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Symmetry and Spaces

In Honor of Gerry Schwarz

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Foreword

I have been blessed with a wonderful group of teachers, students, colleagues, and collaborators during my mathematical career. I would have amounted to little without them. It was a special thrill for me that they organized a conference in celebration of my 60th birthday. I am honored by the wonderful group of participants, their talks, and their contributions to the proceedings.

I thank the gang of four who organized the party; H. E. A. (Eddy) Campbell, Loek Helminck, Hanspeter Kraft, and David Wehlau. We were all pleased when the Fields Institute agreed to host the event. The Fields' financial and logistical support as well as NSERC support through grants of Eddy and David made the presence of the many participants possible. My heartfelt thanks to all of them. Last, but not least, I want to thank Ann Kostant and Birkhäuser Boston for agreeing to publish the proceedings of the conference in the Progress in Mathematics series.

Boston, August 2008 *Gerry Schwarz*

Dedicated to Gerry Schwarz on the occasion of his 60th birthday, with affection from his friends and admirers.

Preface

Gerry Schwarz's many profound contributions to the study of algebraic groups, group actions and invariant theory have had a substantial effect on the progress of these branches of mathematics. This volume contains ten research articles contributed by his friends and colleagues as a tribute to him on the occasion of his sixtieth birthday.

The volume begins with a chapter by Brion. He studies actions of connected, not necessarily linear algebraic groups on normal algebraic varieties, and shows how to reduce them to actions of affine subgroup schemes.

Specifically, let G be a connected algebraic group acting on a normal algebraic variety X. Brion shows that X admits an open cover by G-stable quasi-projective varieties; this generalizes classical results of Sumihiro about actions of linear algebraic groups, and of Raynaud about actions on nonsingular varieties.

Next, if X is quasi-projective and the G-action is faithful, then the existence of a G-equivariant morphism $\psi : X \to G/H$ is shown, where H is an affine subgroup scheme of G, and G/H is an abelian variety. This generalizes a result of Nishi and Matsumura about actions on nonsingular varieties. If, in addition, X contains an open G-orbit, then G/H is the Albanese variey of X, and each fiber of ψ is a normal variety containing an open H-orbit. This reduces the structure of normal, almost homogeneous varieties to the case where the acting group is affine.

Finally, Brion presents a simple proof of an important result due to Sancho de Salas: any complete homogeneous variety X decomposes uniquely into a product $A \times Y$, where A is an abelian variety, and Y is a rational homogeneous variety. Moreover, we have the decomposition of connected automorphism groups $\mathrm{Aut}^0(X) = A \times \mathrm{Aut}^0(Y)$, and $\mathrm{Aut}^0(Y)$ is semi-simple of adjoint type. This is an algebraic analogue of the Borel-Remmert structure theorem for compact homogeneous Kähler manifolds.

The chapter by Broer concerns rings of invariants of a finite group in arbitrary characteristic and the question of when such an invariant ring is a polynomial ring. The well-known theorem, due to Shephard and Todd and independently Chevalley, says that in the non-modular case (when the characteristic of the field $\mathbb{F}$ does not divide $|G|$) the ring of invariants $\mathbb{F}[V]^G$ is a polynomial ring if and only if the action of G is generated by pseudo-reflections. Serre proved that the "only if" direction of this assertion holds in the modular case as well. The question of when the 'if' direction holds is possibly the most important open question in modular invariant theory.

A representation V has the *direct summand property* if $\mathbb{F}[V]^G$ has a G-stable complement in $\mathbb{F}[V]$. In an earlier paper Broer conjectured that $\mathbb{F}[V]^G$ is a polynomial ring if and only if V has the direct summand property and the action of G is generated by pseudo-reflections. The "if" direction of this conjecture follows directly from Serre's Theorem. In another earlier paper Broer showed that his conjecture is true under the additional hypothesis that the group G is abelian. Here Broer proves that if a representation V has the direct summand property then this property is inherited by the point stabilizers of every subspace of V. Using this he proves that his conjecture holds for irreducible G-actions.

The contribution of Daigle and Freudenburg concerns a famous conjecture made by Dolgachev and Weisfeiler in the 1970's. In one formulation this conjecture states, that, if $\phi : \mathbb{A}^n \to \mathbb{A}^m$ is a flat morphism for which each fiber is $\mathbb{A}^{n-m}$ then ϕ is a trivial fibration. This paper gives a construction that unifies the examples given previously by Vénéreau and by Bhatwadekar and Dutta. It is still an open question as to whether the fibrations in these examples are trivial or not. Daigle and Freudenburg use locally nilpotent derivatives to give a systematic construction of a family of examples including the examples due to Vénéreau and to Bhatwadekar and Dutta. They show that many of the known results about these examples follow including the fact that these fibrations are stably trivial. This paper also includes an excellent summary of known results.

The chapter by Elmer and Fleischmann considers rings of invariants of finite modular groups. In the non-modular case, an important theorem due to Eagon and Hochster asserts that rings of invariants of finite groups are Cohen-Macaulay, i.e., that their depth is equal to their Krull dimension. It is well-known that this equality can (and usually does) fail for modular representations. In that setting, formulas for the depth of the ring of invariants are much sought after. Consider a modular representation of a p-group G defined over a field $\mathbb{F}$ of characteristic p. A lower bound on the depth is given by $\min\{\dim V, \dim V^P + \mathrm{cc}(\mathbb{F}[V]) + 1\}$ where P is a p-Sylow subgroup of G and $\mathrm{cc}(\mathbb{F}[V])$ denotes the cohomological connectivity of $\mathbb{F}[V]$. A representation which achieves this lower bound has been called *flat*. Elmer and Fleischmann here introduce a stronger notion *strongly flat* which implies flatness. They proceed to show that certain infinite families of representations of $C_p \times C_p$ are flat and so they have determined the depth of the corresponding rings of invariants. For the case $p = 2$, they are able to determine a complete classification of the strongly flat $C_2 \times C_2$ representations and to determine the depth of every indecomposable modular representation of $C_2 \times C_2$.

In their chapter Greb and Heinzner consider a Kähler manifold X with a Hamiltonian action of a compact Lie group K that extends to a holomorphic action of the complexification $K^{\mathbb{C}}$. The associated set of semistable points is open in X and has a reasonably well-behaved quotient, in particular there exists a complex space Q that parametrises closed $K^{\mathbb{C}}$-orbits of semi-stable points. The group K acts on the zero level set $\mathscr{M} := \mu^{-1}(0)$ of the given moment map $\mu : X \to \mathfrak{k}^*$, and the quotient $\mathscr{M}/K$ is homeomorphic to Q. This homeomorphism endows the complex space Q with a natural Kählerian structure that is smooth along the orbit type stratification. Since the quotient Q will in general be singular, the Kähler structure is locally given by continuous strictly plurisuharmonic functions. The quotient $\mathscr{M}/K$ is called the Kählerian reduction of X by K.

After summarizing the methods and results of the quotient theory outlined above, Greb and Heinzner show that this reduction process is natural in the sense that it can be formed in steps. This means that if L is a closed normal subgroup of K then the Kählerian reduction of X by L is a stratified Hamiltonian Kähler $K^{\mathbb{C}}/L^{\mathbb{C}}$-space. Furthermore the Kählerian reduction of this space by K/L is naturally isomorphic to the Kählerian reduction of X by K.

The chapter by Helminck is directed to the study symmetric k-varieties, providing a survey of known results and giving some open problems. Consider a connected reductive algebraic group G defined over a field k of characteristic different from 2. Let $\theta : G \to G$ be an involution of G also defined over k. Let $H = G^{\theta}$ denote the points of G fixed by θ. Then H is a subgroup of G which is also defined over k. Write G_k (resp. H_k) to denote the k-rational points of G (resp. of H). Then the homogeneous space $X = G_k/H_k$ is what is called a symmetric k-variety. These are a generalization of both real reductive symmetric spaces and symmetric varieties and play an important role in many areas of mathematics, especially representation theory. Helminck considers the study of the orbits on X of parabolic subgroups of G_k. He gives a comprehensive survey of what is known including detailed discussions of the action of the Weyl group, the Bruhat order and the Richardson-Springer involution poset in this setting. He also considers the action on X by the fixed point group of another k-involution σ of G.

The chapter by Kostant develops a comprehensive theory of root systems for a complex semisimple Lie algebra $\mathfrak{g}$, relative to a non-trivial parabolic subalgebra $\mathfrak{q}$, generalizing the classical case, where $\mathfrak{q}$ is a Borel subalgebra. Let $\mathfrak{m}$ be Levi factor of such a $\mathfrak{q}$ and let $\mathfrak{t}$ be its centre: a nonzero element $\nu \in \mathfrak{t}^*$ is called a $\mathfrak{t}$-root if the corresponding adjoint weight space $\mathfrak{g}_\nu$ is not zero. Some time ago, Kostant showed that $\mathfrak{g}_\nu$ is $\operatorname{ad}\mathfrak{m}$ irreducible (and that all $\operatorname{ad}\mathfrak{m}$ irreducibles are of this form). This result is the starting point for his generalization of the classical case, that is, the case when $\mathfrak{t}$ is a Cartan subalgebra. As an application, the author obtains new insight into the structure of the nilradical of $\mathfrak{q}$, and gives new proofs of the main results of Borel-de Siebenthal theory. This chapter contains many interesting and surprising insights into the structure of complex semisimple algebras, a classical subject, central to much of modern mathematics.

Kraft and Wallach study polarization and the nullcone. Suppose V is a complex representation of a reductive group G. Given an invariant $f \in \mathbb{C}[V]^G$, polarization is a classical technique used to generate some new invariants $Pf \subset \mathbb{C}[V^{\oplus k}]^G$ of the direct sum of k copies of V. The *nullcone* of a representation V is the subset $\mathscr{N} = \mathscr{N}_V$ of V on which all homogeneous non-constant invariants vanish. An important classical result, the Hilbert-Mumford criterion says that a point $v \in V$ lies in the nullcone if and only if there is a one-parameter subgroup which annihilates v, i.e., if and only if there is a group homomorphism $\lambda^* : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t)v = 0$. Suppose that $f_1, f_2 \ldots, f_n \in \mathbb{C}[V]^G$ and let m be a positive integer. Kraft and Wallach observe that the polarizations of $f_1, f_2 \ldots, f_n$ will cut out the nullcones $\mathscr{N}_{V^{\oplus k}}$ for all $k \leq m$ if and only for every m dimensional subspace L contained in $\mathscr{N}_V$ there is a one-parameter subgroup λ^* of G which annihilates every point of L.

Specializing to the group $G = \mathrm{SL}_2(\mathbb{C})$ Kraft and Wallach show that if m is any positive integer, V is any representation of $\mathrm{SL}_2(\mathbb{C})$ and $f_1, f_2, \ldots, f_n \in \mathbb{C}[V]^{\mathrm{SL}_2(\mathbb{C})}$ cut out the nullcone of V then the polarizations $Pf_1, Pf_2, \ldots, Pf_n \in \mathbb{C}[V^{\oplus m}]^{\mathrm{SL}_2(\mathbb{C})}$ cut out the nullcone of $V^{\oplus m}$. This result is very surprising since most representations of a general group will not have this property.

They apply their results to study $Q_n := (\mathbb{C}^2)^{\otimes n}$ the tensor product of n copies of the defining representation of $\mathrm{SL}_2(\mathbb{C})$. This space, known as the space of n qubits, plays an important role in the theory of quantum computing.

Shank and Wehlau study the invariants of V_{p+1}, the $p+1$ dimensional indecomposable modular representation of the cyclic group G of order p^2. Much work has been done in the past decade studying the modular invariant theory of the cyclic group of order p but this chapter is the first systematic study of the invariants of a modular representation of a cyclic group of higher order. Shank and Wehlau study the kG-module structure of $k[V_{p+1}]$, using a spectral sequence argument to obtain an explicit description of the ring of invariants. They also obtain a Hilbert series for all indecomposable G-representations of fixed type in $k[V_{p+1}]$ which they then combine to obtain an expression for the Hilbert series of the ring of invariants. This expression is surprisingly simple and compact.

Traves' chapter concerns rings of algebraic differential operators on quotient varieties such as the cone over a Grassmann variety, and actions of reductive groups on differential operators. The topic can be thought of as noncommutative invariant theory. The Weyl algebra of a complex vector space V is the ring $D(\mathbb{C}[V])$ of $\mathbb{C}$-linear differential operators on $\mathbb{C}[V]$. More generally, if $R = \mathbb{C}[V]/I$ is the coordinate ring of an affine variety X then the ring of differential operators on X is

$$D(R) = D(\mathbb{C}[V]/I) = \frac{\{\theta \in D(\mathbb{C}[V]) : \ \theta(I) \subset I\}}{ID(\mathbb{C}[V])}.$$

An action of G on X induces an action on R but also on $D(R)$.

Traves considers the group $G = \mathrm{SL}_k(\mathbb{C})$ and its representation $V^{\oplus n}$ where V is its defining representation (of dimension k). He describes the two rings $D(R)^G$ and $D(R^G)$, through his use of subtle techniques developed by Schwarz.

The Fundamental Theorem of Invariant Theory gives a presentation of the ring of invariants $\mathbb{C}[G(k,n)]^{\mathrm{SL}_k(\mathbb{C})}$ where $G(k,n)$ is the affine cone over the Grassmanian of k planes in n space. In his book, *The Classical Groups. Their Invariants and Representations* Hermann Weyl suggested that the Fundamental Theorem should be extended to $D(R)^G$, giving a presentation of the invariant differential operators on the affine variety $G(k,n)$. Here Traves follows Weyl's suggestion by working with the graded algebra associated to $D(R)^G$. Applying the Fundamental Theorem to the graded algebra, he obtains generators and relations. These then lift to generators of $D(R)^G$ and each of the relations on the graded algebra extends to a relation in $D(R)^G$ as well. He is also able to obtain Hilbert series for the graded algebras.

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Contents

Some Basic Results on Actions of Nonaffine Algebraic Groups

Michel Brion

Summary. We study actions of connected algebraic groups on normal algebraic varieties, and show how to reduce them to actions of affine subgroups.

Key words: Algebraic group action, Albanese morphism, almost homogeneous varieties

Mathematics Subject Classification (2000): 14J50, 14L30, 14M17

Dedicated to Gerry Schwarz on his 60th birthday

1 Introduction

Algebraic group actions have been extensively studied under the assumption that the acting group is affine or, equivalently, linear; see [14, 18, 24]. In contrast, little seems to be known about actions of nonaffine algebraic groups. In this chapter, we show that these actions can be reduced to actions of affine subgroup schemes, in the setting of normal varieties.

Our starting point is the following theorem of Nishi and Matsumura (see [15]). Let G be a connected algebraic group of automorphisms of a nonsingular algebraic variety X and denote by

$$\alpha_X : X \longrightarrow A(X)$$

the *Albanese morphism*, that is, the universal morphism to an abelian variety (see [23]). Then *G acts on $A(X)$ by translations, compatibly with its action on X, and the kernel of the induced homomorphism $G \to A(X)$ is affine*.

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Applied to the case of G acting on itself via left multiplication, this shows that the Albanese morphism

$$\alpha_G : G \longrightarrow A(G)$$

is a surjective group homomorphism having an affine kernel. Since this kernel is easily seen to be smooth and connected, this implies Chevalley's structure theorem: *any connected algebraic group G is an extension of an abelian variety $A(G)$ by a connected affine algebraic group* G_{aff} (see [8] for a modern proof).

