LOCAL PROPERTIES OF ALGEBRAIC GROUP ACTIONS

Friedrich Knop **Hanspeter Kraft** **Domingo Luna**
Thierry Vust

Table of Contents

Introduction

In this article we present a fundamental result due to SUMIHIRO. It states that every normal G-variety X, where G is a connected linear algebraic group, is locally isomorphic to a quasi-projective G-variety, i.e., to a G-stable subvariety of the projective space $\mathbf{P}^n$ with a linear G-action (Theorem 1.1). The central tools for the proof are G-linearization of line bundles (§2) and some properties of the Picard group of a linear algebraic group (§4).

Along the proof, we also need some results about invertible functions on algebraic varieties and groups, which are due to ROSENLICHT. They are given in our second article *"The Picard group of a G-variety"* in this volume; it will be quoted by [Pic].

We work over a field of characteristic zero. Nevertheless, the main results are valid in positive characteristic as well, and our proofs seem to work in the general situation, too. We leave it to the reader to verify the details.

§ 1 The Theorem of SUMIHIRO

We fix an algebraically closed base field k of characteristic zero. Let G be a *connected linear algebraic group* and X a *normal* G-variety. We plan to give a proof of the following fundamental result due to SUMIHIRO [Su74,Su75].

1.1. Theorem. *Let $Y \subset X$ be an orbit in X. There is a finite dimensional rational representation $G \to \mathrm{GL}(V)$ and a G-stable open neighborhood U of Y in X which is G-equivariantly isomorphic to a G-stable locally closed subvariety of the projective space $\mathbf{P}(V)$.*

(As usual, a representation $\rho : G \to \mathrm{GL}(V)$ is called *rational* if ρ is a morphism of algebraic groups.)

Remark. The plane cubic with an ordinary double point admits a k^*-action with two orbits: the singular point as a fixed point and its complement which is isomorphic to k^*. This example shows that the normality assumption in the theorem is necessary. In fact, for every representation V of k^* the closure of an non-trivial orbit in $\mathbf{P}(V)$ always contains two fixed points (cf. [LV83, 1.6] or [Od78]).

1.2. Outline of Proof. Let $U_0 \subset X$ be an affine open subset with $U_0 \cap Y \neq \varnothing$. There is a line bundle L on X and a finite dimensional subspace N of the space $\mathrm{H}^0(X, L)$ of sections of L such that the corresponding rational map

$$\gamma_N : X \dashrightarrow \mathbf{P}(N^\vee)$$

which sends x to the kernel of the map $\varepsilon_x : N \to L_x$, $\sigma \mapsto \sigma(x)$, induces a (biregular) isomorphism of U_0 onto a locally closed subvariety of $\mathbf{P}(N^\vee)$. ($N^\vee$ denotes the dual space of N.) In fact, consider the divisor $D := X \setminus U_0$ and the invertible sheaf $\mathcal{O}(mD)$ of rational functions with poles of order at most m on D. If $f_0 := 1, f_1, \ldots, f_n$ is a system of generators for the subalgebra $k[U_0] \subset k(X)$ and $N := \langle f_0, f_1, \ldots, f_n \rangle$ the linear span of the f_i's, we have $N \subset \mathrm{H}^0(X, \mathcal{O}(mD))$ for all $m \geq m_0$, and the claim follows.

The main step in the proof will be to show that for suitable $m \geq m_0$ the sheaf $\mathcal{O}(mD)$ is *G-linearizable* (Proposition 2.4). Then the linear action of G on $\mathrm{H}^0(X, \mathcal{O}(mD))$ is locally finite and rational (Lemma 2.5). Replacing N by the finite dimensional G-stable subspace $W \subset \mathrm{H}^0(X, \mathcal{O}(mD))$ generated by N we obtain a G-equivariant rational map

$$\gamma_W : X \dashrightarrow \mathbf{P}(W^\vee)$$

which induces an isomorphism of $U := GU_0$ with a G-stable locally closed subvariety of $\mathbf{P}(W^\vee)$.

1.3. In the next two paragraphs we give the details needed in the proof above. In paragraph 4 we offer a different proof based on techniques developed in [LV83].

§ 2 G-Linearization of Line Bundles

2.1. We first recall some results concerning G-linearization of line bundles (cf. [MF82, Chap. I, §3]). As above, G is a linear algebraic group and X a G-variety. We denote by $\varphi : G \times X \to X$ the G-action and by $p_X : G \times X \to X$ the projection. Let $\pi : L \to X$ be a line bundle on X. We do not distinguish between the line bundle L and its sheaf of sections.

