixed subgroup $H \subseteq G_x$ which contains G_x^0. The claims follow. $\square$

5.3. The Luna stratification. Let X be a smooth affine G-variety, and let $O = Gx$ be a closed orbit. Then $N_x := T_xX/T_xO$ is called the *normal space of X at x*. It is a representation of G_x, and we can form the associated bundle $G \times^{G_x} N_x$ which is called the *normal bundle at x*. Clearly, the normal bundles at all $x \in O$ are G-isomorphic.

Denote by $\mathcal{M}_G$ the set of isomorphism classes of associated bundles $G \times^H N$ where $H \subseteq G$ is a closed reductive subgroup and N an H-module. Then we obtain a map $\mu_X \colon X/\!\!/G \to \mathcal{M}_G$ which associates to $z \in X/\!\!/G$ the normal bundle at the closed orbit in the fiber $\pi^{-1}(z)$. The main result is the following.

5.3.1. Theorem. *Let X be a smooth affine G-variety with quotient $\pi_X \colon X \to X/\!\!/G$.*
1. *The image of $\mu_X(X/\!\!/G) \subseteq \mathcal{M}_G$ is finite.*
2. *For any $\lambda \in \mathcal{M}_G$ the subset $(X/\!\!/G)_\lambda := \mu_X^{-1}(\lambda)$ is locally closed and smooth.*
3. *The inverse image $X_\lambda := \pi_X^{-1}((X/\!\!/G)_\lambda)$ is reduced and $\pi_X \colon X_\lambda \to (X/\!\!/G)_\lambda$ is a G-fiber bundle.*

The finite stratification $X/\!\!/G = \bigcup_\lambda (X/\!\!/G)_\lambda$ is called *the* Luna *stratification.*

Proof. (1) Let $\lambda \in \mathcal{M}_G$ be the isomorphism class of $G \times^H N$. Using the canonical identification of $(G \times^H N)/\!\!/G$ with $N/\!\!/H$ we get

$$(N/\!\!/H)_\lambda = N^H/\!\!/H \simeq N^H \quad\text{and}\quad (G \times^H N)_\lambda = G \times^H \pi^{-1}(N^H/\!\!/H).$$

In particular, $(N/\!\!/H)_\lambda \subseteq N/\!\!/H$ is locally closed and smooth, and the inverse image is reduced. Since $\pi_N^{-1}(N^H/\!\!/H) = N^H \times \mathcal{N}_N$ where $\mathcal{N}_N \subseteq N$ is the nullcone we finally get

$$(G \times^H N)_\lambda = G \times^H \pi^{-1}(N^H \times \mathcal{N}_N) = G \times^H \mathcal{N}_N \times N^H.$$

This shows that $(G \times^H N)_\lambda \to (N/\!\!/H)_\lambda$ is a trivial G-fiber bundle with fiber $G \times^H \mathcal{N}_N$.

(2) Let X, Y be two smooth affine G-varieties and let $\varphi\colon X \to Y$ be an excellent morphism. Then, for any $\lambda \in \mathcal{M}_G$, we have $\bar{\varphi}^{-1}((Y/\!\!/G)_\lambda) = (X/\!\!/G)_\lambda$ and $\varphi^{-1}(Y_\lambda) = X_\lambda$. In particular, $(Y/\!\!/G)_\lambda$ is locally closed in $Y/\!\!/G$ if and only if $(X/\!\!/G)_\lambda$ is locally closed in $X/\!\!/G$.

(3) Combining (1) and (2) with Theorem 4.3.3 we get our claims. □

If X is an affine smooth G-variety such that $X/\!\!/G$ is irreducible, then there is unique $\lambda \in \mathcal{M}_G$ such that $(X/\!\!/G)_\lambda$ is open. This stratum is called the *principal stratum*. It has the following characterization.

5.3.2. Corollary. *Let $Gx \subseteq X$ be a closed orbit and put $z = \pi_X(x) \in X/\!\!/G$. Then the following conditions are equivalent.*

(1) *z belongs to the principal stratum;*
(2) *$\mathcal{N}_{N_x}$ is the canonical G_x-stable complement of $N_x^{G_x}$ in N_x;*
(3) *π_X is smooth in Gx;*
(4) *For every closed orbit $O \subseteq X$ there is a G-equivariant morphism $Gx \to O$.*