The Nishi–Matsumura theorem may be reformulated as follows: for any faithful action of G on a nonsingular variety X, the induced homomorphism $G \to A(X)$ factors through a homomorphism $A(G) \to A(X)$ having a finite kernel (see [15] again). This easily implies the existence of a G-equivariant morphism

$$\psi : X \longrightarrow A,$$

where A is an abelian variety which is the quotient of $A(G)$ by a finite subgroup scheme (see Section 4 for details). Since $A(G) \cong G/G_{\mathrm{aff}}$, we have $A \cong G/H$ where H is a closed subgroup scheme of G containing G_{aff} with H/G_{aff} finite; in particular, H is affine, normalized by G, and uniquely determined by A. Then there is a G-equivariant isomorphism

$$X \cong G \times^H Y,$$

where the right-hand side denotes the homogeneous fiber bundle over G/H associated with the scheme-theoretic fiber Y of ψ at the base point.

In particular, given a faithful action of an abelian variety A on a nonsingular variety X, there exist a positive integer n and a closed A_n-stable subscheme Y of X such that $X \cong A \times^{A_n} Y$, where A_n denotes the kernel of the multiplication by n in A. For free actions, (i.e., abelian torsors), this result is due to Serre (see [22, Proposition 17]).

Next, consider a faithful action of G on a possibly singular variety X. Then, in addition to the Albanese morphism, we have the *Albanese map*

$$\alpha_{X,r} : X - - \rightarrow A(X)_r,$$

that is, the universal rational map to an abelian variety. Moreover, the regular locus U of X is G-stable, and $A(U) = A(U)_r = A(X)_r$. Thus, G acts on $A(X)_r$ via a homomorphism $A(G) \to A(X)_r$ such that the canonical homomorphism

$$h_X : A(X)_r \longrightarrow A(X)$$

is equivariant; h_X is surjective, but generally not an isomorphism; see [23] again. Applying the Nishi–Matsumura theorem to U, we see that the kernel of the G-action on $A(X)_r$ is affine, and there exists a G-equivariant rational map $\psi_r : X - - \rightarrow A$ for some abelian variety A as above.

However, G may well act trivially on $A(X)$; then there exists no morphism ψ as above, and hence X admits no equivariant embedding into a nonsingular G-variety. This happens indeed for several classes of examples constructed by Raynaud; see

[20, XII 1.2, XIII 3.2] or Examples 6.2, 6.3, 6.4; in the latter example, X is normal, and G is an abelian variety acting freely. Yet we show that such an equivariant embedding (in particular, such a morphism ψ) exists locally for any normal G-variety.

To state our results in a precise way, we introduce some notation and conventions. We consider algebraic varieties and schemes over an algebraically closed field k; morphisms are understood to be k-morphisms. By a variety, we mean a separated integral scheme of finite type over k; a point always means a closed point.

As a general reference for algebraic geometry, we use the book [12], and use [9] for algebraic groups.

We fix a connected algebraic group G, that is, a k-group variety; in particular, G is smooth. A *G-variety* is a variety X equipped with an algebraic G-action

$$\varphi : G \times X \longrightarrow X, \quad (g,x) \longmapsto g \cdot x.$$

The *kernel* of φ is the largest subgroup scheme of G that acts trivially. We say that φ is *faithful* if its kernel is trivial. The G-variety X is *homogeneous* (resp. *almost homogeneous*) if it contains a unique orbit (resp. an open orbit). Finally, a *G-morphism* is an equivariant morphism between G-varieties.

We may now formulate our results, and discuss their relations to earlier work.

Theorem 1.1. *Any normal G-variety admits an open cover by G-stable quasi-projective varieties.*

When G is affine, this fundamental result is due to Sumihiro (see [14, 24]). On the other hand, it has been obtained by Raynaud for actions of smooth, connected group schemes on smooth schemes over normal bases; in particular, for actions of connected algebraic groups on nonsingular varieties (see [20, Corollary V 3.14]).

Theorem 1.1 implies readily the quasi-projectivity of all homogeneous varieties and, more generally, of all normal varieties where a connected algebraic group acts with a unique closed orbit. For the latter result, the normality assumption cannot be omitted in view of Example 6.2.

Next, we obtain an analogue of the Nishi–Matsumura theorem.

Theorem 1.2. *Let X be a normal, quasi-projective variety on which G acts faithfully. Then there exists a G-morphism $\psi : X \longrightarrow A$, where A is the quotient of $A(G)$ by a finite subgroup scheme. Moreover, X admits a G-equivariant embedding into the projectivization of a G-homogeneous vector bundle over A.*

The second assertion generalizes (and builds on) another result of Sumihiro: when G is affine, any normal quasi-projective G-variety admits an equivariant embedding into the projectivization of a G-module (see [14, 24] again). It implies that any normal quasi-projective G-variety has an equivariant embedding into a nonsingular G-variety of a very special type. Namely, a vector bundle E over an abelian variety A is homogeneous with respect to a connected algebraic group G (acting transitively on A) if and only if $a^*(E) \cong E$ for all $a \in A$; see [17] for a description of all homogeneous vector bundles on an abelian variety.

Clearly, a morphism ψ as in Theorem 1.2 is not unique, as we may replace A with any finite quotient. In characteristic zero, we can always impose that the fibers of ψ are normal varieties, by using the Stein factorization. Under that assumption, there may still exist several morphisms ψ, but no universal morphism (see Example 6.1).

Returning to arbitary characteristics, there does exist a universal morphism ψ when X is normal and almost homogeneous, namely, the Albanese morphism.

Theorem 1.3. *Let X be a normal, almost homogeneous G-variety. Then $A(X) = A(X)_r = G/H$, where H is a closed subgroup scheme of G containing G_{aff}, and the quotient group scheme H/G_{aff} is finite. Moreover, each fiber of the Albanese morphism is a normal variety, stable under H and almost homogeneous under G_{aff}.*

The existence of this *Albanese fibration* is well known in the setting of complex Lie groups acting on compact Kähler manifolds (this is the Remmert–van de Ven theorem, see e.g., [1, Section 3.9]), and is easily obtained for nonsingular varieties (see [6, Section 2.4]).

In particular, Theorem 1.3 can be applied to any normal *G-equivariant embedding* of G, that is, to a normal G-variety X containing an open orbit isomorphic to G; then $A(X) = A(G) = G/G_{\mathrm{aff}}$ and hence

$$X \cong G \times^{G_{\mathrm{aff}}} Y\,,$$

where Y is a normal G_{aff}-equivariant embedding of G_{aff}. (Here again, the normality assumption cannot be omitted; see Example 6.2.) If G is a *semi-abelian variety*, that is, G_{aff} is a torus, then Y is a toric variety, and the G-equivariant embedding X is called a *semi-abelic variety*. In that case, the above isomorphism has been obtained by Alexeev under the assumption that X is projective (see [2, Section 5]).

As a further application of Theorem 1.3, consider a normal algebraic monoid X with unit group G. Then X can be regarded as a $(G \times G)$-equivariant embedding of G, and the above isomorphism yields another proof of the main result of [6].

For a complete homogeneous variety, it is known that the Albanese fibration is trivial. In the setting of compact homogeneous Kähler manifolds, this is the classical Borel–Remmert theorem (see e.g., [1, Section 3.9]); in our algebraic setting, this is part of a structure result due to Sancho de Salas.

Theorem 1.4 ([21, Theorem 5.2]). *Let X be a complete variety, homogeneous under a faithful action of G. Then there exists a canonical isomorphism of algebraic groups*

$$G \cong A(G) \times G_{\mathrm{aff}}\,,$$

and G_{aff} is semi-simple of adjoint type. Moreover, there exists a canonical isomorphism of G-varieties

$$X \cong A(G) \times Y\,,$$

where Y is a complete, homogeneous G_{aff}-variety; the Albanese morphism of X is identified with the projection $A(G) \times Y \to A(G)$.

Here we present a short proof of that result by analyzing first the structure of G and then the Albanese morphism of X, whereas [21] proceeds by constructing the projection $X \to Y$ via the study of the $G_{\rm aff}$-action on X. In characteristic zero, this projection is the *Tits morphism* that assigns to any point of X its isotropy Lie algebra; this yields a very simple proof of Theorem 1.4, see [6, Section 1.4]. But this approach does not extend to positive characteristics, as the G-action on X may well be nonsmooth; equivalently, the orbit maps can be nonseparable.

In fact, there exist many complete varieties Y that are homogeneous under a nonsmooth action of a semi-simple group $G_{\rm aff}$. These *varieties of unseparated flags* are classified in [11, 21, 25]. Any such variety Y is projective (e.g., by Theorem 1.1) and rational. Indeed, Y contains only finitely many fixed points of a maximal torus of $G_{\rm aff}$; thus, Y is paved by affine spaces (see [4]).

In positive characteristics again, the homogeneity assumption of Theorem 1.4 cannot be replaced with the assumption that the tangent bundle is globally generated. Indeed, there exist nonsingular projective varieties which have a trivial tangent bundle, but fail to be homogeneous (see [13] or Example 6.5). Also, for G-varieties of unseparated flags, the subbundle of the tangent bundle generated by $\mathrm{Lie}(G)$ is a proper direct summand (see Example 6.6).

The proofs of Theorems 1–4 are given in the corresponding sections. The final Section 6 presents examples illustrating the assumptions of these theorems. Three of these examples are based on constructions, due to Raynaud, of torsors under abelian schemes, which deserve to be better known to experts in algebraic groups.

In the opposite direction, it would be interesting to extend our results to group schemes. Our methods yield insight into actions of abelian schemes (e.g., the proof of Theorem 1.2 adapts readily to that setting), but the general case requires new ideas.

2 Proof of Theorem 1.1

As in the proof of the projectivity of abelian varieties (see e.g., [16, Theorem 7.1]), we first obtain a version of the theorem of the square. We briefly present the setting of this theorem, referring to [20, Chapter IV] and [5, Section 6.3] for further developments.

Consider an invertible sheaf L on a G-variety X. For any $g \in G$, let $g^*(L)$ denote the pull-back of L under the automorphism of X induced by g. Loosely speaking, L satisfies the theorem of the square if there are compatible isomorphisms

$$(gh)^*(L) \otimes L \cong g^*(L) \otimes h^*(L) \tag{1}$$

for all g, h in G. Specifically, consider the action

$$\varphi : G \times X \to X\,,$$

the projection

$$p_2 : G \times X \to X\,,$$

and put

$$\mathscr{L} := \varphi^*(L) \otimes p_2^*(L)^{-1} \,. \tag{2}$$

This is an invertible sheaf on $G \times X$, satisfying

$$\mathscr{L}|_{\{g\}\times X} \cong g^*(L) \otimes L^{-1}$$

for all $g \in G$.

Next, consider the morphisms

$$\mu, \pi_1, \pi_2 : G \times G \longrightarrow G$$

given by the multiplication and the two projections. Define an invertible sheaf $\mathscr{M}$ on $G \times G \times X$ by

$$\mathscr{M} := (\mu \times \mathrm{id}_X)^*(\mathscr{L}) \otimes (\pi_1 \times \mathrm{id}_X)^*(\mathscr{L})^{-1} \otimes (\pi_2 \times \mathrm{id}_X)^*(\mathscr{L})^{-1} \,. \tag{3}$$

Then we have isomorphisms

$$\mathscr{M}|_{\{(g,h)\}\times X} \cong (gh)^*(L) \otimes L \otimes g^*(L)^{-1} \otimes h^*(L)^{-1}$$

for all g, h in G. We say that L satisfies the theorem of the square, if $\mathscr{M}$ is the pull-back of some invertible sheaf under the projection

$$f : G \times G \times X \longrightarrow G \times G \,.$$

Then each $\mathscr{M}|_{\{(g,h)\}\times X}$ is trivial, which implies (1).

Also, recall the classical notion of a *G-linearization* of the invertible sheaf L, that is, a G-action of the total space of the associated line bundle which is compatible with the G-action on X and commutes with the natural $\mathbb{G}_m$-action (see [14, 18]). The isomorphism classes of G-linearized invertible sheaves form a group denoted by $\mathrm{Pic}^G(X)$. We use the following observation (see [18, p. 32]).

Lemma 2.1. *Let $\pi : X \to Y$ be a torsor under G for the fppf topology. Then the pull-back under π yields an isomorphism* $\mathrm{Pic}(Y) \cong \mathrm{Pic}^G(X)$.

Proof. By assumption, π is faithfully flat and fits into a Cartesian square

$$\begin{array}{ccc} G \times X & \xrightarrow{\varphi} & X \\ {\scriptstyle p_2}\downarrow & & {\scriptstyle \pi}\downarrow \\ X & \xrightarrow{\pi} & Y \,. \end{array}$$

Moreover, a G-linearization of an invertible sheaf L on X is exactly a descent datum for L under π, see [18, Section 1.3]. So the assertion follows from faithfully flat descent, see e.g., [5, Section 6.1]. $\square$

We now come to our version of the theorem of the square.

Lemma 2.2. *Let L be a G_{aff}-linearizable invertible sheaf on a G-variety X. Then L satisfies the theorem of the square.*

Proof. Consider the action of G on $G \times X$ via left multiplication on the first factor. Then φ is equivariant, and p_2 is invariant. Hence the choice of a G_{aff}-linearization of L yields a G_{aff}-linearization of the invertible sheaf $\mathscr{L}$ on $G \times X$ defined by (2). Note that the map

$$\alpha_G \times \mathrm{id}_X : G \times X \longrightarrow A(G) \times X$$

is a G_{aff}-torsor. By Lemma 2.1, there exists a unique invertible sheaf $\mathscr{L}'$ on $A(G) \times X$ such that

$$\mathscr{L} = (\alpha_G \times \mathrm{id}_X)^*(\mathscr{L}') \tag{4}$$

as G_{aff}-linearized sheaves. Let $\mathscr{M}$ be the invertible sheaf on $G \times G \times X$ defined by (3). Similarly, let $\mathscr{M}'$ be the invertible sheaf on $A(G) \times A(G) \times X$ given by

$$\mathscr{M}' = (\mu' \times \mathrm{id}_X)^*(\mathscr{L}') \otimes (\pi_1' \times \mathrm{id}_X)^*(\mathscr{L}')^{-1} \otimes (\pi_2' \times \mathrm{id}_X)^*(\mathscr{L}')^{-1},$$

where $\mu', \pi_1', \pi_2' : A(G) \times A(G) \to A(G)$ denote the addition and the two projections. Then $\mathscr{M} = (\alpha_G \times \alpha_G \times \mathrm{id}_X)^*(\mathscr{M}')$. Thus, it suffices to show that $\mathscr{M}'$ is the pull-back of an invertible sheaf under the projection

$$f' : A(G) \times A(G) \times X \longrightarrow A(G) \times A(G).$$

Choose $x \in X$ and put

$$\mathscr{M}_x' := \mathscr{M}'|_{A(G) \times A(G) \times \{x\}}.$$

We consider $\mathscr{M}_x'$ as an invertible sheaf on $A(G) \times A(G)$, and show that the invertible sheaf $\mathscr{M}' \otimes f'^*(\mathscr{M}_x')^{-1}$ is trivial. By a classical rigidity result (see [16, Theorem 6.1]), it suffices to check the triviality of the restrictions of this sheaf to the subvarieties $\{0\} \times A(G) \times X$, $A(G) \times \{0\} \times X$, and $A(G) \times A(G) \times \{x\}$. In view of the definition of $\mathscr{M}_x'$, it suffices in turn to show that

$$\mathscr{M}'|_{\{0\} \times A(G) \times X} \cong \mathscr{O}_{A(G) \times X}. \tag{5}$$

For this, note that $\mathscr{L}|_{G_{\mathrm{aff}} \times X} \cong \mathscr{O}_{G_{\mathrm{aff}} \times X}$ since L is G_{aff}-linearized. By Lemma 2.1, it follows that

$$\mathscr{L}'|_{\{0\} \times X} \cong \mathscr{O}_{\{0\} \times X}. \tag{6}$$

Thus, $(\pi_1' \times \mathrm{id}_X)^*(\mathscr{L}')$ is trivial; on the other hand, μ' and π_2' have the same restriction to $\{0\} \times A(G)$. These facts imply (5). □

Next, recall that for any invertible sheaf L on a normal G-variety X, some positive power L^n admits a G_{aff}-linearization; specifically, the Picard group of G_{aff} is finite, and we may take for n the order of that group (see [14, p. 67]). Together with Lemma 2.2, this yields the following.