Definition. A *G-linearization* of L is a G-action

$$\Phi : G \times L \to L$$

on L such that (a) $\pi : L \to X$ is G-equivariant and (b) the action is linear on the fibers, i.e., for every $g \in G$ and $x \in X$ the map $\Phi_x : L_x \to L_{gx}$ is linear.

Example. Let H be a closed subgroup of G. We denote by $\pi : G \to G/H$ the projection and by $\mathcal{X}(H)$ the character group of H. For every character $\chi \in \mathcal{X}(H)$ we define a line bundle E_χ on G/H in the following way: It is the quotient of $G \times k$ by the action of H given by

$$h(g, x) := (gh^{-1}, \chi(h)\cdot x), \qquad (h \in H, g \in G, x \in k).$$

(Of course one has to show that this quotient exists.) This defines a homomorphism

$$\mathcal{E} : \mathcal{X}(H) \longrightarrow \operatorname{Pic}(G/H) : \chi \mapsto E_\chi.$$

The image of this map is the subgroup consisting of the G-linearizable line bundles on G/H. In fact, by construction, E_χ is equipped with a G-action, which is linear in the fibres. On the other hand, given a G-linearized line bundle L on G/H, the group H acts on the fibre $L_H \simeq k$ over $H = eH \in G/H$ by a character χ, and the canonical map $G \times L_H \to L$, $(g, l) \mapsto gl$ induces an G-isomorphism $E_\chi \simeq L$. In particular, *every G-linearizable line bundle on G is trivial.*

2.2. It is clear that for any G-linearization we obtain a commutative diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\Phi} & L \\ \big\downarrow{\scriptstyle \mathrm{id}\times\pi} & & \big\downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\varphi} & X \end{array} \tag{1}$$

which is a pull-back diagram, i.e., it induces an isomorphism

$$G \times L = p_X^*(L) \xrightarrow{\sim} \varphi^*(L)$$

of line bundles on $G \times X$. In fact, the commutativity of the diagram is equivalent to the G-equivariance of $\pi : L \to X$, and the induced morphism $p_X^*(L) \to \varphi^*(L)$ is a bijective homomorphism of line bundles, since the action is linear on the fibers. In addition, the restriction of Φ to $\{e\} \times L$ is the identity. We claim that the converse is true:

Lemma. *Let $\Phi : G \times L \to L$ be a morphism. Assume that the diagram*

$$\begin{array}{ccc} G \times L & \xrightarrow{\Phi} & L \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \pi} & & \Big\downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\varphi} & X \end{array}$$

is a pull-back diagram, that $\Phi(e, z) = z$ for all $z \in L$ and that $\Phi(g, ?)$ maps the zero section of L into itself for all $g \in G$. Then Φ is a G-linearization of L.

PROOF: By assumption, the morphisms $\Phi(g, ?) : L_x \to L_{gx}$ are all bijective and send 0 to 0; hence they are linear isomorphisms. It follows that there is an invertible function $f : G \times G \times L \longrightarrow k^*$ such that

$$\Phi(gh, z) = f(g, h, z)\Phi(g, \Phi(h, z)) \quad \text{for all } g, h \in G \text{ and } z \in L.$$

(The existence of such a map f is clear; we leave it to the reader to check that f is regular.) By a result of ROSENLICHT's (see [Pic, Proposition 1.1]), the function f is of the form

$$f(g, h, z) = r(g)s(h)t(z) \quad (g, h \in G, z \in L)$$

with invertible regular functions r, s on G and t on L. Since $\Phi(e, z) = z$ for every $z \in L$ we obtain

$$r(e)s(h)t(z) = 1 \qquad (h \in G, z \in L),$$

and similarly

$$r(g)s(e)t(z) = 1 \qquad (g \in G, z \in L).$$

Hence

$$\begin{aligned} f(g, h, z) &= r(g)s(h)t(z) = (r(g)s(h)t(z))\,(r(e)s(e)t(z)) \\ &= (r(g)s(e)t(z))\,(r(e)s(h)t(z)) = 1 \end{aligned}$$

for every $g, h \in G$, $z \in L$, and the claim follows. □

2.3. Lemma. *The line bundle L is G-linearizable if and only if the two bundles $\varphi^*(L)$ and $p_X^*(L)$ on $G \times X$ are isomorphic.*

PROOF: We have already seen that a G-linearization of L induces an isomorphism $p_X^*(L) \xrightarrow{\sim} \varphi^*(L)$. Conversely, such an isomorphism gives rise to a pull-back diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\Phi} & L \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \pi} & & \Big\downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\varphi} & X \end{array}$$

with the property that every $\Phi(g, ?)$ maps the zero section of L into itself. The restriction of Φ to $\{e\} \times L$ is an automorphism of the line bundle L, hence given by a regular function $\lambda : X \to k^*$, defined by $\lambda(\pi(z)) \cdot z = \Phi(e, z)$ $(z \in L)$. Replacing Φ by $\lambda^{-1}\Phi$ we obtain a pull-back diagram satisfying the assumptions of the previous Lemma 2.2, and so L is G-linearizable. □

In the proof of the next proposition we shall need two results from paragraph 4.