5.3.3. Corollary ([**Lun72**]). *Let X be an smooth affine G-variety. Assume that for every $x \in X$ the tangent space T_xX admits a G_x-invariant, non-degenerate symmetric bilinear form. Then X contains a non-empty open set consisting of closed orbits, i.e. the fibers of the principal stratum are orbits.*

Proof. We will show that the fibers over the principal stratum $(X/\!\!/G)_{\text{pr}}$ are orbits, i.e. that $N_x = N_x^{G_x}$. For this we use the following fact: If V is a G_x-module and $U \subseteq V$ a G_x-submodule and if both admit a G_x-invariant, non-degenerate symmetric bilinear form, then the same holds for V/U. Using this, we can conclude that $N_x/N_x^{G_x}$ admits such a form. Since this space is equal to $\mathcal{N}_{N_x}$, by the previous corollary, and since every G_x-invariant function on $\mathcal{N}_{N_x}$ is a constant we get $N_x/N_x^{G_x} = 0$, and the claim follows. □

5.4. Fix-pointed actions. Fix-pointed actions of reductive groups have been introduced and studied by Bass and Haboush in the paper [**BH85**]. Recall that an action of an algebraic group on a variety is called *fix-pointed* if every closed orbit is a fixed point.

Bibliography

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. MR 0242802 (39 #4129)

[Art91] Michael Artin, *Algebra*, Prentice Hall Inc., Englewood Cliffs, NJ, 1991. MR 1129886 (92g:00001)

[BH85] H. Bass and W. Haboush, *Linearizing certain reductive group actions*, Trans. Amer. Math. Soc. **292** (1985), no. 2, 463–482. MR 808732 (87d:14039)

[Bor91] Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.

[Bre72] Glen E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York-London, 1972, Pure and Applied Mathematics, Vol. 46. MR 0413144 (54 #1265)

[DC70] Jean A. Dieudonné and James B. Carrell, *Invariant theory, old and new*, Advances in Math. **4** (1970), 1–80 (1970). MR 0255525 (41 #186)

[DC71] ———, *Invariant theory, old and new*, Academic Press, New York, 1971. MR 0279102 (43 #4828)

[Der01] Harm Derksen, *Polynomial bounds for rings of invariants*, Proc. Amer. Math. Soc. **129** (2001), no. 4, 955–963 (electronic). MR MR1814136 (2002b:13008)

[Eis95] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. MR 1322960 (97a:13001)

[Fis66] Ch.S. Fisher, *The death of a mathematical theory: a study in the sociology of knowledge*, Arch. History Exact Sci. **3** (1966), 137–159.

[FK16] Jean-Philippe Furter and Hanspeter Kraft, *On the geometry of automorphism groups of affine varieties*, in preparation, 2015/16.

[Fre06] Gene Freudenburg, *Algebraic theory of locally nilpotent derivations*, Encyclopaedia of Mathematical Sciences, vol. 136, Springer-Verlag, Berlin, 2006, Invariant Theory and Algebraic Transformation Groups, VII. MR 2259515 (2008f:13049)

[Gro58] A. Grothendieck, *Torsion homologique et sections rationelle*, Séminaire Chevalley (1958), exposé n°5.

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York, 1977.

[Hil90] David Hilbert, *Ueber die Theorie der algebraischen Formen*, Math. Ann. **36** (1890), no. 4, 473–534. MR MR1510634

[Hil93] ———, *Ueber die vollen Invariantensysteme*, Math. Ann. **42** (1893), no. 3, 313–373. MR MR1510781

[Hum75] James E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975, Graduate Texts in Mathematics, No. 21.

[Hur97] A. Hurwitz, *Über die Erzeugung der Invarianten durch Integration*, Nachr. Akad. Wiss. Göttingen (1897).

[Igu73] Jun-ichi Igusa, *Geometry of absolutely admissible representations*, Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki, Kinokuniya, Tokyo, 1973, pp. 373–452. MR 0367077 (51 #3319)

[KKV89] Friedrich Knop, Hanspeter Kraft, and Thierry Vust, *The Picard group of a G-variety*, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 77–87. MR MR1044586

[KR14] Hanspeter Kraft and Peter Russell, *Families of group actions, generic isotriviality, and linearization*, Transform. Groups **19** (2014), no. 3, 779–792. MR 3233525

[Kra78] Hanspeter Kraft, *Parametrisierung von Konjugationsklassen in $\mathfrak{sl}_n$*, Math. Ann. **234** (1978), no. 3, 209–220. MR MR0491855 (58 #11047)

[Lun72] Domingo Luna, *Sur les orbites fermées des groupes algébriques réductifs*, Invent. Math. **16** (1972), 1–5. MR MR0294351 (45 #3421)