Lemma 2.3. *Let L be an invertible sheaf on a normal G-variety. Then some positive power L^n satisfies the theorem of the square; we may take for n the order of* $\mathrm{Pic}(G_{\mathrm{aff}})$.

From this, we deduce an ampleness criterion on normal G-varieties, analogous to a result of Raynaud about actions of group schemes on smooth schemes over a normal base (see [20, Theorem V 3.10]).

Lemma 2.4. *Let X be a normal G-variety, and D an effective Weil divisor on X. If* $\mathrm{Supp}(D)$ *contains no G-orbit, then some positive multiple of D is a Cartier divisor generated by its global sections. If, in addition, $X \setminus \mathrm{Supp}(D)$ is affine, then D is ample.*

Proof. Consider the regular locus X_0 of X, and the restricted divisor $D_0 := D \cap X_0$. Then X_0 is G-stable, and the sheaf $\mathscr{O}_{X_0}(D_0)$ is invertible; moreover, $g^*(\mathscr{O}_{X_0}(D_0)) = \mathscr{O}_{X_0}(g \cdot D_0)$ for all $g \in G$. By Lemma 2.3, there exists a positive integer n such that $\mathscr{O}_{X_0}(nD_0)$ satisfies the theorem of the square. Replacing D with nD, we obtain, in particular, isomorphisms

$$\mathscr{O}_{X_0}(2D_0) \cong \mathscr{O}_{X_0}(g \cdot D_0 + g^{-1} \cdot D_0)$$

for all $g \in G$. Since X is normal, it follows that

$$\mathscr{O}_X(2D) \cong \mathscr{O}_X(g \cdot D + g^{-1} \cdot D) \tag{7}$$

for all $g \in G$.

Next, let

$$U := X \setminus \mathrm{Supp}(D) \ .$$

By (7), the Weil divisor $2D$ restricts to a Cartier divisor on every open subset

$$V_g := X \setminus \mathrm{Supp}(g \cdot D + g^{-1} \cdot D) = g \cdot U \cap g^{-1} \cdot U \ ,$$

where $g \in G$. Moreover, these subsets form a covering of X. (Indeed, given $x \in X$, the subset

$$W_x := \{g \in G \mid g \cdot x \in U\} \subset G$$

is open and nonempty since U contains no G-orbit. Thus, W_x meets its image under the inverse map of G; that is, there exists $g \in G$ such that U contains both points $g \cdot x$ and $g^{-1} \cdot x$.) It follows that $2D$ is a Cartier divisor on X. Likewise, the global sections of $\mathscr{O}_X(2D)$ generate this divisor on each V_g, and hence everywhere.

If U is affine, then each V_g is affine as well. Hence X is covered by affine open subsets X_s, where s is a global section of $\mathscr{O}_X(2D)$. Thus, $2D$ is ample. □

We may now prove Theorem 1.1. Let $x \in X$ and choose an affine open subset U containing x. Then $G \cdot U$ is a G-stable open subset of X; moreover, the complement $(G \cdot U) \setminus U$ is of pure codimension 1 and contains no G-orbit. Thus, $G \cdot U$ is quasi-projective by Lemma 2.4. □

3 Proof of Theorem 1.2

First, we gather preliminary results about algebraic groups and their actions.

Lemma 3.1. (i) *Let $\pi : X \to Y$ be a torsor under a group scheme H. Then the morphism π is affine if and only if H is affine.*
(ii) *Let X be a variety on which G acts faithfully. Then the orbit map*

$$\varphi_x : G \longrightarrow G \cdot x, \quad g \longmapsto g \cdot x$$

is an affine morphism for any $x \in X$.
(iii) *Let $C(G)$ denote the center of G, and $C(G)^0$ its reduced neutral component. Then $G = C(G)^0 G_{\mathrm{aff}}$.*
(iv) *Let M be an invertible sheaf on $A(G)$, and $L := \alpha_G^*(M)$ the corresponding G_{aff}-linearized invertible sheaf on G. Then L is ample if and only if M is ample.*

Proof. (i) follows by faithfully flat descent, like Lemma 2.1.

(ii) Since φ_x is a torsor under the isotropy subgroup scheme $G_x \subset G$, it suffices to check that G_x is affine or, equivalently, admits an injective representation in a finite-dimensional k-vector space. Such a representation is afforded by the natural action of G_x on the quotient $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ for $n \gg 0$, where $\mathcal{O}_{X,x}$ denotes the local ring of X at x, with maximal ideal $\mathfrak{m}_x$; see [15, Lemma p. 154] for details.

(iii) By [22, Lemma 2], α_G restricts to a surjective morphism $C(G) \to A(G)$. Hence the restriction $C(G)^0 \to A(G) = G/G_{\mathrm{aff}}$ is surjective as well.

(iv) Since the morphism α_G is affine, the ampleness of M implies that of L. The converse holds by [20, Lemma XI 1.11.1]. □

Next, we obtain our main technical result.

Lemma 3.2. *Let X be a variety on which G acts faithfully. Then the following conditions are equivalent.*

(i) *There exists a G-morphism $\psi : X \to A$, where A is the quotient of $A(G)$ by a finite subgroup scheme.*
(ii) *There exists a G_{aff}-linearized invertible sheaf L on X such that $\varphi_x^*(L)$ is ample for any $x \in X$.*
(iii) *There exists a G_{aff}-linearized invertible sheaf L on X such that $\varphi_{x_0}^*(L)$ is ample for some $x_0 \in X$.*

Proof. (i) $\Rightarrow$ (ii) Choose an ample invertible sheaf M on the abelian variety A. We check that $L := \psi^*(M)$ satisfies the assertion of (ii). Indeed, by the universal property of the Albanese morphism α_G, there exists a unique G-morphism $\alpha_x : A(G) \to A$ such that the square

$$\begin{array}{ccc} G & \xrightarrow{\varphi_x} & X \\ {\scriptstyle \alpha_G}\downarrow & & \downarrow{\scriptstyle \psi} \\ A(G) & \xrightarrow{\alpha_x} & A \end{array}$$

is commutative. By rigidity (see, e.g., [16, Corollary 3.6]), α_x is the composite of a homomorphism and a translation. On the other hand, α_x is G-equivariant; thus, it is the composite of the quotient map $A(G) \to A$ and a translation. So α_x is finite, and hence $\alpha_x^*(M)$ is ample. By Lemma 3.1 (iv), it follows that $\varphi_x^*(L) = \alpha_G^*(\alpha_x^*(M))$ is ample.

(ii) $\Rightarrow$ (iii) is left to the reader.

(iii) $\Rightarrow$ (i) Consider the invertible sheaves $\mathscr{L}$ on $G \times X$, and $\mathscr{L}'$ on $A(G) \times X$, defined by (2) and (4). Recall from (6) that $\mathscr{L}'$ is equipped with a rigidification along $\{0\} \times X$. By [5, Section 8.1], it follows that $\mathscr{L}'$ defines a morphism

$$\psi : X \longrightarrow \operatorname{Pic} A(G), \quad x \longmapsto \mathscr{L}'|_{A(G)\times\{x\}} .$$

Moreover, $\operatorname{Pic} A(G)$ is a reduced group scheme, locally of finite type, and its connected components are exactly the cosets of the dual abelian variety, $A(G)^\vee$. Since X is connected, its image under ψ is contained in a unique coset.

By construction, ψ maps every point $x \in X$ to the isomorphism class of the unique invertible sheaf M_x on $A(G)$ such that

$$\alpha_G^*(M_x) = \varphi_x^*(L)$$

as G_{aff}-linearized sheaves on G. When $x = x_0$, the invertible sheaf $\varphi_x^*(L)$ is ample, and hence M_x is ample by Lemma 3.1(iv). It follows that the points of the image of ψ are exactly the ample classes $a^*(M_{x_0})$, where $a \in A(G)$.

Moreover, $\varphi_{g\cdot x} = \varphi_x \circ \rho(g)$ for all $g \in G$ and $x \in X$, where ρ denotes right multiplication in G. Thus, $\alpha_G^*(M_{g\cdot x}) = \rho(g)^*(\alpha_G^*(M_x))$; that is, $\psi(g\cdot x) = \alpha_G(g)^*\psi(x)$. In other words, ψ is G-equivariant, where G acts on $\operatorname{Pic} A(G)$ via the homomorphism $\alpha_G : G \to A(G)$ and the $A(G)$-action on $\operatorname{Pic} A(G)$ via pull-back. Hence the image of ψ is the G-orbit of $\psi(x_0)$, that is, the quotient $A(G)/F$, where F denotes the scheme-theoretic kernel of the polarization homomorphism

$$A(G) \longrightarrow A(G)^\vee, \quad a \longmapsto a^*(M_{x_0}) \otimes M_{x_0}^{-1} . \qquad \square$$

We now are in a position to prove Theorem 1.2. Under the assumptions of that theorem, we may choose an ample invertible sheaf L on X. Then $\varphi_x^*(L)$ is ample for any $x \in X$, as follows from Lemma 3.1(ii). Moreover, replacing L with some positive power, we may assume that L is G_{aff}-linearizable. Now Lemma 3.2 yields a G-morphism

$$\psi : X \longrightarrow A,$$

where $A \cong G/H$ for an affine subgroup scheme H of G. So $X \cong G \times^H Y$, where Y is a closed H-stable subscheme of X. To complete the proof, it suffices to show that X (and hence Y) admits an H-equivariant embedding into the projectivization of an H-module. Equivalently, X admits an ample, H-linearized invertible sheaf. By [24], this holds when H is smooth and connected; the general case can be reduced to that one as follows.

We may consider H as a closed subgroup scheme of some GL_n. Then G_{aff} embeds into GL_n as the reduced neutral component of H. The homogeneous fiber bundle $\mathrm{GL}_n \times^{G_{\mathrm{aff}}} X$ is a normal GL_n-variety, (see [22, Proposition 4]); it is quasi-projective by [18, Proposition 7.1]. The finite group scheme H/G_{aff} acts on that variety, and this action commutes with the GL_n-action. Hence the quotient

$$(\mathrm{GL}_n \times^{G_{\mathrm{aff}}} X)/(H/G_{\mathrm{aff}}) = \mathrm{GL}_n \times^H X$$

is a normal, quasi-projective variety as well (see, e.g., [19, §12]). So we may choose an ample, GL_n-linearized invertible sheaf L on that quotient. The pull-back of L to $X \subset \mathrm{GL}_n \times^H X$ is the desired ample, H-linearized invertible sheaf. $\square$

4 Proof of Theorem 1.3

We begin with some observations about the Albanese morphism of a G-variety X, based on the results of [23]. Since the Albanese morphism of $G \times X$ is the product morphism $\alpha_G \times \alpha_X$, there exists a unique morphism of varieties

$$\beta : A(G) \times A(X) \to A(X)$$

such that the following square is commutative,

$$\begin{array}{ccc} G \times X & \xrightarrow{\varphi} & X \\ {\scriptstyle \alpha_G \times \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_X} \\ A(G) \times A(X) & \xrightarrow{\beta} & A(X). \end{array}$$

Then β is the composite of a homomorphism and a translation. Moreover, $\beta(0,z) = z$ for all z in the image of α_X. Since this image is not contained in a translate of a smaller abelian variety, it follows that $\beta(0,z) = z$ for all $z \in A(X)$. Thus,

$$\beta(y,z) = \alpha_\varphi(y) + z\,,$$

where

$$\alpha_\varphi : A(G) \longrightarrow A(X)$$

is a homomorphism. In other words, the G-action on X induces an action of $A(G)$ on $A(X)$ by translations via α_φ.

Given an abelian variety A, the datum of a morphism $X \to A$ up to a translation in A is equivalent to that of a homomorphism $A(X) \to A$. Thus, the datum of a G-equivariant morphism

$$\psi : X \longrightarrow A$$

up to a translation in A, is equivalent to that of a homomorphism

$$\alpha_\psi : A(X) \longrightarrow A = A(G)/F$$

such that the composite $\alpha_\psi \circ \alpha_\varphi$ equals the quotient morphism

$$q : A(G) \to A(G)/F$$

up to a translation. Then the kernel of α_φ is contained in F, and hence is finite.

Conversely, if α_φ has a finite kernel, then there exist a finite subgroup scheme F of $A(G)$ and a homomorphism α_ψ such that $\alpha_\psi \circ \alpha_\varphi = q$. Indeed, this follows from Lemma 3.2 applied to the image of α_φ, or, alternatively, from the Poincaré complete reducibility theorem (see, e.g., [16, Proposition 12.1]). We also see that for a fixed subgroup scheme $F \subset A(G)$ (or, equivalently, $H \subset G$), the set of homomorphisms α_ψ is a torsor under $\mathrm{Hom}(A(X)/\alpha_\varphi(A(G)), A(G)/F)$, a free abelian group of finite rank.

So we have shown that the existence of a G-morphism ψ as in Theorem 1.2 is equivalent to the kernel of the G-action on $A(X)$ being affine, and also to the kernel of α_φ being finite.

Next, we assume that X is normal and quasi-homogeneous, and prove Theorem 1.3. Choose a point x in the open orbit X_0 and denote by $G_x \subset G$ its isotropy subgroup scheme, so that X_0 is isomorphic to the homogeneous space G/G_x. Then G_x is affine by Lemma 3.1; therefore, the product $H_0 := G_x G_{\mathrm{aff}}$ is a closed affine subgroup scheme of G, and the quotient $F := H_0/G_{\mathrm{aff}}$ is finite. As a consequence, the homogeneous space

$$G/H_0 \cong (G/G_{\mathrm{aff}})/(H_0/G_{\mathrm{aff}}) = A(G)/F$$

is an abelian variety, isogenous to $A(G)$.

We claim that the Albanese morphism of X_0 is the quotient map $G/G_x \to G/H_0$. Indeed, let $f : X_0 \to A$ be a morphism to an abelian variety. Then the composite

$$G \xrightarrow{\varphi_x} X_0 \xrightarrow{f} A$$

factors through a unique morphism $A(G) = G/G_{\mathrm{aff}} \to A$ which must be invariant under G_x. Thus, f factors through a unique morphism $G/H_0 \to A$.

Next, we claim that the Albanese morphism of X_0 extends to X. Of course, such an extension is unique if it exists; hence we may assume that X is quasi-projective, by Theorem 1.1. Let $\psi : X \to A$ be a morphism as in Theorem 1.2. Then ψ factors through a G-morphism $\alpha_\psi : A(X) \to A$, and the composite $\alpha_\psi \circ \alpha_\varphi : A(G) \to A$ is an isogeny. Thus, α_φ is an isogeny, and hence the canonical morphism $h_X : A(X)_r \to A(X)$ is an isogeny as well (since $A(X)_r = A(X_0) = A(G)/F$). Together with Zariski's main theorem, it follows that the rational map $\alpha_{X,r} : X - - \to A(X)_r$ is a morphism; that is, h_X is an isomorphism. $\square$

5 Proof of Theorem 1.4

Choose $x \in X$ so that $X \cong G/G_x$. The radical $R(G_{\rm aff})$ fixes some point of X, and is a normal subgroup of G; thus, $R(G_{\rm aff}) \subset G_x$. By the faithfulness assumption, it follows that $R(G_{\rm aff})$ is trivial, that is, $G_{\rm aff}$ is semi-simple. Moreover, the reduced connected center $C(G)^0$ satisfies $C(G)^0_{\rm aff} \subset R(G_{\rm aff})$. Thus, $C(G)^0$ is an abelian variety, and hence $C(G)^0 \cap G_{\rm aff}$ is a finite central subgroup scheme of G.