2.4. Proposition. *Let L be a line bundle on a normal G-variety X. There is a number $n > 0$ such that $L^{\otimes n}$ is G-linearizable.*

PROOF: As before we denote by $\varphi : G \times X \to X$ the G-action on X and by p_X, p_G the two projections $G \times X \to X$ and $G \times X \to G$. It follows from Lemma 4.2 that

$$\varphi^*(L) \simeq p_G^*(M) \otimes p_X^*(N)$$

with a line bundle M on G and with

$$N := \varphi^*(L)|_{\{e\}\times X} \simeq L.$$

(Here we use the normality of X!) Since $\operatorname{Pic} G$ is finite (Proposition 4.5) we obtain

$$\varphi^*(L^{\otimes n}) \simeq p_X^*(L^{\otimes n}),$$

for a suitable $n > 0$, and the claim follows from Lemma 2.3. □

Remark. We have seen in the proof above that the number n in the proposition can be chosen to be the order of $\operatorname{Pic} G$. In particular, if G is factorial then every line bundle on X is G-linearizable.

2.5. End of proof. To finish the proof along the lines indicated in 1.2 we need the following result about the action of G on the space $\mathrm{H}^0(X, L)$ of sections of a G-linearized line bundle L.

Lemma. *Let L be a G-linearized line bundle on X. Then the action of G on $\mathrm{H}^0(X, L)$ given by*

$$^g\sigma(x) := g(\sigma(g^{-1}x)) = \Phi(g, \sigma(g^{-1}x))$$

for $g \in G$, $\sigma \in \mathrm{H}^0(X, L)$, $x \in X$, is locally finite and rational.

(A linear action of G on a vector space W is called *locally finite and rational* if every $w \in W$ is contained in a finite dimensional G-stable subspace V such that the corresponding homomorphism $G \to \mathrm{GL}(V)$ is a rational representation of G.)

PROOF: We first remark that there is a canonical isomorphism

$$k[G] \otimes \mathrm{H}^0(X, L) \xrightarrow{\sim} \mathrm{H}^0(G \times X, p_X^*(L))$$

(see [Ha77, Chap. III, Proposition 9.3]), which associates to $f \otimes \tau$ the section $(g,x) \mapsto (g, f(g)\cdot\tau(x))$. The G-linearization $\Phi : G \times L \to L$ of L induces a linear map

$$\Phi^* : \mathrm{H}^0(X,L) \longrightarrow \mathrm{H}^0(G \times X, p_X^*(L)) \simeq k[G] \otimes \mathrm{H}^0(X,L)$$

which sends the section σ to the map

$$\tilde{\sigma} : G \times X \longrightarrow L, \quad (g,x) \mapsto {}^{g^{-1}}\sigma(x) = \Phi(g^{-1}, \sigma(gx)).$$

(This follows immediately from the pull-back diagram (1) in 2.2.) If we write $\Phi^*(\sigma)$ in the form

$$\Phi^*(\sigma) = \textstyle\sum_i f_i \otimes \sigma_i, \quad f_i \in k[G],\ \sigma_i \in \mathrm{H}^0(X,L),$$

we see that ${}^g\sigma = \sum f_i(g^{-1})\sigma_i$, and the claim follows easily. □

As a consequence of the previous results we obtain the following corollary:

2.6. Corollary. *Let X be a quasi-projective normal G-variety. There is a finite dimensional rational representation $G \to \mathrm{GL}(V)$ and a G-equivariant isomorphism of X with a locally closed G-stable subvariety of the projective space $\mathbf{P}(V)$.*

PROOF: By assumption, X is a locally closed subvariety of some projective space $\mathbf{P}(M)$, and the inclusion $X \hookrightarrow \mathbf{P}(M)$ is of the form γ_N as in 1.2, where L is the line bundle associated to the invertible sheaf $\mathcal{O}(1)|_X$ and $N = M^\vee \subset \mathrm{H}^0(X,L)$. By Proposition 2.4 the line bundle $L^{\otimes m}$ is G-linearizable for a suitable $m > 0$, and we proceed as above to obtain a G-equivariant inclusion of X into a projective space $\mathbf{P}(V)$ with a linear G-action. □

§ 3 Another Proof of SUMIHIRO's Theorem

We give a second proof of Theorem 1.1 which is based on techniques developed by LUNA and VUST in [LV83, §8].