[Lun73] ______, *Slices étales*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33. MR MR0342523 (49 #7269)

[Lun75] D. Luna, *Adhérences d'orbite et invariants*, Invent. Math. **29** (1975), no. 3, 231–238. MR MR0376704 (51 #12879)

[Mat60] Yozô Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. J **16** (1960), 205–218. MR 0109854 (22 #739)

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR MR1011461 (90i:13001)

[Mey99] Franz Meyer, *Invariantentheorie*, Enzyklopädie der Mathematischen Wissenschaften, vol. I, Teil IB2, 1899, pp. 320–403.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR MR1304906 (95m:14012)

[Mon36] Deane Montgomery, *Continuity in topological groups*, Bull. Amer. Math. Soc. **42** (1936), no. 12, 879–882. MR 1563458

[Mum95] David Mumford, *Algebraic geometry. I*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Complex projective varieties, Reprint of the 1976 edition. MR MR1344216 (96d:14001)

[Mum99] ______, *The red book of varieties and schemes*, expanded ed., Lecture Notes in Mathematics, vol. 1358, Springer-Verlag, Berlin, 1999, Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello. MR MR1748380 (2001b:14001)

[Nag59] Masayoshi Nagata, *On the 14-th problem of Hilbert*, Amer. J. Math. **81** (1959), 766–772.

[Nag64] ______, *Invariants of a group in an affine ring*, J. Math. Kyoto Univ. **3** (1963/1964), 369–377. MR 0179268 (31 #3516)

[Noe15] Emmy Noether, *Der Endlichkeitssatz der Invarianten endlicher Gruppen*, Math. Ann. **77** (1915), no. 1, 89–92. MR 1511848

[Pal78] Richard S. Palais, *Some analogues of Hartogs' theorem in an algebraic setting*, Amer. J. Math. **100** (1978), no. 2, 387–405. MR 0480509 (58 #670)

[Pro78] C. Procesi, *Positive symmetric functions*, Adv. in Math. **29** (1978), no. 2, 219–225. MR 506891 (80d:15034)

[PS85] Claudio Procesi and Gerald Schwarz, *Inequalities defining orbit spaces*, Invent. Math. **81** (1985), no. 3, 539–554. MR MR807071 (87h:20078)

[Sch68] Issai Schur, *Vorlesungen über Invariantentheorie*, Bearbeitet und herausgegeben von Helmut Grunsky. Die Grundlehren der mathematischen Wissenschaften, Band 143, Springer-Verlag, Berlin, 1968. MR MR0229674 (37 #5248)

[Sch75] Gerald W. Schwarz, *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), 63–68. MR MR0370643 (51 #6870)

[Sch89] Barbara J. Schmid, *Generating invariants of finite groups*, C. R. Acad. Sci. Paris Sér. I Math. **308** (1989), no. 1, 1–6. MR MR980311 (90d:20014)

[Ser58] Jean-Pierre Serre, *Espaces fibrés algébriques*, Séminaire Chevalley (1958), exposé n° 5.

[Sha94a] Igor R. Shafarevich, *Basic algebraic geometry. 1*, second ed., Springer-Verlag, Berlin, 1994, Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid. MR 1328833 (95m:14001)

[Sha94b] ______, *Basic algebraic geometry. 2*, second ed., Springer-Verlag, Berlin, 1994, Schemes and complex manifolds, Translated from the 1988 Russian edition by Miles Reid. MR 1328834 (95m:14002)

[Slo89] Peter Slodowy, *Der Scheibensatz für algebraische Transformationsgruppen*, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 89–113. MR 1044587

[Spr77] T. A. Springer, *Invariant theory*, Lecture Notes in Mathematics, Vol. 585, Springer-Verlag, Berlin, 1977. MR 0447428 (56 #5740)

[Spr89] Tonny A. Springer, *Aktionen reduktiver Gruppen auf Varietäten*, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 3–39. MR 1044583

[Vus76] Thierry Vust, *Sur la théorie des invariants des groupes classiques*, Ann. Inst. Fourier (Grenoble) **26** (1976), no. 1, ix, 1–31. MR MR0404280 (53 #8082)

[Wey39] Hermann Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton University Press, Princeton, N.J., 1939. MR MR0000255 (1,42c)

[Wey97] ______, *The classical groups*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Their invariants and representations, Fifteenth printing, Princeton Paperbacks. MR MR1488158 (98k:01049)