We claim that $C(G)^0 \cap G_{\rm aff}$ is trivial. Consider indeed the reduced neutral component P of G_x. Then $P \subset G_{\rm aff}$ by Lemma 3.1(ii). Moreover, the natural map

$$\pi : G/P \longrightarrow G/G_x \cong X$$

is finite, and hence G/P is complete. It follows that $G_{\rm aff}/P$ is complete as well. Thus, P is a parabolic subgroup of $G_{\rm aff}$, and hence contains $C(G)^0 \cap G_{\rm aff}$. In particular, $C(G)^0 \cap G_{\rm aff} \subset G_x$; arguing as above, this implies our claim.

By that claim and the equality $G = C(G)^0 G_{\rm aff}$ (Lemma 3.1 (iii)), we obtain that $C(G)^0 = A(G)$ and $G \cong A(G) \times G_{\rm aff}$, where $G_{\rm aff}$ is semi-simple and adjoint. Moreover,

$$G/P \cong A(G) \times (G_{\rm aff}/P).$$

Next, recall from Section 4 that the Albanese morphism α_X is the natural map $G/G_x \to G/H$, where $H := G_x G_{\rm aff}$ is affine. So we may write $H = F \times G_{\rm aff}$, where F is a finite subgroup scheme of $A(G)$, and hence a central subgroup scheme of G. Then

$$X \cong A(G) \times^F Y$$

and $A(X) \cong A(G)/F$, where $Y := H/G_x \cong G_{\rm aff}/(G_{\rm aff} \cap G_x)$. We now show that F is trivial; equivalently, F acts trivially on Y. As above, it suffices to prove that $F \subset G_x$.

Choose a maximal torus T of P; then x lies in the fixed-point subscheme X^T. The latter is nonsingular and stable under $A(G)$. Moreover, the restriction

$$\pi^T : (G/P)^T = A(G) \times (G_{\rm aff}/P)^T \longrightarrow X^T$$

is surjective, and hence $A(G)$ acts transitively on each component of X^T; the Weyl group of $(G_{\rm aff}, T)$ transitively permutes these components. Let Z be the component containing x. To complete the proof, it suffices to show that the morphism $\alpha_X : X \to A(G)/F$ restricts to an isomorphism $Z \cong A(G)/F$.

By the Bialynicki–Birula decomposition (see [4]), Z admits a T-stable open neighborhood U in X together with a T-equivariant retraction $\rho : U \to Z$, a locally trivial fibration in affine spaces. It follows that the Albanese morphisms of U and Z satisfy $\alpha_U = \alpha_Z \circ \rho$. Since α_U is the restriction of α_X, we obtain that $\alpha_Z = \alpha_X|_Z$. On the other hand, α_Z is the identity, since Z is homogeneous under $A(G)$. Thus, α_X maps Z isomorphically to $A(X) \cong A(G)/F$. $\square$

6 Examples

Example 6.1. Let A be an abelian variety. Choose distinct finite subgroups $F_1,\dots,F_n$ of A such that $F_1\cap\cdots\cap F_n=\{0\}$ (as schemes), and let

$$X := A/F_1\times\cdots\times A/F_n\,.$$

Then A acts faithfully on X via simultaneous translation on all factors, and each projection $p_i : X\to A/F_i$ satisfies the assertion of Theorem 1.2. Since the fibers of p_i are varieties, this morphism admits no nontrivial factorization through an A-morphism $\psi : X\to A/F$, where F is a finite subgroup scheme of A.

Assume that the only endomorphisms of A are the multiplications by integers, and the orders of $F_1,\dots,F_n$ have a nontrivial common divisor. Then there exists no A-morphism $\psi : X\to A$ (otherwise, we would obtain endomorphisms $u_1,\dots,u_n$ of A such that each u_i is zero on F_i and $u_1+\cdots+u_n=\mathrm{id}_A$, which contradicts our assumptions). Equivalently, A is not a direct summand of X for its embedding via the diagonal map $a\mapsto(a \mod F_1,\dots,a \mod F_n)$. Thus, there exists no universal morphism ψ satisfying the assertions of Theorem 1.2.

Example 6.2. Following [20, XIII 3.1], we construct a complete equivariant embedding X of a commutative nonaffine group G, for which the assertions of Theorems 1, 2, 3 do not hold. Of course, X will be non-normal.

Consider an abelian variety A, and a non-zero point $a\in A$. Let X_a be the scheme obtained from $A\times\mathbb{P}^1$ by identifying every point $(x,0)$ with $(x+a,\infty)$. (That X_a is indeed a scheme follows e.g., from the main result of [10].) Then X_a is a complete variety, and the canonical map

$$f_a : A\times\mathbb{P}^1\longrightarrow X_a$$

is the normalization. In particular, X_a is *weakly normal*, i.e., every finite birational bijective morphism from a variety to X_a is an isomorphism.

The connected algebraic group

$$G := A\times\mathbb{G}_m$$

acts faithfully on $A\times\mathbb{P}^1$ via $(x,t)\cdot(y,u)=(x+y,tu)$, and this induces a faithful G-action on X_a such that f_a is equivariant. Moreover, X_a consists of two G-orbits: the open orbit,

$$f_a(A\times(\mathbb{P}^1\setminus\{0,\infty\}))\cong G,$$

and the closed orbit,

$$f_a(A\times\{0\})=f_a(A\times\{\infty\})\cong A\cong G/\mathbb{G}_m\,,$$

which is the singular locus of X_a.

The projection $A \times \mathbb{P}^1 \to A$ induces a morphism

$$\alpha_a : X_a \longrightarrow A/a\,,$$

where A/a denotes the quotient of A by the closure of the subgroup generated by a. We claim that α_a is the Albanese morphism: indeed, any morphism $\beta : X_a \to B$, where B is an abelian variety, yields a morphism $\beta \circ f_a : A \times \mathbb{P}^1 \to B$. By rigidity (see e.g., [16, Corollary 2.5, Corollary 3.9]), there exists a morphism $\gamma : A \to B$ such that

$$(\beta \circ f_a)(y,z) = \gamma(y)$$

for all $y \in A$ and $z \in \mathbb{P}^1$. Conversely, a morphism $\gamma : A \to B$ yields a morphism $\beta : X_a \to B$ if and only if $\gamma(y+a) = \gamma(y)$ for all $y \in A$, which proves our claim.

Also, note that $\alpha_G : G \to A(G)$ is just the projection $A \times \mathbb{G}_m \to A$. Thus, the kernel of the homomorphism $A(G) \to A(X_a)$ is the closed subgroup generated by a.

In particular, if the order of a is infinite, then X_a does not admit any G-morphism to a finite quotient of A, so that the assertions of Theorems 1.2 and 1.3 are not satisfied. Moreover, X_a is not projective (as follows, e.g., from Lemma 3.2 applied to the action of A), so that the assertion of Theorem 1.1 does not hold as well.

On the other hand, if a has finite order n, then the fibers of α_a are unions of n copies of $\mathbb{P}^1$ on which the origins are identified. In that case, X_a is projective, but does not satisfy the assertions of Theorem 1.3 as $n \geq 2$.

We may also consider X_a as an A-variety via the inclusion $A \subset G$. The projection $A \times \mathbb{P}^1 \to \mathbb{P}^1$ induces an A-invariant morphism

$$\pi : X_a \longrightarrow C\,,$$

where C denotes the nodal curve obtained from $\mathbb{P}^1$ by identifying 0 with ∞; one checks that π is an A-torsor. If a has infinite order, then X_a is not covered by A-stable quasi-projective open subsets: otherwise, by Lemma 3.2 again, any such subset U would admit an A-morphism to a finite quotient of A, i.e., the kernel of the A-action on $A(U)$ would be finite. However, when U meets the singular locus of X_a, one may show as above that the Albanese morphism of U is the restriction of α_a, which yields a contradiction.

Example 6.3. Given any abelian variety A, we construct (after [20, XII 1.2]) an A-torsor $\pi : X \to Y$ such that A acts trivially on $A(X)$.

Let X be the scheme obtained from $A \times A \times \mathbb{P}^1$ by identifying every point $(x,y,0)$ with $(x+y,y,\infty)$. Then again, X is a complete variety, and the canonical map

$$f : A \times A \times \mathbb{P}^1 \longrightarrow X$$

is the normalization; X is weakly normal but not normal.

Let A act on $A \times A \times \mathbb{P}^1$ via translation on the first factor; this induces an A-action on X such that f is equivariant. Moreover, the projection $p_{23} : A \times A \times \mathbb{P}^1 \to A \times \mathbb{P}^1$ induces an A-invariant morphism

$$\pi : X \longrightarrow A \times C,$$

where C denotes the rational curve with one node, as in the above example. One checks that π is an A-torsor, and the Albanese morphism α_X is the composite of π with the projection $A \times C \to A$. Thus, α_X is A-invariant.

So Theorem 1.2 does not hold for the A-variety X. Given any A-stable open subset U of X which meets the singular locus, one may check as above that α_U is A-invariant; as a consequence, Theorem 1.1 does not hold as well.

Example 6.4. Given an elliptic curve E, we construct an example of an E-torsor $\pi : X \to Y$, where Y is a normal affine surface and the Albanese variety of X is trivial. (In particular, X is a normal E-variety which does not satisfy the assertions of Theorem 1.2.) For this, we adapt a construction from [20, XIII 3.2].

Let $\dot{E} := E \setminus \{0\}$ and $\dot{\mathbb{A}}^1 := \mathbb{A}^1 \setminus \{0\}$. We claim that there exist a normal affine surface Y having exactly two singular points y_1, y_2, and a morphism

$$f : \dot{E} \times \dot{\mathbb{A}}^1 \longrightarrow Y$$

such that f contracts $\dot{E} \times \{1\}$ to y_1, $\dot{E} \times \{-1\}$ to y_2, and restricts to an isomorphism on the open subset $\dot{E} \times (\dot{\mathbb{A}}^1 \setminus \{1, -1\})$.

Indeed, we may embed E in $\mathbb{P}^2$ as a cubic curve with homogeneous equation $F(x,y,z) = 0$, such that the line $(z = 0)$ meets C with order 3 at the origin. Then $\dot{E}$ is identified with the curve in $\mathbb{A}^2$ with equation $F(x,y,1) = 0$. Now one readily checks that the claim holds for the surface

$$Y \subset \mathbb{A}^2 \times \dot{\mathbb{A}}^1$$

with equation $F(x,y,z^2 - 1) = 0$, and the morphism

$$f : (x,y,z) \longmapsto (x(z^2 - 1), y(z^2 - 1), z).$$

The singular points of Y are $y_1 := (0,0,1)$ and $y_2 := (0,0,-1)$.

Next, let $U_1 = Y \setminus \{y_2\}$, $U_2 := Y \setminus \{y_1\}$, and $U_{12} := U_1 \cap U_2$. Then U_{12} is nonsingular and contains an open subset isomorphic to $\dot{E} \times (\dot{\mathbb{A}}^1 \setminus \{1, -1\})$. Thus, the projection $\dot{E} \times (\dot{\mathbb{A}}^1 \setminus \{1, -1\}) \to \dot{E}$ extends to a morphism

$$p : U_{12} \longrightarrow E.$$

We may glue $E \times U_1$ and $E \times U_2$ along $E \times U_{12}$ via the automorphism

$$(x,y) \longmapsto (x + p(y), y)$$

to obtain a torsor $\pi : X \to Y$ under E. Arguing as in Example 6.2, one checks that the Albanese variety of X is trivial.

Example 6.5. Following Igusa (see [13]), we construct nonsingular projective varieties for which the tangent bundle is trivial and the Albanese morphism is a nontrivial fibration.

We assume that the ground field k has characteristic 2, and consider two abelian varieties A, B such that A has a point a of order 2. Let X be the quotient of $A \times B$ by the involution

$$\sigma : (x,y) \longmapsto (x+a,-y)$$

and denote by $\pi : A \times B \to X$ the quotient morphism. Since σ has no fixed point, X is a nonsingular projective variety, and the natural map between tangent sheaves yields an isomorphism

$$\mathscr{T}_{A\times B} \cong \pi^*(\mathscr{T}_X).$$

This identifies $\Gamma(X, \mathscr{T}_X)$ with the invariant subspace

$$\Gamma(A \times B, \mathscr{T}_{A\times B})^\sigma = \mathrm{Lie}(A \times B)^\sigma.$$

Moreover, the action of σ on $\mathrm{Lie}(A\times B)$ is trivial, by the characteristic 2 assumption. This yields

$$\Gamma(X, \mathscr{T}_X) \cong \mathrm{Lie}(A \times B).$$

As a consequence, the natural map $\mathscr{O}_X \otimes \Gamma(X, \mathscr{T}_X) \to \mathscr{T}_X$ is an isomorphism, since this holds after pull-back via π. In other words, the tangent sheaf of X is trivial.

The action of A on $A \times B$ via translation on the first factor induces an A-action on X, which is easily seen to be faithful. Moreover, the first projection $A \times B \to A$ induces an A-morphism

$$\psi : X \longrightarrow A/a,$$

a homogeneous fibration with fiber B. One checks as in Example 6.2 that ψ is the Albanese morphism of X.

Let G denote the reduced neutral component of the automorphism group scheme $\mathrm{Aut}(X)$. We claim that the natural map $A \to G$ is an isomorphism. Indeed, G is a connected algebraic group, equipped with a morphism $G \to A/a$ having an affine kernel (by the Nishi–Matsumura theorem) and such that the composite $A \to G \to A/a$ is the quotient map. Thus, $G = AG_{\mathrm{aff}}$, and G_{aff} acts faithfully on the fibers of ψ, i.e., on B. This implies our claim.

In particular, X is not homogeneous, and the fibration ψ is nontrivial (otherwise, X would be an abelian variety). In fact, the sections of ψ correspond to the 2-torsion points of B, and coincide with the local sections; in particular, ψ is not locally trivial.

Example 6.6. We assume that k has characteristic $p > 0$. Let X be the hypersurface in $\mathbb{P}^n \times \mathbb{P}^n$ with bihomogeneous equation

$$f(x_0,\dots,x_n,y_0,\dots,y_n) := \sum_{i=0}^{n} x_i^p\, y_i = 0,$$

where $n \geq 2$. The simple algebraic group

$$G := \mathrm{PGL}(n+1)$$

acts on $\mathbb{P}^n \times \mathbb{P}^n$ via

$$[A] \cdot ([x,], [y]) = ([Ax], [F(A^{-1})^T y]),$$

where $A \in \mathrm{GL}(n+1)$ and F denotes the Frobenius endomorphism of $\mathrm{GL}(n+1)$ obtained by raising matrix coefficients to their pth power. This induces a G-action on X which is faithful and transitive, but not smooth; the isotropy subgroup scheme of any point $\xi = ([x], [y])$ is the intersection of the parabolic subgroup $G_{[x]}$ with the nonreduced parabolic subgroup scheme $G_{[y]}$. So X is a variety of unseparated flags.

Denote by

$$\pi_1, \pi_2 : X \longrightarrow \mathbb{P}^n$$

the two projections. Then X is identified via π_1 to the projective bundle associated with $F^*(\mathscr{T}_{\mathbb{P}^n})$ (the pull-back of the tangent sheaf of $\mathbb{P}^n$ under the Frobenius morphism). In particular, π_1 is smooth. Also, π_2 is a homogeneous fibration and $(\pi_2)_* \mathscr{O}_X = \mathscr{O}_{\mathbb{P}^n}$, but π_2 is not smooth.