3.1. As before, let G be a connected linear algebraic group and X a normal G-variety. We assume that $k[G]$ is factorial. This is no restriction since every algebraic group G has a finite covering $\tilde{G} \to G$ such that $k[\tilde{G}]$ factorial (Proposition 4.6).

We consider the following two actions of G on $G \times X$:

- A *left action* defined by $t\cdot(s,x) := (ts, x)$,
- A *right action* defined by $(s,x)\cdot t := (st, t^{-1}\cdot x)$,

where $s,t \in G$, $x \in X$. Clearly, these two actions commute. We denote by $k(G \times X)^G$ the field of those rational functions on $G \times X$ which are invariant under the right action of G. The G-action $\varphi : G \times X \to X$ on X is equivariant with respect to the left action of G, and φ^* induces an isomorphism $k(X) \xrightarrow{\sim} k(G \times X)^G$.

3.2. Let $\mathcal{O}_{X,Y} \subset k(X)$ denote the local ring of $Y \subset X$ and $\mathfrak{m}_{X,Y}$ its maximal ideal. We plan to show that there exist a finite dimensional subspace M of $k[G] \otimes k(X)$ which is stable under the left action of G, and an element $h \in M$, $h \neq 0$, satisfying the following properties:

(i) $\frac{1}{h}M$ is contained in $\varphi^*(\mathcal{O}_{X,Y})$; in particular $\frac{1}{h}M \subset k(G \times X)^G$.

(ii) $\mathcal{O}_{X,Y}$ is the localization of $k[\frac{1}{h}M]$ (considered as a subalgebra of $k(X)$) at the ideal $k[\frac{1}{h}M] \cap \mathfrak{m}_{X,Y}$.

We claim that this implies Theorem 1.1. In fact, the inclusion of M into the field $k(G \times X)$ corresponds to a rational map

$$\mu : G \times X \dashrightarrow \mathbf{P}(M^\vee)$$

which is equivariant with respect to the left action of G on $G \times X$ and the linear action of G on $\mathbf{P}(M^\vee)$. We denote by X' the closure of the image of μ and by X'_h the intersection of X' with the affine open subset

$$\mathbf{P}(M^\vee)_h := \{x \in \mathbf{P}(M^\vee) \mid h(x) \neq 0\} \subset \mathbf{P}(M^\vee).$$

X'_h is affine and the algebra $k[X'_h]$ coincides with the subalgebra $k[\frac{1}{h}M]$ of $k(G \times X)$. According to (i) the map μ factors through φ, inducing a rational map (again denoted by μ)

$$\mu : X \dashrightarrow X'$$

which is regular in a neighbourhood of Y. Now it follows from (ii) that μ induces an isomorphism of an open subset U containing Y with a locally closed subvariety of $\mathbf{P}(M^\vee)$. □

3.3. Construction of M. To simplify notations we set $A := k[G] \otimes k(X)$; this is a factorial ring (see 3.5) whose field of fractions is $k(G \times X)$. Let $f \in k(X)$. We write $\varphi^*(f) = \frac{a}{b}$ where $a, b \in A$ are relatively prime. Since $\varphi^*(f) \in k(G \times X)^G$, we get

$$a^t = \gamma(t)a \quad \text{and} \quad b^t = \gamma(t)b \quad (t \in G)$$

where $a^t(s, x) := a(st^{-1}, tx)$ is the translate of a with respect to the right action of G and $\gamma : G \to k(X)^*$ is a cocycle with values in $k(X)^*$ by Lemma 3.6.

We choose a finite dimensional subspace N_1 of $\mathcal{O}_{X,Y}$ containing the constants such that $\mathcal{O}_{X,Y}$ is the localization of $k[N_1]$ at $k[N_1] \cap \mathfrak{m}_{X,Y}$. It follows from what we have seen above that there are a finite dimensional subspace N of A, an element $h \in N$ and a cocycle $\gamma : G \to k(X)^*$ such that $\varphi^*(N_1) = \frac{1}{h}N$ and $a^t = \gamma(t)a$ for every $a \in N$ and $t \in G$. Of course, we can assume that the elements of N do not have a common divisor in A.

Claim. *For every $a \in N$ and $t \in G$ we have* $\dfrac{{}^t a}{h} \in \varphi^*(\mathcal{O}_{X,Y})$.

(The function ${}^t a$ is the translate of a with respect to the left action of the group G: ${}^t a(s,x) := a(t^{-1}s, x)$.)

It is clear now that the G-submodule M of A generated by N satisfies the conditions (i) and (ii) of 3.2. It remains to prove the claim above.