Let $\mathscr{T}_{\pi_1}$, $\mathscr{T}_{\pi_2}$ denote the relative tangent sheaves (i.e., $\mathscr{T}_{\pi_i}$ is the kernel of the differential $d\pi_i : \mathscr{T}_X \to \pi_i^*(\mathscr{T}_{\mathbb{P}^n})$). We claim that

$$\mathscr{T}_X = \mathscr{T}_{\pi_1} \oplus \mathscr{T}_{\pi_2}; \tag{8}$$

in particular, $\mathscr{T}_{\pi_2}$ has rank n. Moreover, $\mathscr{T}_{\pi_2}$ is the subsheaf of $\mathscr{T}_X$ generated by the global sections.

For this, consider the natural map

$$\mathrm{op}_X : \mathrm{Lie}(G) \longrightarrow \Gamma(X, \mathscr{T}_X)$$

and the induced map of sheaves

$$op_X : \mathscr{O}_X \otimes \mathrm{Lie}(G) \longrightarrow \mathscr{T}_X.$$

For any $\xi = ([x], [y]) \in X$, the kernel of the map

$$op_{X,\xi} : \mathrm{Lie}(G) \to \mathscr{T}_{X,\xi}$$

is the isotropy Lie algebra $\mathrm{Lie}(G)_\xi$, that is, $\mathrm{Lie}(G)_{[x]}$ (since $\mathrm{Lie}(G)$ acts trivially on the second copy of $\mathbb{P}^n$). Thus, $(d\pi_1)_\xi : \mathscr{T}_{X,\xi} \longrightarrow \mathscr{T}_{\mathbb{P}^n,[x]}$ restricts to an isomorphism $\mathrm{Im}(op_{X,\xi}) \cong \mathscr{T}_{\mathbb{P}^n,[x]}$. In other words, $d\pi_1$ restricts to an isomorphism $\mathrm{Im}(op_X) \cong \pi_1^*(\mathscr{T}_{\mathbb{P}^n})$. So π_1 is the Tits morphism of X. Moreover, we have a decomposition

$$\mathscr{T}_X = \mathscr{T}_{\pi_1} \oplus \mathrm{Im}(op_X),$$

and $\mathrm{Im}(op_X) \subset \mathscr{T}_{\pi_2}$. Since $\mathscr{T}_{\pi_1} \cap \mathscr{T}_{\pi_2} = 0$, this implies the equalities (8) and $\mathrm{Im}(op_X) = \mathscr{T}_{\pi_2}$.

To complete the proof of the claim, we show that op_X is an isomorphism. Consider indeed the bihomogeneous coordinate ring

$$k[x_0, \ldots, x_n, y_0, \ldots, y_n]/(x_0^p y_0 + \cdots + x_n^p y_n)$$

of X. Its homogeneous derivations of bidegree $(0,0)$ are those given by $x \mapsto Ax$, $y \mapsto ty$, where A is an $n+1 \times n+1$ matrix, and t a scalar; this is equivalent to our assertion.

Also, X admits no nontrivial decomposition into a direct product, by [21, Corollary 6.3]. This can be seen directly: if $X \cong X_1 \times X_2$, then the G-action on X induces actions on X_1 and X_2 such that both projections are equivariant (as follows, e.g., by linearizing the pull-backs to X of ample invertible sheaves on the nonsingular projective varieties X_1, X_2). So $X_1 \cong G/H_1$ and $X_2 \cong G/H_2$, where H_1, H_2 are parabolic subgroup schemes of G. Since the diagonal G-action on $G/H_1 \times G/H_2$ is transitive, it follows that $G = P_1 P_2$, where P_i denotes the reduced scheme associated with H_i (so that each P_i is a proper parabolic subgroup of G). But the simple group G cannot be the product of two proper parabolic subgroups, a contradiction.

Since X is rational and hence simply connected, this shows that a conjecture of Beauville (relating decompositions of the tangent bundle of a compact Kähler manifold to decompositions of its universal cover; see [3]) does not extend to nonsingular projective varieties in positive characteristics.

More generally, let G be a simple algebraic group in characteristic $p \geq 5$, and $X \cong G/G_x$ a complete variety which is homogeneous under a faithful G-action. By [25], there exists a unique decomposition

$$G_x = \bigcap_{i=1}^{r} P_i \, G_{(n_i)} \,,$$

where $P_1, \ldots, P_r$ are pairwise distinct maximal parabolic subgroups of G, and $(n_1, \ldots, n_r)$ is an increasing sequence of nonnegative integers. Here $G_{(n)}$ denotes the nth Frobenius kernel of G. Since G acts faithfully on X, we must have $n_1 = 0$. Let

$$Q_1 := \bigcap_{i, n_i = 0} P_i \,, \quad Q_2 := \bigcap_{i, n_i \geq 1} P_i \, G_{(n_i)} \,.$$

Then Q_1 is a parabolic subgroup of G, Q_2 is a parabolic subgroup scheme, $G_x = Q_1 \cap Q_2$, and $\mathrm{Lie}(G_x) = \mathrm{Lie}(Q_1)$. Thus, the Tits morphism of X is the canonical map

$$\pi_1 : G/G_x \to G/Q_1 \,,$$

and $d\pi_1$ restricts to an isomorphism $\mathrm{Im}(op_X) \cong \pi_1^*(\mathscr{T}_{G/Q_1})$. This implies the decomposition $\mathscr{T}_X = \mathrm{Im}(op_X) \oplus \mathscr{T}_{\pi_1}$, which is nontrivial unless $G_x = Q_1$ (i.e., G_x is reduced). Moreover, $\mathrm{Im}(op_X) \subset \mathscr{T}_{\pi_2}$ (where $\pi_2 : G/G_x \to G/Q_2$ denotes the canonical map), since G_1 acts trivially on G/Q_2. As $\pi_1 \times \pi_2$ is a closed immersion, it follows again that

$$\mathscr{T}_X = \mathscr{T}_{\pi_1} \oplus \mathscr{T}_{\pi_2} \quad \text{and} \quad \mathrm{Im}(op_X) = \mathscr{T}_{\pi_2} \,.$$

On the other hand, the variety X is indecomposable, since G is simple (see [21, Corollary 6.3] again, or argue directly as above). This yields many counterexamples to the analogue of the Beauville conjecture.

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On Chevalley–Shepherd–Todd's Theorem in Positive Characteristic

Abraham Broer

Summary. Let G be a finite group acting linearly on the vector space V over a field of arbitrary characteristic. The action is called *coregular* if the invariant ring is generated by algebraically independent homogeneous invariants and the *direct summand property* holds if there is a surjective $k[V]^G$-linear map $\pi : k[V] \to k[V]^G$. The following Chevalley–Shepherd–Todd type theorem is proved. Suppose V is an irreducible kG-representation, then the action is coregular if and only if G is generated by pseudo-reflections and the direct summand property holds.

Key words: Reflection groups, modular invariant theory, direct summand, Dedekind different

Mathematics Subject Classification (2000): 13A50, 20H15

To Gerald Schwarz on the occasion of his 60th anniversary

Introduction

Let V be a vector space of dimension n over a field k. A linear transformation $\tau : V \to V$ is called a *pseudo-reflection*, if its fixed-points space $V^\tau = \{v \in V; \tau(v) = v\}$ is a linear subspace of codimension one. Let $G < \mathrm{GL}(V)$ be a finite group acting linearly on V. Then G acts by algebra automorphisms on the coordinate ring $k[V]$, which is by definition the symmetric algebra on the dual vector space V^*. We say that G is a *pseudo-reflection group* if G is generated by pseudo-reflections; it is called a *nonmodular* group if $|G|$ is not divisible by the characteristic of the field. The action is called *coregular* if the invariant ring is generated by n algebraically independent homogeneous invariants.

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A well-known theorem of Chevalley–Shephard–Todd [2, Chapter 6] says that if the group is nonmodular then G is a pseudo-reflection group if and only if the action is coregular.

By a theorem of Serre [2, Theorem 6.2.2] the implication that coregularity of the action implies that G is a pseudo-reflection group is true even without the condition of nonmodularity. This is not true for the other implication: there are pseudo-reflection groups whose action is not coregular.

Coxeter, Shephard and Todd classified all pseudo-reflection groups in characteristic zero. More recently the irreducible pseudo-reflection groups were classified over any characteristic by Kantor, Wagner, Zaleskiĭ, and Serežkin. Using this classification Kemper–Malle [6] decided which irreducible pseudo-reflection groups possess the coregular property and which do not. They observed that those irreducible pseudo-reflection groups that possess the coregularity property are exactly those such that the actions are coregular for all the point-stabilizers of nontrivial subspaces.

We say that the *direct summand property* holds if there is a surjective $k[V]^G$-linear map $\pi : k[V] \to k[V]^G$ respecting the gradings. For a nonmodular group the direct summand property always holds, because in that case we can take the transfer Tr^G as projection, defined by

$$\mathrm{Tr}^G : k[V] \to k[V]^G : \ \mathrm{Tr}^G(f) = \sum_{\sigma \in G} \sigma(f),$$

since for any invariant f we have $\mathrm{Tr}^G(|G|^{-1}f) = f$. Also the coregular property implies the direct summand property.

In this chapter, we show first that if the direct summand property holds for G then the direct summand property holds for all the point-stabilizers of subspaces of V, (cf. Theorem 1.1). Then using this and the results of Kemper–Malle we show for irreducible G-actions that the action is coregular if and only if G is a pseudo-reflection group and the direct summand property holds, (cf. Theorem 2.1). We conjectured before that this also holds without the irreduciblity condition, (cf. [3]). Elsewhere we show that the converse is also true if G is abelian, (cf. [4]).

In the first section we show that the direct summand property is inherited by point-stabilizers. In the second section we recall Kemper–Malle's classification of irreducible pseudo-reflection groups that are not coregular, and describe the other tools used in the proof of the main theorem. In the last section we give the details of the calculations.

1 The direct summand property and point-stabilizers

For elementary facts on the invariant theory of finite groups we refer to [2], for a discussion of the direct summand property and the different see [3]. The transfer map extends to the quotient field of $k[V]$. We recall that the (Dedekind) different θ_G of the G-action on V can be defined as the largest degree homogeneous form

$\theta_G \in k[V]$ such that $\mathrm{Tr}^G(f/\theta_G)$ is without denominator, i.e., $\mathrm{Tr}^G(f/\theta_G) \in k[V]^G$, for all $f \in k[V]$; it is unique up to a multiplicative scalar. The direct summand property holds if and only if there exists a $\tilde{\theta}_G \in k[V]$ such that $\mathrm{Tr}^G(\tilde{\theta}_G/\theta_G) = 1$ and then we can take as $k[V]^G$-linear projection

$$\pi : k[V] \to k[V]^G : \ \pi(f) := \mathrm{Tr}^G(\frac{\tilde{\theta}_G f}{\theta_G}).$$

In Kemper–Malle's classification Steinberg's classical result is often used saying that the coregular property is inherited by point-stabilizers of linear subspaces. We prove that the direct summand property is also inherited by point-stabilizers of linear subspaces.

The key point in the proof of both results is that the affine group V^G acts on V by translations, namely $\tau_u : v \mapsto v + u$ ($u \in V^G$, $v \in V$), commuting with the linear G-action.

Theorem 1.1. *Let the finite group G act linearly on the vector space V over the field k and let H be the point-stabilizer of a linear subspace $U \subset V$.*

If the G action on V has the direct summand property then the H action on V also has the direct summand property.

Proof. We write $A := k[V]$, $C := k[V]^H$, and $B = A^G$. The prime ideal generated by the linear forms vanishing on U is denoted by $\mathfrak{P} \subset A$; its intersection with C is the prime ideal $\mathfrak{q} = \mathfrak{P} \cap C$. The inertia subgroup of $\mathfrak{P}$ coincides with H:

$$H = \{\sigma \in G; (\sigma - 1)(A) \subseteq \mathfrak{P}\}.$$

Let θ_G and θ_H be the two (Dedekind) differents with respect to the G-action and the H-action on V. In particular $\mathrm{Tr}^G(A/\theta_G) \subseteq B$, and $\mathrm{Tr}^H(A/\theta_H) \subseteq C$.

Let $V^\alpha \subset V$ be a linear subspace of codimension one, defined as the zero set of the linear form x_α. Then x_α divides θ_G if and only if there is a pseudo-reflection in G with reflecting hyperplane V^α, or in other words there exists a $g \in G$ such that for all $a \in A$, $g(a) - a \in x_\alpha A$. Now for such a pseudo-reflection g we have

$$g \in H \iff V^\alpha \supseteq U \iff x_\alpha \in \mathfrak{P}.$$

It follows that θ_H is the part of θ_G involving the powers of linear forms x_α, such that $x_\alpha \in \mathfrak{P}$. Let $\theta_{G/H}$ be the part of θ_G involving the powers of linear forms x_α, such that $x_\alpha \notin \mathfrak{P}$; then $\theta_G = \theta_{G/H} \cdot \theta_H$. So θ_H and $\theta_{G/H}$ are relatively prime, and more important $\theta_{G/H} \notin \mathfrak{P}$

The homogeneous element θ_G is a G-semi-invariant for some character $\chi : G \to k^\times$. Similarly θ_H is an H-semi-invariant. The quotient $\theta_{G/H} = \theta_G/\theta_H$ is an element of A, and is also an H-semi-invariant. So there is a power $\theta^e_{G/H}$ that is an absolute H-invariant (i.e., $\theta^e_{G/H} \in C$), but

$$\theta^e_{G/H} \notin \mathfrak{P} \cap C = \mathfrak{q}.$$

Assume now that B is a direct summand of A as a graded B-module; hence there exists a homogeneous $\tilde{\theta} \in A$ such that $\mathrm{Tr}^G(\tilde{\theta}/\theta_G) = 1$. We have to prove that the action of H also has the direct summand property, or that the ideal

$$I_H := \mathrm{Tr}^H(\frac{A}{\theta_H}) \subseteq C$$

is in fact equal to C.

We first show that

$$\theta_{G/H}^e \in I_H.$$

Since $\theta_{G/H}^e \notin \mathfrak{q}$, it follows that $I_H \not\subseteq \mathfrak{q}$.

Let $g_1, \ldots, g_s$ be right coset representatives of H in G; i.e., we have a disjoint union $G = \cup_{i=1}^s H g_i$. Then

$$\begin{aligned}
\theta_{G/H}^e &= \theta_{G/H}^e \cdot \mathrm{Tr}^G(\frac{\tilde{\theta}}{\theta_G}) \\
&= \theta_{G/H}^e \cdot \mathrm{Tr}^H\left(\sum_{i=1}^s g_i(\frac{\tilde{\theta}}{\theta_G})\right) \\
&= \mathrm{Tr}^H\left(\theta_{G/H}^e \cdot \sum_{i=1}^s g_i(\frac{\tilde{\theta}}{\theta_G})\right) \\
&= \mathrm{Tr}^H\left(\theta_{G/H}^e \cdot \frac{\sum_{i=1}^s \chi^{-1}(g_i) g_i(\tilde{\theta})}{\theta_G})\right) \\
&= \mathrm{Tr}^H\left(\frac{\theta_{G/H}^{e-1} \cdot \sum_{i=1}^s \chi^{-1}(g_i) g_i(\tilde{\theta})}{\theta_H})\right) \in \mathrm{Tr}^H(\frac{A}{\theta_H}) = I_H.
\end{aligned}$$

Suppose now that I_H is a proper ideal. Since it is a homogeneous ideal of C it is then contained in the maximal homogeneous ideal $\mathfrak{M}_0$ of A, the ideal of polynomials all vanishing at the origin $0 \in V$. We show that then even $I_H \subseteq \mathfrak{P}$, which is a contradiction.

To prove this we can assume that k is algebraically closed. If $u \in U$, then the affine transformation $\tau_u : v \mapsto v + u$ commutes with the linear H action, since

$$\tau_u(h \cdot v) = h \cdot v + u = h \cdot (v + u) = h \cdot (\tau_u(v)).$$

So it induces an algebra automorphism $\alpha = \tau_u^*$ of A commuting with the H-action, by

$$\alpha(f)(v) = (\tau_u^* \cdot f)(v) := f(v - u)$$

moving the maximal ideal $\mathfrak{M}_0$ into the maximal ideal $\mathfrak{M}_u$ of polynomials in A vanishing at u. It also commutes with Tr^H, and fixes the linear forms of $\mathfrak{P}$, so it fixes θ_H. But then

$$I_H = \mathrm{Tr}^H(\frac{A}{\theta_H}) = \mathrm{Tr}^H(\frac{\alpha(A)}{\theta_H}) = \alpha\left(\mathrm{Tr}^H\left(\frac{A}{\theta_H}\right)\right) \subseteq \alpha(\mathfrak{M}_0) \subseteq \mathfrak{M}_u.$$

So $I_H \subseteq \cap_{u\in U}\mathfrak{M}_u$. By Hilbert's Nullstellensatz $\mathfrak{P} = \cap_{u\in U}\mathfrak{M}_u$ and

$$I_H \subseteq \mathfrak{P} \cap C = \mathfrak{q}.$$

This is a contradiction, so I_H is not a proper ideal, i.e.,

$$\mathrm{Tr}^H(\frac{A}{\theta_H}) = C,$$

which implies that the direct summand property holds for the H action.

2 Main result and tools for the proof

In this section we describe the tools we used to prove our main theorem.

Theorem 2.1. *Let G be an irreducible pseudo-reflection group acting on V. Then the action is coregular if and only if G is a pseudo-reflection group and the direct summand property holds.*

It is already known that if the action is coregular then G is a pseudo-reflection group and the direct summand property holds; it follows from Serre's theorem [2, Theorem 6.2.2]. For the other direction we use Kemper–Malle's classification of irreducible pseudo-reflection groups not having the coregular property. We use their notation.

Theorem 2.2 (Kemper–Malle [6]). *Let G be an irreducible pseudo-reflection group group. Then it does not have the coregular property if and only if it occurs in the following list.*

(I) (Unitary pseudo-reflection groups) $\mathrm{SU}_n(q) \leq G \leq \mathrm{GU}_n(q)$, $n \geq 4$, *and* $\mathrm{SU}_3(q) \leq G < \mathrm{GU}_3(q)$.

(II) (Symplectic pseudo-reflection groups) $\mathrm{Sp}_n(q)$, $n \geq 4$ *and* $n = 2m$ *even.*

(IIIa) (Orthogonal reflection groups of odd characteristic) q odd: $\Omega_n^{(\pm)}(q) < G \leq \mathrm{GO}_n^{(\pm)}(q)$, *except* $\mathrm{GO}_3(q)$, $\mathrm{R}^+\mathrm{O}_3(q)$, $\mathrm{GO}_4^-(q)$.

(IIIb) (Orthogonal pseudo-reflection groups of even characteristic) q even: $\mathrm{SO}_{2m}^{(\pm)}(q)$, $2m \geq 4$, *except* $\mathrm{SO}_4^-(q)$.

(IV) (Symmetric groups) $\mathfrak{S}_{n+2}$, $p|(n+2)$, $n \geq 3$.

(V) (Exceptional cases) (i) $W_3(G_{30}) = W_3(H_4)$, *(ii)* $W_3(G_{31})$, *(iii)* $W_5(G_{32})$, *(iv)* $W_3(G_{36}) = W_3(E_7)$, *(v)* $W_3(G_{37}) = W_3(E_8)$, *(vi)* $W_5(G_{37}) = W_5(E_8)$ *and (vii)* $W_2(G_{34}) = 3 \cdot U_4(3) \cdot 2_2$.

Remark 2.1. In comparing Kemper–Malle's calculations with ours the reader should be aware that they work with the symmetric algebra of V and we with the coordinate ring of V. See also the comments in [5] on Kemper–Malle's article.

2.1 Tools

To prove our theorem we exhibit for every pseudo-reflection group in Kemper–Malle's list an explicit point-stabilizer H such that for the H-action on V the direct summand property does not hold. Then by Theorem 1.1 the G-action on V does not have the direct summand property either.

In most cases we found a point-stabilizer H that is a p-group. Then we can use the following tools to show that the H-action does not have the direct summand property.

If H is a p-group acting on V and the direct summand property holds then H is generated by its transvections, (cf. [3, Corollary 4]). So if H is not generated by transvections then the direct summand property does not hold.

Often H is abelian. Then we can use that for abelian pseudo-reflection groups the direct summand property holds if and only if the action is coregular, (cf. [4]).

For induction purposes the following trivial remark is useful. Let H be a group. We say that two kH-modules V_1 and V_2 are *equivalent* if one is obtained from the other by adding a trivial direct summand; for example, $V_2 \simeq V_1 \oplus k^r$. Then the action on one is coregular (has the direct summand property, etc.) if and only if the other is coregular (has the direct summand property, etc.).

Sometimes the following is useful to disprove coregularity. If the action is coregular with fundamental degrees $d_1, \dots, d_n$, then $\prod_{i=1}^n d_i = |G|$ and $\sum_{i=1}^n d_i = \delta_G + n$, e.g., [3, §2.5], where δ_G is the differential degree, i.e., the degree of the different θ_G. We give two examples.

Example 2.1. (i) Let $k = \mathbb{F}_{q^2}$, $q = p^r$ and $V = k^{2n}$, $2n \geq 4$, with standard basis $e_1, \dots, e_{2n}$. Consider the group

$$G_n := \{ \begin{pmatrix} I & O \\ B & I \end{pmatrix} ; \overline{B} = -B^T \},$$

where B is an anti-Hermitian $n \times n$ matrix with coefficients in $\mathbb{F}_{q^2}$, and where $\overline{B}_{ij} := B_{ij}^q$. It is normalized by

$$N := \{ \begin{pmatrix} A & O \\ O & \overline{A}^{-T} \end{pmatrix} ; A \in \mathrm{GL}(n, \mathbb{F}_{q^2}) \},$$

and N acts transitively on the $\frac{q^{2n}-1}{q^2-1}$ hyperplanes containing the subspace $\langle e_{n+1}, \dots, e_{2n} \rangle$. Take the hyperplane $\langle e_2, \dots, e_{2n} \rangle$; its point-stabilizer H consists of all matrices in G_n where all coefficients of B are 0 except possibly b_{11}. Its invariant ring has differential degree $q-1$. So the differential degree of the G_n action is

$$\delta_{G_n} = \frac{q^{2n}-1}{q^2-1}(q-1).$$

Let K be the point-stabilizer of the subspace $\langle e_3,\dots,e_{2n}\rangle$. Then $K \simeq G_2$ and the action of K on V is equivalent to the G_2 action on k^4. Suppose the G_2 action is coregular with fundamental degrees $d_1,\dots,d_4$. Then $d_1 = d_2 = 1$, since the first two coordinate functions are invariants. And $d_1d_2d_3d_4 = |G_2| = q^4$. So $(d_1,d_2,d_3,d_4) = (1,1,p^r,p^s)$ for some $r \geq 1, s \geq 1$ and $\sum_i d_i = \delta_{G_2} + n$, i.e., $1+1+p^r+p^s = q^3 - q^2 + q - 1 + 4$, implying that $2 \equiv 3$ modulo p, which is a contradiction. So the action of K is not coregular. Since it is abelian it does not have the direct summand property either.

Conclusion: the action of transvection group G_n on k^{2n}, $n \geq 2$, is not coregular and does not have the direct summand property.

(ii) Let $k = \mathbb{F}_q$, $q = p^r$, $V = k^{2n}$, $2n \geq 4$, with standard basis $e_1,\dots,e_{2n}$. Consider the group

$$G_n := \{ \begin{pmatrix} I & O \\ B & I \end{pmatrix} ;\ B = B^T \},$$

where B is a symmetric $n \times n$ matrix with coefficients in $\mathbb{F}_q$. It is normalized by

$$N := \{ \begin{pmatrix} A & O \\ O & A^{-T} \end{pmatrix} ;\ A \in \mathrm{GL}(n,\mathbb{F}_q) \}.$$

As in (i) we can calculate the differential degree of the G_n-action, it is

$$\delta_{G_n} = \frac{q^n - 1}{q - 1}(q-1) = q^n - 1.$$

Then as in (i) we can consider the point-stablizer H of $\langle e_3,\dots,e_{2n}\rangle$ and obtain the same conclusion. The action of G_n on k^{2n}, $n \geq 2$ is not coregular and does not have the direct summand property.

And the last tool we use to prove that the direct summand property does not hold is the following. If the direct summand property holds for the action of G on V and $J \subset k[V]^G$ an ideal, then $J = (J \cdot k[V]) \cap k[V]^G$ (cf. [3, Proposition 6(ii)]). We give an example of its use.

Example 2.2. Let $k = \mathbb{F}_{q^2}$ and $V = k^3$ with standard basis e_1,e_2,e_3 and coordinate functions x_1,x_2,x_3. Let H be the point-stabilizer of $\langle e_3\rangle$ inside $\mathrm{GU}_3(q)$, and $\tilde{H}$ the point-stabilizer of $\langle e_3\rangle$ inside $\mathrm{SU}_3(q)$. Or explicitly,

$$H = \{ \begin{pmatrix} 1 & 0 & 0 \\ a & b & 0 \\ c & d & 1 \end{pmatrix} ;\ b^{q+1} = 1, d = -ba^q, c + c^q + a^{q+1} = 0 \}$$

and $\tilde{H}$ is the normal subgroup where $b = 1$. Let η be a primitive $q+1$st root of unity and

$$\tau := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $H = \langle \tilde{H}, \tau \rangle$.

Both point-stabilizer groups have the algebraically independent invariants

$$x_1,\ F := x_1x_3^q + x_2^{q+1} + x_3x_1^q \quad \text{and} \quad N(x_3) := \prod_{\sigma\in H/\operatorname{Stab}_H(x_3)} \sigma(x_3),$$

of degrees $1, q+1$, and q^3 respectively. Since $|H| = (q+1)q^3$ they form a generating set of the invariant ring $k[V]^H$ (cf. [6, Proof of Proposition 3.1]).

Let

$$h := N(x_2) = \prod_{\sigma\in \tilde{H}/\operatorname{Stab}_{\tilde{H}}(x_2)} \sigma(x_2) = x_2\left(x_2^{q^2-1} - x_1^{q^2-1}\right).$$

Then by construction h is $\tilde{H}$-invariant and $\tau\cdot h = \eta h$. So h^{q+1} is the smallest power of h that is H-invariant. Let f be any $\tilde{H}$-invariant such that $\tau\cdot f = \eta f$. Since τ is a pseudo-reflection we have that $\tau(f) - f = (\eta - 1)f$ is divisible by x_2. Since f is also $\tilde{H}$-invariant it is also divisible by every $\sigma(x_2)$, $\sigma \in \tilde{H}$, so is divisible by h. Using powers h^i we get similar results for other H-semi-invariants. We get

$$k[V]^{\tilde{H}} = \oplus_{i=0}^q k[V]^H h^i$$

and so

$$k[V]^{\tilde{H}} = k[x_1, F, N(x_3), h];$$

in particular it is a hypersurface ring. Similarly for $H_m = \langle \tilde{H}, \tau^m\rangle$, for $m|(q+1)$, we get $k[V]^{H_m} = k[x_1, F, N(x_3), h^{(q+1)/m}]$. In any case

$$(x_1, F, N(x_3), h^{(q+1)/m})k[V] = (x_1, x_2^{q+1}, x_3^{q^3})k[V] = (x_1, F, N(x_3))k[V].$$

If the direct summand property holds for H_m acting on V, then for any ideal $J \subset k[V]^{H_m}$ we have $J = (J\cdot k[V]) \cap k[V]^{H_m}$. In particular, it follows for the maximal homogeneous ideal $k[V]_+^{H_m}$ of $k[V]^{H_m}$ that

$$k[V]_+^{H_m} = (x_1, F, N(x_3), h^{(q+1)/m})k[V]^{H_m} = (x_1, F, N(x_3))k[V]^{H_m}.$$

So x_1, F and $N(x_3)$ generate the maximal homogeneous ideal $k[V]_+^{H_m}$ and also the algebra $k[V]^{H_m}$. But this is a contradiction if $m \neq q+1$. Conclusion: the action of H_m on $V = k^3$ does not satisfy the direct summand property if $m|(q+1)$ and $m \neq q+1$.

3 Details

In this last section we establish explicitly for every pseudo-reflection group not having the coregular property in Kemper–Malle's list in Theorem 2.2 a point-stabilizer not having the direct summand property. For more information on some of the involved classical groups, for example, Witt's theorem, see [1].

3.1 Families

(I) (Unitary pseudo-reflection groups) $\mathrm{SU}_n(q) \le G \le \mathrm{GU}_n(q)$, $n \ge 4$, and $\mathrm{SU}_3(q) \le G < \mathrm{GU}_3(q)$.

Let first $n = 2m \ge 4$ be even. Then there is a basis $e_1, \ldots, e_{2m}$ of $V = \mathbb{F}_{q^2}^n$ such that the associated Gram matrix is

$$J = \begin{pmatrix} O & I \\ I & O \end{pmatrix},$$

where I is the identity $m \times m$ matrix and O the zero $m \times m$ matrix. So an $n \times n$ matrix g with coefficients in $\mathbb{F}_{q^2}$ is in $\mathrm{GU}_n(q)$ if and only if $g^T J \overline{g} = J$, where $\overline{g}$ is the matrix obtained from g by raising all its coefficients to the qth power. Let H be the point-stabilizer in $GU_n(q)$ of the maximal isotropic subspace $U = \langle e_{m+1}, \ldots, e_n \rangle$, so

$$H = \{ \begin{pmatrix} I & O \\ B & I \end{pmatrix} ; \overline{B} = -B^T \}.$$

If $\mathrm{SU}_n(q) \le G \le \mathrm{GU}_n(q)$ then H is also the point-stabilizer in G of U, since the index of G in $\mathrm{GU}_n(q)$ is relatively prime to p and H is a p-group. We encountered this group in example 2.1(i), and we conclude that the direct summand property does not hold for H.

If $n = 2m+1 \ge 5$ is odd, then the stabilizer in $\mathrm{SU}_n(q) \le G \le \mathrm{GU}_n(q)$ of a non-singular vector is a reflection group $\mathrm{SU}_{2m}(q) \le G_1 \le \mathrm{GU}_{2m}(q)$, so we can reduce to the even case, which we just handled.

For $\mathrm{SU}_3(q) \le G < \mathrm{GU}_3(q)$, see example 2.2. This is one of the rare cases where no point-stablizer could be found that was a p-group not having the direct summand property.

(II) (Symplectic pseudo-reflection groups) $\mathrm{Sp}_n(q)$, $n \ge 4$ and $n = 2m$ even. There is a basis $e_1, \ldots, e_{2m}$ of $\mathbb{F}_q^n$ such that the associated Gram-matrix is

$$J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

where I is the identity $m \times m$ matrix and O the zero $m \times m$ matrix. So an $n \times n$ matrix g with coefficients in $\mathbb{F}_q$ is in $\mathrm{Sp}_n(q)$ if and only if $g^T J g = J$. Let H be the point-stabilizer of the maximal isotropic subspace $U = \langle e_{m+1}, \ldots, e_n \rangle$, so

$$H = \{ \begin{pmatrix} I & O \\ B & I \end{pmatrix} ; B = B^T \}.$$

We encountered this group in Example 2.1(ii), and conclude that the direct summand property does not hold for H.

(IIIa) (Orthogonal reflection groups of odd characteristic) q odd: $\Omega_n^{(\pm)}(q) < G \le \mathrm{GO}_n^{(\pm)}(q)$, except $\mathrm{GO}_3(q)$, $\mathrm{R}^+\mathrm{O}_3(q)$, $\mathrm{GO}_4^-(q)$.