3.4. Proof of the Claim. Since the right and the left action of G commute and, in addition, the left action is trivial on $k(X)$, we obtain $({}^t a)^s = \gamma(s)({}^t a)$ ($s, t \in G$, $a \in N$). Therefore we have $\frac{{}^t a}{h} \in \varphi^*(k(X)) = k(G \times X)^G$.

Up to now we have not used the normality of X. This assumption implies that $\mathcal{O}_{X,Y}$ is a Krull-ring whose essential valuations ν_Z are those associated to the local rings $\mathcal{O}_{X,Z}$ where Z is an irreducible closed subvariety of codimension 1 containing Y.

Let Z_0 be such a subvariety. Then $\varphi^{-1}(Z_0)$ is an irreducible subvariety of $G \times X$ of codimension 1 and the corresponding valuation $\nu_{\varphi^{-1}(Z_0)}$ of $k(G \times X)$ extends ν_{Z_0}. (Recall that $k(X) = k(G \times X)^G \subset k(G \times X)$.) If Z_0 is G-invariant then $\nu_{\varphi^{-1}(Z_0)}$ is G-invariant, too, i.e., $\nu_{\varphi^{-1}(Z_0)}({}^t f) = \nu_{\varphi^{-1}(Z_0)}(f)$. If Z_0 is not G-invariant then $\nu_{\varphi^{-1}(Z_0)}$ is improper on the subfield $k(X)$ of $k(G \times X)$ and is positive on $k[G]$. It follows that $\nu_{\varphi^{-1}(Z_0)}$ is an essential valuation of the factorial $k(X)$-algebra (hence Krull-algebra) $A = k[G] \otimes k(X)$.

Now let ν_Z be any essential valuation of $\varphi^*(\mathcal{O}_{X,Y})$, $f \in N$ and $t \in G$. We have to show that $\nu_Z(\frac{{}^t f}{h}) \geq 0$. If Z is G-stable we find

$$\begin{aligned}
\nu_Z(\frac{{}^t f}{h}) &= \nu_{\varphi^{-1}(Z)}(\frac{{}^t f}{h}) = \nu_{\varphi^{-1}(Z)}({}^t f) - \nu_{\varphi^{-1}(Z)}(h) \\
&= \nu_{\varphi^{-1}(Z)}(f) - \nu_{\varphi^{-1}(Z)}(h) = \nu_{\varphi^{-1}(Z)}(\frac{f}{h}) \\
&= \nu_Z(\frac{f}{h}) \geq 0.
\end{aligned}$$

If Z is not G-stable we have $\nu_{\varphi^{-1}(Z)}(f') \geq 0$ for all $f' \in N$ because $N \subset A$ and $\nu_{\varphi^{-1}(Z)}$ is essential for this algebra. Also,

$$\nu_Z(\frac{f'}{h}) = \nu_{\varphi^{-1}(Z)}(f') - \nu_{\varphi^{-1}(Z)}(h) \geq 0.$$

By assumption, the elements of N do not have a common divisor and so $\nu_{\varphi^{-1}(Z)}(h) = 0$. Hence

$$\nu_Z(\frac{{}^t f}{h}) = \nu_{\varphi^{-1}(Z)}({}^t f) \geq 0,$$

since ${}^t f \in A$. This finishes the proof of the theorem. □

3.5. In 3.3 we have used the result that for an algebraic group G and a field extension K/k the K-algebra $k[G] \otimes K$ is factorial in case $k[G]$ is factorial. This follows from the lemma below and the fact that G is a rational variety (see 4.1).

Lemma. *Let Y be an affine rational variety. If $k[Y]$ is factorial then for every field extension K/k the algebra $k[Y] \otimes_k K$ is also factorial.*

PROOF: Since Y is rational there is an $f \in k[Y]$ such that the localisation $k[Y]_f$ is isomorphic to $k[x_1, x_2, \ldots, x_n]_h$ for a suitable $h \in k[x_1, x_2, \ldots, x_n]$, $n = \dim Y$. Clearly, $K[x_1, x_2, \ldots, x_n]_h$ is factorial, and so $K[Y]_{f\otimes 1}$ is factorial, too. (We put $K[Y] := k[Y] \otimes_k K$.) Consider a primary decomposition $f = \prod f_i$. Since

$$K[Y]/(f_i \otimes 1)K[Y] \simeq (k[Y]/f_i k[Y]) \otimes_k K$$

is an integral domain (k is algebraically closed), the ideal of $K[Y]$ generated by $f_i \otimes 1$ is prime. This implies that $K[Y]$ is factorial ([BAC7, §3, n°4, proposition 3]). □

3.6. Finally we prove the second result used in 3.3.

Lemma. *Every cocycle of G with values in the group A^* of units of $A = k[G] \otimes k(X)$ takes its values in $k(X)^*$.*