Let $V = \mathbb{F}_q^n$. If $n = 2m$ is even, then V admits two equivalence classes of non-degenerate quadratic forms distinguished by their sign; they are not similar. We get two orthogonal groups $\mathrm{GO}^{\pm}_{2m}(q)$. If $n = 2m+1$ is odd then there are also two equivalence classes of quadratic forms, but they are similar. For our purposes we need not distinguish the two (classes of) orthogonal groups, we write $\mathrm{GO}_{2m+1}(q)$. In any case the orthogonal group does not contain transvections and contains two types of reflections (i.e., pseudo-reflections of order two). If σ is a reflection, then its center $(\sigma-1)(V)$ is a one-dimensional nonsingular subspace $\langle u\rangle$. Conversely, to any one-dimensional nonsingular subspace $\langle u\rangle$ there corresponds a unique reflection. The orthogonal complement $\langle u\rangle^{\perp}$ is an irreducible orthogonal space and there are two possibilities, so by Witt's lemma there are exactly two conjugacy classes of nonsingular subspaces $\langle u\rangle$, hence two conjugacy classes of reflections. Each conjugacy class generates a normal reflection subgroup of the full orthogonal group of index 2. These are the three reflection groups we consider.

Let us first consider $n = 2m \geq 4$ and the reflection subgroups $G < \mathrm{GO}^{+}_{2m}(q)$. So there is a basis $e_1, \dots, e_{2m}$ of V such that the associated Gram matrix is

$$J = \begin{pmatrix} O & I \\ I & O \end{pmatrix},$$

where I is the identity $m \times m$ matrix and O the zero $m \times m$ matrix. So an $n \times n$ matrix g with coefficients in $\mathbb{F}_q$ is in $\mathrm{GO}^{+}_n(q)$ if and only if $g^T J g = J$. Let H be the point-stabilizer in $GO^{+}_n(q)$ of the maximal isotropic subspace $U = \langle e_{m+1}, \dots, e_n\rangle$, then

$$H = \{ \begin{pmatrix} I & O \\ B & I \end{pmatrix} ;\ B = -B^T \}.$$

If G is any of the reflection groups associated to $\mathrm{GO}^{+}_n(q)$ then its index is 1 or 2, so H is also the point-stabilizer in G of U. Since H is a nontrivial p-group and does not contain pseudo-reflections it follows that the direct summand property does not hold for H.

If $n = 2m+1 \geq 5$ is odd, then there is a nonsingular vector u such that the point-stabilizer of $\langle u\rangle$ in the reflection group $G < \mathrm{GO}_{2m+1}(q)$ of index ≤ 2 is a reflection group $G_1 < \mathrm{GO}^{+}_{2m}$ of index ≤ 2 acting irreducibly on $u^{\perp}$. We can use induction.

Consider now $n = 2m > 4$ and the reflection subgroups $G < \mathrm{GO}^{-}_{2m}(q)$ of index ≤ 2. There are two linearly independent nonsingular vectors u_1, u_2 such that the point-stabilizer of $\langle u_1, u_2\rangle$ in the reflection group $G < \mathrm{GO}^{-}_{2m}(q)$ of index ≤ 2 is a reflection group $G_1 < \mathrm{GO}^{+}_{2m-2}$ of index ≤ 2 acting irreducibly on $\langle u_1, u_2\rangle^{\perp}$. We can reduce to the earlier case.

Consider $\mathrm{GO}_3(q)$, the orthogonal group with respect to the quadratic form $2x_1x_3 + x_2^2$. Let H be the point-stabilizer of $\langle e_3\rangle$; then

$$H = \{ \begin{pmatrix} 1 & 0 & 0 \\ -b & a & 0 \\ \frac{-b^2}{2} & ab & 1 \end{pmatrix} ;\ a^2 = 1, b \in \mathbb{F}_q \}$$

The point-stabilizer H^- of $\mathrm{GO}_3^-(q)$ is the subgroup of H where the coefficient $a = 1$. So H^- is a p-group without transvections, so the direct summand property does not hold for H^-.

Let H be the point-stabilizer in $\mathrm{GO}_4^-(q)$ of an anisotropic line. Then H is isomorphic to $\mathrm{GO}_3(q)$. So for at least one of the two reflection subgroups of $\mathrm{GO}_4^-(q)$ the point-stabilizer H' of the anisotropic line is $\mathrm{GO}_3^-(q)$. So for that one the direct summand property does not hold. But both reflection subgroups of index two in $\mathrm{GO}_4^-(q)$ are conjugate inside the conformal orthogonal group; thus neither of them has the direct summand property.

(IIIb) (Orthogonal pseudo-reflection groups of even characteristic) q even: $\mathrm{SO}_{2m}^{(\pm)}(q)$, $2m \geq 4$, except $\mathrm{SO}_4^-(q)$. Let $V = \mathbb{F}_q^n$, where $n = 2m \geq 4$ is even. Then V admits two equivalence classes of nondegenerate quadratic forms distinguished by their sign. We get two orthogonal groups $\mathrm{GO}_{2m}^{\pm}(q)$. Now the orthogonal groups are generated by transvections and do not contain reflections.

First consider $n = 2m \geq 4$ and a quadratic form with maximal Witt index. Then there is a basis $e_1, \ldots, e_n$ with dual basis $x_1, \ldots, x_n$ such that the quadratic form becomes $Q = \sum_{i=1}^m x_i x_{m+i}$ and the Gram matrix is

$$J = \begin{pmatrix} O & I \\ I & O \end{pmatrix},$$

where I is the identity $m \times m$ matrix and O the zero $m \times m$ matrix. So an $n \times n$ matrix g with coefficients in $\mathbb{F}_q$ is in $\mathrm{GO}_n^+(q)$ if and only if $Q = Q \circ g$ (and so $g^T J g = J$). Let H be the point-stabilizer in $GO_n^+(q)$ of the maximal isotropic subspace $U = \langle e_{m+1}, \ldots, e_n \rangle$, so H is the collection of matrices

$$\begin{pmatrix} I & O \\ B & I \end{pmatrix}$$

such that $B_{ij} = B_{ji}$ if $1 \leq i \neq j \leq m$ and $B_{ii} = 0$, for $1 \leq i \leq m$. Since H is a p-group without pseudo-reflections, the direct summand property does not hold for H.

Next consider $n = 2m \geq 6$ and a quadratic form with nonmaximal Witt index. Then there are two nonsingular vectors u_1, u_2 such that the point-stabilizer in $\mathrm{GO}_n^-(q)$ of $\langle u_1, u_2 \rangle$ is $\mathrm{GO}_{n-2}^+(q)$ acting irreducibly on $\langle u_1, u_2 \rangle^\perp$. And we can reduce to that case.

(IV) (Symmetric groups) $\mathfrak{S}_{n+2}$, $p | (n+2)$, $n \geq 3$. Let $W = k^m$ be a vector space over a field of characteristic $p > 0$ with basis $e_1, \ldots, e_m$; we assume $m \geq 5$. The symmetric group $\mathfrak{S}_m$ acts on W by permuting the basis elements. The submodule of codimension one

$$\tilde{V} = \langle e_i - e_j ; 1 \leq i < j \leq m \rangle$$

contains the submodule spanned by $v = \sum_{i=1}^m e_i$ if and only if p divides m. We assume this; so $m = pm'$ for some integer m' and we define V to be the quotient module $\tilde{V}/\langle v \rangle$ with dimension $n := m - 2 \geq 3$.

For $1 \leq j \leq m'$ define $v_j := \sum_{i=1}^p e_{p(j-1)+i}$, then $v = \sum_{j=1}^{m'} v_j$ and each $v_j \in \tilde{V}$. Write $\tilde{U_1} = \langle v_1, \ldots, v_{m'} \rangle \subset \tilde{V}$ with image U_1 in V.

We remark that if for $\sigma \in \mathfrak{S}_m$ and i it holds that $\sigma(v_i) \neq v_i$, then since $m \geq 5$ we have $\sigma(v_i) - v_i \notin \langle v \rangle$. So the point-stabilizer of U_1 is the natural subgroup

$$H := \mathfrak{S}_{\{1,2,\ldots,p\}} \times \mathfrak{S}_{\{p+1,p+2,\ldots,2p\}} \times \cdots \times \mathfrak{S}_{\{(m'-1)p+1,\ldots,m'p\}} \simeq \mathfrak{S}_p \times \mathfrak{S}_p \times \cdots \times \mathfrak{S}_p.$$

Suppose p odd or $p = 2$ and m' is even. Then $w := \sum_{i=1}^m ie_i \in \tilde{V}$ and we define $\tilde{U} = \tilde{U}_1 + \langle w \rangle$ with image $U \subset V$. Let $\pi \in H$ such that $\pi(w) = w$ so if $\pi(e_i) = e_j$ then i and j are congruent modulo p, but this is only possible for $\pi \in H$ if π is trivial. And if $\pi(w) - w \in \langle v \rangle$, or equivalently if there is a $c \in k$ such that

$$\pi(w) - w = \sum_{i=1}^m ie_{\pi(i)} - \sum_{i=1}^m ie_i = \sum_{i=1}^m (\pi^{-1}(i) - i)e_i = c\sum_{i=1}^m e_i$$

so $\pi^{-1}(i) = i + c$ for all i. So $c \in \mathbb{F}_p$ and π is a power of

$$\sigma := (1,2,3,\ldots,p)(p+1,p+2,\ldots,2p)\cdots((m'-1)p+1,(m'-1)p+2,\ldots,m).$$

We conclude that the point-stabilizer in G of $\tilde{U}$ is now trivial, but the point-stabilizer in G of U is not, it is generated by σ. On the other hand, the fixed-point space of σ is U. Since the dimension of U is m', its codimension is

$$m - 2 - m' = (p-1)m' - 2 > 1$$

(if $p = 3$ then $m' \geq 2$ and if $p = 2$ then $m' \geq 4$, by our assumptions), so σ is not a pseudo-reflection. So we found a linear subspace whose point-stabilizer is a cyclic p-group not containing a pseudo-reflection. So the direct summand property does not hold.

Let now $p = 2$ and m' odd and we can assume $k = \mathbb{F}_2$. The point-stabilizer H is now an elementary abelian 2-group of order $2^{m'}$ generated by the m' transpositions $(1,2),(3,4),\ldots,(m-1,m)$. These are all transvections and the only pseudo-reflections contained in H. We shall show that its invariant ring is not polynomial. Take as basis $f_1,\ldots,f_{m-2}$ the images in V of the vectors $e_1+e_2, e_2+e_3,\ldots,e_{m-2}+e_{m-1}$. Let $y_1,\ldots,y_m$ be the dual basis. Then the fixed point set V^H is spanned by $f_1, f_3, f_5, \ldots, f_{m-3}$ and the fixed-point set $(V^*)^H$ by $y_2, y_4, \ldots, y_{m-2}$. Suppose $k[V]^H$ is a polynomial ring, and its fundamental degrees $d_1, d_2, \ldots, d_{m-2}$. We must have $|H| = 2^{m'} = d_1 d_2 \ldots d_{m-2}$ and the number of reflections must be $d_1 + d_2 + \cdots + d_{m'-2} - (m'-2)$. Since we have exactly $m'-1$ independent linear invariants the fundamental invariant degrees there must be $m'-2$ quadratic generating invariants and one of degree 4. So the number of reflections is $m'-2+3 = m'+1$. But we have only m' reflections: a contradiction. So we found a linear subspace whose point-stabilizer is an abelian p-group, whose ring of invariants is not a polynomial ring. Therefore the direct summand property does not hold either.

3.2 Exceptional cases

Kemper–Malle [6] made some explicit calculations to show that several exceptional irreducible reflection groups have a linear subspace whose point-stabilizer is not generated by pseudo-reflections or at least its invariant ring is not polynomial. Using MAGMA we checked all these calculations and obtained the more precise result that all exceptional irreducible reflection groups without polynomial ring of invariants have in fact a linear subspace whose point-stabilizer is an abelian p-group with an invariant ring that is not a polynomial ring, and so the direct summand property does not hold. In fact in most cases the point-stabilizer is not even generated by pseudo-reflections.

(i) $W_3(G_{30}) = W_3(H_4)$. According to [6, p. 76] there is a point-stabilizer of a two-dimensional linear subspace that is cyclic of order 3. Since it was already known that the full pseudo-reflection group has no transvections, it follows that the point-stabilizer is not generated by pseudo-reflections. Indeed, we checked that there is a two dimensional linear subspace with point-stabilizer of order three and whose generator has two Jordan blocks of size 2, hence this point-stabilizer is an abelian p-group not generated by pseudo-reflections.

(ii) $W_3(G_{31})$. According to [6, p. 76] there is a point-stabilizer of a linear subspace that is not generated by pseudo-reflections and of order 48, which is not enough for our purposes. We checked that there is a unique orbit of length 960; fix a point v in this orbit and let H be its stabilizer (it is indeed of order 48). Now H has 18 orbits of length 16. We took one of them and took the stabilizer, say $K = H_w$. Then it turned out that K had order 3, generated by a 4×4 matrix whose Jordan form had two blocks of size 2, so K was the point-stabilizer of $\langle v, w \rangle$ and was a p-group not generated by pseudo-reflections.

(iii) $W_5(G_{32})$. According to [6, p. 79] there is a one-dimensional linear subspace with point-stabilizer a cyclic group of order 5. Since the full pseudo-reflection group was known to have no transvections it followed that this point-stabilizer is not generated by pseudo-reflections. Indeed we checked that there is a unique orbit whose stabilizers have order 5 and are generated by a 4×4 matrix whose Jordan form has one block of size 4. So these stabilizers are not generated by pseudo-reflections, not even by 2-reflections. So by Kemper's theorem [5, Theorem 3.4.6], the invariant ring is not even Cohen–Macaulay.

(iv) $W_3(G_{36}) = W_3(E_7)$. According to [6, p. 78] there is a linear subspace whose point-stabilizer of order 24 is not generated by pseudo-reflections. There is a unique orbit of length 672; let N be the stabilizer of one of its points, say v_1. Now this group N has several orbits of length 180, but only one of them has stablizers not generated by pseudo-reflections. Take v_2 in that orbit and take its stabilizer $N_1 = N_{V_1}$ (so its order is 24 and is not generated by pseudo-reflections). Now this group N_1 has orbits whose stabilizers have order 3. We take v_3 in one of those orbits, and take its stabilizer N_2; N_2 is cyclic of order 3, whose Jordan form has two blocks of size 2 and one of size three, so N_2 is the point-stabilizer of $\langle v_1, v_2, v_3 \rangle$ and its generator is not a pseudo-reflection, not even a 2-reflection. So the invariant ring is not even Cohen–Macaulay, [5, Theorem 3.4.6].

(v) $W_3(G_{37}) = W_3(E_8)$. According to [6, p. 78] there is a linear subspace having $W_3(E_7)$ as point-stabilizer, so by the previous case it also has a linear subspace whose point-stabilizer is a cyclic group of order 3, whose Jordan form has two blocks of size 2 and one each of sizes one and three. So the invariant ring is not even Cohen–Macaulay, [5, Theorem 3.4.6].

(vi) $W_5(G_{37}) = W_5(E_8)$. According to [6, p. 78] there is linear subspace whose point-stabilizer is cyclic of order 5. Since it was already known that the pseudo-reflection group does not contain any transvections it follows that this point-stabilizer is not generated by pseudo-reflections. There is a unique orbit whose stabilizers have order 14400; let v_1 be one of its points and H its stabilizer. Now H has a unique orbit with stabilizers of order 5; let v_2 be one of its points and N its stabilizer. Then N is indeed cyclic of order 5 and the Jordan form of its generator has two blocks of size 4, so N is the point-stabilizer of $\langle v_1, v_2 \rangle$. So the invariant ring is not even Cohen–Macaulay, [5, Theorem 3.4.6]. Larger linear subspaces have a point-stabilizer with polynomial ring of invariants.