PROOF: Let U be an open subvariety of X. By results of ROSENLICHT ([Pic, 1.1, 1.2]) the group $(k[G] \otimes k[U])^*$ is generated by $k[G]^*$ and $k[U]^*$, and $k[G]^* = k^* \times \mathcal{X}(G)$ where $\mathcal{X}(G)$ denotes the character group of G. From this we obtain

$$\begin{aligned} A^* &= \left(\bigcup_{U \text{ open in } X} k[G] \otimes k[U] \right)^* = \bigcup_{U \subset X} (k[G] \otimes k[U])^* \\ &= \bigcup_{U \subset X} \mathcal{X}(G) \times k[U]^* \quad \simeq \quad \mathcal{X}(G) \times k(X)^*. \end{aligned}$$

Now consider an element a of A^* and write a in the form $a = \chi p$ with $\chi \in \mathcal{X}(G)$ and $p \in k(X)^*$. One easily sees that

$$a^t = \chi(\chi(t)^{-1} p^t) \quad (t \in G).$$

Let γ be a cocycle with values in A^* and decompose $\gamma(s)$ in the form $\gamma(s) = \chi_s p_s$ with $\chi_s \in \mathcal{X}(G)$ and $p_s \in k(X)^*$. Then the cocycle condition $\gamma(st) = \gamma(s)^t \gamma(t)$ becomes

$$\begin{aligned} \chi_{st} p_{st} &= (\chi_s p_s)^t \chi_t p_t \\ &= (\chi_s \chi_t)(\chi_s(t)^{-1} p_s^t p_t). \end{aligned}$$

In particular one sees that the map $s \mapsto \chi_s$ is a group homomorphism $G \to \mathcal{X}(G)$. But every such homomorphism is trivial: We can clearly assume that G is commutative which implies that G is divisible (being a product of additive and multiplicative groups), whereas $\mathcal{X}(G)$ is a finitely generated abelian group. Hence we obtain $\gamma(G) \subset k(X)^*$. □

§ 4 Picard Group of a Linear Algebraic Group

Let G be a connected linear algebraic group. In this paragraph we explain some classical results about the Picard group $\operatorname{Pic} G$ (cf. [FI74], [Po74], [Iv76]). In particular, we give the proofs of several results which have been used in the previous paragraphs.

4.1. We start by recalling some well-known results about the structure of the underlying variety of a linear algebraic group G. If G is unipotent then G is isomorphic—as a variety—to k^n: This is clear for $\dim G = 1$ (see [Hu75, Theorem 20.5] or [Kr85, III.1.1 Beispiel 2]); the general case follows by induction using the fact that every principal k^+-bundle over an affine variety is trivial (cf. [Gr58, proposition 1]). If G is connected and solvable, then G is isomorphic—again as a variety—to $k^{*p} \times k^q$, because G is a semidirect product of a torus and a unipotent group ([Hu75, Theorem 19.3b]).

Now let G be a connected reductive group. Then G contains an affine open subset U, the *big cell*, which is isomorphic to $k^{*p} \times k^q$ ([Hu75, Proposition 28.5], Bruhat-decomposition). In general, a connected algebraic group G is isomorphic—as a variety—to $(G/G_u) \times G_u$ where G_u is the unipotent radical of G ([Gr58, loc. cit.]), hence also contains an affine open subvariety isomorphic to $k^{*p} \times k^q$.

4.2. Lemma. *Let X be a normal algebraic variety. For every line bundle L on $G \times X$ we have*

$$L \simeq p_G^*(L|_{G\times\{x_0\}}) \otimes p_X^*(L|_{\{e\}\times X}).$$

PROOF: (a) We first assume that X is smooth. Then the Picard group $\operatorname{Pic}(G \times X)$ can be identified with the group $\operatorname{Cl}(G \times X)$ of divisor classes on $G \times X$. By [Ha77, Chap. II, Proposition 6.6] the claim is true if we replace G by the variety k or k^* or more generally by $k^{*p} \times k^q$. We know that G contains an open subset U isomorphic to $k^{*p} \times k^q$ (4.1). Hence, the line bundle

$$M := L \otimes \left(p_G^*(L|_{G\times\{x_0\}}) \otimes p_X^*(L|_{\{e\}\times X})\right)^{-1}$$

is trivial on $U \times X$. Therefore, the corresponding divisor class can be represented by a divisor $D \subset (G \setminus U) \times X$. It follows that $D = p_G^{-1}(\overline{D})$ with a divisor $\overline{D} \subset G$ and so

$$M \simeq p_G^*(M|_{G\times\{x_0\}}).$$

Since $M|_{G\times\{x_0\}}$ is trivial, M is trivial, too.

(b) For a normal variety X the open subset X_{reg} of regular points of X has a complement of codimension at least 2, and every function defined on X_{reg} extends to a regular function on X. We have just seen in (a) that $M|_{X_{\text{reg}}}$ is trivial. Hence M is trivial, too. □

4.3. Lemma. *Let $L \in \operatorname{Pic} G$ and denote by L^* the complement of the zero section in L. Then L^* has the structure of a linear algebraic group such that the following holds:*

(a) *The projection $p : L \to G$ induces a group homomorphism $L^* \to G$ with central kernel isomorphic to k^*.*

(b) *The line bundle L is L^*-linearizable.*

PROOF: We denote by $m : G \times G \to G$ the multiplication in G and by $p_1, p_2 : G \times G \to G$ the two projections. By 4.2 the two line bundles $m^*(L)$ and $p_1^*(L) \otimes p_2^*(L)$ are isomorphic. Choosing such an isomorphism ψ we obtain a "bilinear" morphism $\mu : L \times L \to L$ via the following commutative diagram:

$$\begin{array}{ccccccc} L \times L & \longrightarrow & p_1^*(L) \otimes p_2^*(L) & \longrightarrow & m^*(L) & \longrightarrow & L \\ \Big\downarrow{\scriptstyle p \times p} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ G \times G & = & G \times G & = & G \times G & \xrightarrow{m} & G \end{array}$$

We want to modify ψ in such a way that μ defines a product on L^*. First we fix an identification of k with the fiber L_e of L over the unit element e of G; we denote by $1 \in L_e$ the multiplicative unit of $L_e = k$. Now consider the composition:

$$\begin{array}{ccccc} L & \longrightarrow & L \times \{1\} & \xrightarrow{\mu} & L \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ G & \longrightarrow & G \times \{e\} & \longrightarrow & G \end{array}$$

It is an isomorphism of L over G, inducing the identity on G, hence given by a invertible function $r \in k[G]^*$:

$$\mu(u, 1) = r(p(u))u, \quad (u \in L).$$

Similarly, we see that there is a $s \in k[G]^*$ such that

$$\mu(1, v) = s(p(v))v, \quad (v \in L).$$

Replacing ψ by $\psi \circ (r^{-1} \otimes s^{-1})$ the element $1 \in L$ becomes a left and right unit for μ. We claim that μ is associative. In fact, $\mu(\mathrm{id} \times \mu)$ and $\mu(\mu \times \mathrm{id})$ are two "trilinear" morphisms $L \times L \times L \longrightarrow L$ over the same map $G \times G \times G \longrightarrow G$. Hence there is an $t \in k[G \times G \times G]^*$ such that

$$\mu(\mathrm{id} \times \mu)(u, v, w) = t(p(u), p(v), p(w))\mu(\mu \times \mathrm{id})(u, v, w),$$

$(u, v, w \in L)$. There are invertible functions $t_i \in k[G]^*$, $i = 1, 2, 3$, such that

$$t(g, h, l) = t_1(g)t_2(h)t_3(l), \quad (g, h, l \in G)$$

(see [Pic, 1.1]). Since $t(e, e, e) = 1$ we may assume that $t_i(e) = 1$ $(i = 1, 2, 3)$. It follows that

$$1 = t(g, e, e) = t_1(g)t_2(e)t_3(e) = t_1(g) \quad \text{for all } g \in G,$$

because $1 \in L$ is a unit for μ. Similarly, we get $t_2 = t_3 = 1$, i.e., μ is associative.

Since L^* is the subset of "invertible" elements of L with respect to μ, the first assertion (a) follows. Furthermore, the restriction of μ to $L^* \times L$ defines a L^*-linearization of L, hence (b). □

4.4. Lemma. *Let $L \in \operatorname{Pic} G$. There is a finite covering $\pi : G' \to G$ of algebraic groups such that L is G'-linearizable and $\pi^*(L) = 0$.*

PROOF: Consider the exact sequence

$$1 \longrightarrow T \longrightarrow L^* \xrightarrow{p} G \longrightarrow 1$$

where L^* is as in Lemma 4.3, and $T := \ker p$ is isomorphic to k^*. Let $\rho : L^* \longrightarrow \mathrm{GL}(V)$ be a finite dimensional rational representation such that $\rho(T) \neq \{\mathrm{id}\}$. Replacing V by a suitable submodule on which T acts by scalar multiplication, we may assume that $\rho(T) \not\subseteq \mathrm{SL}(V)$. Denote by G' the identity component of $\rho^{-1}(\mathrm{SL}(V))$. Then the restriction π of p to G' is a finite covering of G. Since L is L^*-linearizable (4.3b) it is also G'-linearizable. Finally, the line bundle $L' := \pi^*(L)$ on G' is G'-linearizable, hence trivial (see Example 2.1). □

4.5. Proposition. $\operatorname{Pic} G$ *is a finite group.*

PROOF: Let $L \in \operatorname{Pic} G$ and $\pi : G' \to G$ as in the previous Lemma 4.4. The (finite) kernel H of π acts on the fibers of L, hence acts trivially on $L^{\otimes d}$ where d is the order of H. As a consequence, $L^{\otimes d}$ is G-linearizable, hence trivial (Example 2.1). This shows that $\operatorname{Pic} G$ is a torsion group.

On the other hand, $\operatorname{Pic} G$ is finitely generated. In fact, G contains an affine open subset U whose coordinate ring is factorial (see 4.1). It follows that the divisor class group $\operatorname{Cl} G$ is generated by the irreducible components of $G \setminus U$ ([Ha77, Chap. II, 6.5]). This implies the claim because $\operatorname{Pic} G$ coincides with $\operatorname{Cl} G$. □

4.6. Proposition. *There exists a finite covering $\tilde{G} \to G$ of algebraic groups such that* $\operatorname{Pic} \tilde{G} = 0$.

PROOF: By Lemma 4.4 and Proposition 4.5 it suffices to show that for every finite covering $\alpha : G_1 \to G$ the induced map $\alpha^* : \operatorname{Pic} G \to \operatorname{Pic} G_1$ is surjective.

Let $L_1 \in \operatorname{Pic} G_1$, and let $\pi : G' \to G_1$ be a finite covering such that L_1 is G'-linearizable as in Lemma 4.4. Then $G_1 = G'/H_1$ where H_1 is the kernel of π, and $L_1 = E_{\chi_1}$ with a suitable character χ_1 of H_1 (Example 2.1). Let $H := \ker(\alpha \circ \pi) \supset H_1$. Since H is (finite and) commutative there is a character χ of H extending χ_1. Now it follows that L_1 is the pull back of the line bundle $L := E_\chi$ on G. □

References

[BAC7] Bourbaki, N.: *Algèbre commutative, chap. 7.* Hermann, Paris 1965

[FI74] Fossum, R., Iversen, B.: *On Picard groups of algebraic fibre spaces.* J. Pure Appl. Algebra **3** (1974), 269–280

[Gr58] Grothendieck, A.: *Torsion homologique et sections rationelles.* In: Anneaux de Chow et application, exposé 5. Séminaire C. Chevalley, E. N. S., 1958

[Ha77] Hartshorne, R.: *Algebraic Geometry.* Graduate Texts in Math. **52**. Springer-Verlag, 1977

[Hu75] Humphreys, J. E.: *Linear algebraic groups.* Graduate Texts in Math. **21**. Springer-Verlag, 1975

[Is74] Ischebeck, F.: *Zur Picard-Gruppe eines Produktes.* Math. Z. **139** (1974), 141–157

[Iv76] Iversen, B.: *The geometry of algebraic groups.* Adv. in Math. **20** (1976), 57–85

[Pic] Knop, F.; Kraft, H.; Vust, Th.: *The Picard group of a G-variety.* In these DMV Seminar Notes

[Kr85] Kraft, H.: *Geometrische Methoden in der Invariantentheorie.* Aspekte der Mathematik **D1**. Vieweg Verlag, Braunschweig/Wiesbaden 1985

[Kr89a] Kraft, H.: *Algebraic automorphisms of affine space.* In: Topological methods in algebraic transformation groups. Progress in Math. **80**, 81–106. Birkhäuser Verlag, Boston Basel 1989

[Kr89b] Kraft, H.: *G-vector bundles and the linearization problem.* Proceedings of a Conference on "Group Actions and Invariant Theory," Montreal, 1988. To appear in Contemp. Math.

[LV83] Luna, D., Vust, Th.: *Plongements d'espaces homogènes.* Comment. Math. Helv. **58** (1983), 186–245

[MF82] Mumford, D., Fogarty, J.: *Geometric Invariant Theory.* Ergeb. Math. Grenzgeb. **34**. Springer-Verlag, 1982

[Od78] Oda, T.: *Torus embeddings and applications.* Tata Lectures on Mathematics **57**, Tata Institute of Fundamental Research Bombay. Springer-Verlag, 1978

[Po74] Popov, V. L.: *Picard groups of homogeneous spaces of linear algebraic groups and one dimensional homogeneous vector bundles.* Math. USSR Izv. **8** (1974), 301–327

[Su74] Sumihiro, H.: *Equivariant completion.* J. Math. Kyoto Univ. **14** (1974), 1–28

[Su75] Sumihiro, H.: *Equivariant completion, II.* J. Math. Kyoto Univ. **15** (1975), 573–605