(vii) $W_2(G_{34}) = 3 \cdot U_4(3) \cdot 2_2$ (it has half the order of G_{34}). According to [6, p. 80] there is an explicit three-dimensional linear subspace whose point-stabilizer K is a 2-group of order 32. The point-stabilizer is abelian and generated by transvections, but we checked using MAGMA that the K-invariant rings of both V and V^* are nonpolynomial (compare [5, p.107]).

As shown in [6, p.73] the remaining exceptional cases are all isomorphic as reflection groups to members of one of the families we already considered above. This finishes the proof of our main theorem.

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Families of Affine Fibrations

Daniel Daigle[1] and Gene Freudenburg[2]

Summary. This paper gives a method of constructing affine fibrations for polynomial rings. The method can be used to construct the examples of $\mathbb{A}^2$-fibrations in dimension 4 due to Bhatwadekar and Dutta (1994) and Vénéreau (2001). The theory also provides an elegant way to prove many of the known results for these examples.

Key words: Locally nilpotent derivations,polynomial rings, stably polynomial rings

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To Gerry Schwarz on his 60th birthday

1 Introduction

One version of the famous Dolgachev–Weisfeiler conjecture asserts that, if $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ is a flat morphism of affine spaces in which every fiber is isomorphic to $\mathbb{A}^{n-m}$, then it is a trivial fibration [6]. In his 2001 thesis, Vénéreau constructed a family of fibrations $\varphi_n : \mathbb{C}^4 \to \mathbb{C}^2$ $(n \geq 1)$ whose status relative to the Dolgachev–Weisfeiler conjecture could not be determined. These examples attracted wide interest in the intervening years, and have been investigated in several papers, for example, [8, 10, 11, 13]. A less well-known example of an affine fibration, due to Bhatwadekar and Dutta, appeared in 1992 [5], and it turns out that this older example is quite similar to the fibration φ_1 of Vénéreau. Bhatwadekar and Dutta asked if their fibration is trivial, a question which remains open.

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In this chapter, we prove the following result, which is a tool for building affine fibrations. Here, it should be noted that statements (ii)–(iv) follow by combining (i) with known results.

Proposition 1.1. *Let k be a field, and let $B = k[x, y_1, \dots, y_r, z_1, \dots, z_m] = k^{[r+m+1]}$. Suppose that $v_1, \dots, v_m \in B$ are such that*

$$k[x, y_1, \dots, y_r, v_1, \dots, v_m]_x = B_x.$$

Pick any $\phi_1, \dots, \phi_r \in k[x, v_1, \dots, v_m]$, and define $f_i = y_i + x\phi_i$ ($1 \le i \le r$) and $A = k[x, f_1, \dots, f_r]$. Then:

(i) *B is an $\mathbb{A}^m$-fibration over A.*
(ii) *B is a stably polynomial algebra over A, i.e., $B^{[n]} = A^{[m+n]}$ for some $n \in \mathbb{N}$.*

If moreover $m = 2$ and $\mathrm{char}(k) = 0$:

(iii) $\ker D = A^{[1]} = k^{[r+2]}$ *for every nonzero $D \in \mathrm{LND}_A(B)$.*
(iv) *If I is an ideal of A such that A/I is a PID, then $B/IB = (A/I)^{[2]}$.*

As a consequence, we obtain a family of $\mathbb{A}^2$-fibrations in dimension 4 to which the examples of Vénéreau and Bhatwadekar-Dutta belong. In addition, many of the known results about these examples follow from the more general theory, for example, that the fibrations are stably trivial.

An excellent summary of known results for affine fibrations is found in [5], and the reader is referred to this article for further background.

2 Preliminaries

2.1 Notation and definitions

Throughout, k denotes a field, and rings are assumed to be commutative. For any field F, affine n-space over F is denoted by $\mathbb{A}^n_F$, or simply $\mathbb{A}^n$ when the ground field is understood. For a ring A and positive integer n, $A^{[n]}$ is the polynomial ring in n variables over A. If B is a domain, and A is a subring of B, then $\mathrm{tr.deg}_A(B)$ denotes the transcendence degree of the field $\mathrm{frac}(B)$ over $\mathrm{frac}(A)$. If $x \in A$ is nonzero, then A_x is the localization of A at the set $\{x^n | n \in \mathbb{N}\}$. Likewise, if $\mathfrak{p}$ is a prime ideal of A, then $A_{\mathfrak{p}}$ is the localization of A determined by $\mathfrak{p}$, and $\kappa(\mathfrak{p})$ denotes the field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Here is the definition of the main object under consideration (following [5]).

Definition. Let B be an algebra over a ring A. Then B is an *$\mathbb{A}^m$-fibration over A* if and only if B is finitely generated as an A-algebra, flat as an A-module, and for every $\mathfrak{p} \in \mathrm{Spec}\,A$, $\kappa(\mathfrak{p}) \otimes_A B \cong \kappa(\mathfrak{p})^{[m]}$.

Geometrically, in this case if $X = \mathrm{Spec}\,B$, $Y = \mathrm{Spec}\,A$, and $\varphi : X \to Y$ is the morphism induced by the inclusion $A \to B$, then φ will be called an $\mathbb{A}^m$-fibration of X over Y. For convenience, we also introduce the following terminology.

Definition. The ring B is an $\mathbb{A}^m$*-prefibration* over the subring A if and only if for every $\mathfrak{p} \in \operatorname{Spec} A$, $\kappa(\mathfrak{p}) \otimes_A B \cong \kappa(\mathfrak{p})^{[m]}$.

Note that if B is an $\mathbb{A}^m$-fibration over A then B is faithfully flat over A (by flatness and surjectivity of $\operatorname{Spec} B \to \operatorname{Spec} A$) so the homomorphism $A \to B$ is injective. Thus, every fibration is a prefibration. The converse is valid in certain situations, for instance, we observe the following.

Lemma 2.1. *Let $A \subset B$ be polynomial rings over a field. If B is an $\mathbb{A}^m$-prefibration over A, then it is an $\mathbb{A}^m$-fibration over A.*

Proof. We have $A = k^{[n]}$ and $B = k^{[n+m]}$ for some field k and some $n \in \mathbb{N}$.

If k is algebraically closed then $\operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism of nonsingular algebraic varieties such that every fiber has dimension equal to $\dim B - \dim A$, so B is flat over A and B is an $\mathbb{A}^m$-fibration over A.

For the general case, let $\bar{k}$ be the algebraic closure of k, $\bar{A} = \bar{k} \otimes_k A = \bar{k}^{[n]}$, and $\bar{B} = \bar{k} \otimes_k B = \bar{k}^{[n+m]}$. One can see that $\bar{B}$ is an $\mathbb{A}^m$-prefibration over $\bar{A}$; by the first paragraph, it follows that $\bar{B}$ is an $\mathbb{A}^m$-fibration over $\bar{A}$ so in particular $\bar{B}$ is faithfully flat over $\bar{A}$.

$$\begin{array}{ccccc} \bar{k} & \to & \bar{A} & \to & \bar{B} \\ \uparrow & & \uparrow & & \uparrow \\ k & \to & A & \to & B\,. \end{array}$$

It follows that $\bar{B}$ is faithfully flat over B and also over A; consequently B is faithfully flat over A (descent property). Thus B is an $\mathbb{A}^m$-fibration over A. $\square$

Suppose B is an affine fibration over A (i.e., B is an $\mathbb{A}^m$-fibration for some nonnegative integer m). This fibration is said to be *trivial* if $B = A^{[m]}$, i.e., B is a polynomial algebra over A. Likewise, the fibration is *stably trivial* if $B^{[n]} = A^{[m+n]}$ for some $n \geq 0$, and we say that B is a stably polynomial algebra over A.

For $i = 1, 2$, suppose B_i is an affine fibration over A_i, with inclusion map $j_i : A_i \to B_i$. These two fibrations are said to be *equivalent* if there exist isomorphisms $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ such that $\beta j_1 = j_2 \alpha$.

2.2 *Some known results on affine fibrations*

This section lays out certain known results on affine fibrations which are needed in the rest of the chapter. The module of Kähler differentials of B over A is denoted by $\Omega_{B/A}$.

Theorem 2.1 (Asanuma, [2], Theorem 3.4). *Let A be a noetherian ring and B an $\mathbb{A}^m$-fibration over A. Then $\Omega_{B/A}$ is a projective B-module of rank m, and there exists $n \geq 0$ such that $A \subset B \subset A^{[n]}$. If $\Omega_{B/A}$ is a free B-module, then $B^{[n]} = A^{[m+n]}$.*

In view of the Quillen–Suslin Theorem, there follows this corollary.

Corollary 2.1. *Consider $A \subset B$ where A is a Noetherian ring and B is a polynomial ring over a field. If B is an $\mathbb{A}^m$-fibration over A, then $B^{[n]} = A^{[m+n]}$ for some $n \geq 0$.*

It was proved by Hamann [9] that if A is a noetherian ring containing $\mathbb{Q}$ then the conditions $A \subset B$ and $B^{[n]} = A^{[n+1]}$ imply $B = A^{[1]}$. Combining this with the above result of Asanuma gives the following theorem.

Theorem 2.2 ([5], Theorem 3.4). *Let A be a noetherian ring containing a field of characteristic zero, and let B be an $\mathbb{A}^1$-fibration over A. If $\Omega_{B/A}$ is a free B-module, then $B = A^{[1]}$.*

From the results of Sathaye [16] and Bass, Connell, and Wright [4], one derives the following.

Theorem 2.3 ([5], Corollary 4.8). *Let A be a PID containing a field of characteristic zero. If B is an $\mathbb{A}^2$-fibration over A, then $B = A^{[2]}$.*

Corollary 2.2. *Suppose that B is an $\mathbb{A}^2$-fibration over a ring A which contains $\mathbb{Q}$. If I is an ideal of A such that A/I is a PID, then $B/IB = (A/I)^{[2]}$.*

Proof. As $B/IB = A/I \otimes_A B$, the ring homomorphism $A/I \to B/IB$ makes B/IB an $\mathbb{A}^2$-fibration over A/I, so the desired conclusion follows from Theorem 2.3. $\square$

There are many other papers that discuss affine fibrations. For example, results concerning morphisms with $\mathbb{A}^1$-fibers are due to Kambayashi and Miyanishi [14] and to Kambayashi and Wright [15]. Likewise, Asanuma and Bhatwadekar [3] and Kaliman and Zaidenberg [12] give important facts about $\mathbb{A}^2$-fibrations. For an overview of affine fibrations, the reader is referred to the aforementioned survey article [5].

For affine spaces, the first case where few results are known is the case of $\mathbb{A}^2$-fibrations $\mathbb{A}^4 \to \mathbb{A}^2$, and consequently these receive special attention.

2.3 Locally nilpotent derivations

By a *locally nilpotent derivation* of a commutative ring B of characteristic 0, we mean a derivation $D : B \to B$ such that, to each $b \in B$, there is a positive integer n with $D^n b = 0$. The kernel of D is denoted $\ker D$. Let $D : B \to B$ be a nonzero locally nilpotent derivation, where B is an integral domain of characteristic zero. If $K = \ker D$, then it is known that K is factorially closed in B, $B^* \subset K$, and $\mathrm{tr.deg}_K B = 1$. An element $s \in B$ is a slice for D if $Ds = 1$, and in this case $B = K[s]$. The notation $\mathrm{LND}(B)$ denotes the set of all locally nilpotent derivations of B. Likewise, if A is a subring of B, then $\mathrm{LND}_A(B)$ denotes the set of locally nilpotent derivations D of B with $D(A) = 0$. If $D \in \mathrm{LND}(B)$, then $\exp D$ is an automorphism of B. A reference for locally nilpotent derivations is [7].

An important fact about locally nilpotent derivations which we need is the following.

Proposition 2.1. *Let B be a UFD of characteristic zero, and let $A \subset B$ be a subring such that $B^{[n]} = A^{[n+2]}$ for some $n \geq 0$. Then* $\ker D = A^{[1]}$ *for every nonzero* $D \in \mathrm{LND}_A(B)$.

Proof. From $B^{[n]} = A^{[n+2]}$, it follows that A is a UFD and that $\mathrm{tr.deg}_A(B) = 2$.

Let $D \in \mathrm{LND}_A(B)$ be given, $D \neq 0$. As $\ker D$ is factorially closed in B, it is a UFD. We also have $\mathrm{tr.deg}_{\ker D}(B) = 1$, so

$$A \subset \ker D \subset A^{[n+2]},$$

where A and $\ker D$ are UFDs and $\mathrm{tr.deg}_A(\ker D) = 1$. It now follows from a classical result of Abhyankar, Eakin, and Heinzer that $\ker D = A^{[1]}$ ([1], Theorem 4.1). $\square$

By *Proposition 2.1* and *Corollary 2.1*, we obtain the following.

Corollary 2.3. *Let A and B be polynomial rings over a field k of characteristic* 0 *such that $A \subset B$, and B is an $\mathbb{A}^2$-fibration over A. Then:*

(i) $\ker D = A^{[1]}$ *for every nonzero* $D \in \mathrm{LND}_A(B)$.
(ii) $B = A^{[2]}$ *if and only if there exists* $D \in \mathrm{LND}_A(B)$ *with a slice.*

Remark 2.1. The corollary above is of interest since when B is a polynomial ring, the kernel of D is not a polynomial ring for most $D \in \mathrm{LND}(B)$. So the fact that $\ker D = A^{[1]} = k^{[d-1]}$, $d = \dim_k B$, for all nonzero $D \in \mathrm{LND}_A(B)$ when A is a polynomial ring means that these subrings are quite special.

3 A criterion for affine fibrations

Lemma 3.1. *Consider a triple (S, B, x) and an $m \in \mathbb{N}$ satisfying:*

(i) *B is a domain, S is a subring of B, and $x \in B$ is transcendental over S;*
(ii) *$S \cap xB = 0$;*
(iii) *B_x is an $\mathbb{A}^m$-prefibration over A_x, where $A = S[x]$;*
(iv) *$\bar{B}$ is an $\mathbb{A}^m$-prefibration over $\bar{A}$, where $\bar{B} = B/xB$, and $\bar{A} \subseteq \bar{B}$ is the image of A via the canonical epimorphism $B \to \bar{B}$.*

Then B is an $\mathbb{A}^m$-prefibration over A.

Proof. Let $\mathfrak{p} \in \mathrm{Spec}\,A$ and consider the fiber of $f : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$ over $\mathfrak{p}$.

If $x \notin \mathfrak{p}$, let $\mathfrak{q} = \mathfrak{p}A_x \in \mathrm{Spec}(A_x)$. Then the fiber of f over $\mathfrak{p}$ is the same thing as that of $\mathrm{Spec}(B_x) \to \mathrm{Spec}(A_x)$ over $\mathfrak{q}$, and by (iii) this is $\mathbb{A}^m_{\kappa(\mathfrak{q})}$ $(= \mathbb{A}^m_{\kappa(\mathfrak{p})})$.

If $x \in \mathfrak{p}$, let $\mathfrak{q} = \mathfrak{p}\bar{A} \in \mathrm{Spec}\,\bar{A}$. As (ii) implies $A \cap xB = xA$, we may identify $A/xA \to B/xB = \bar{B}$ with $\bar{A} \to \bar{B}$ and consequently the fiber of f over $\mathfrak{p}$ is the same thing as that of $\mathrm{Spec}\,\bar{B} \to \mathrm{Spec}\,\bar{A}$ over $\mathfrak{q}$, which is $\mathbb{A}^m_{\kappa(\mathfrak{q})}$ $(= \mathbb{A}^m_{\kappa(\mathfrak{p})})$ by (iv). $\square$

Remark 3.1. If $B_x = A_x^{[m]}$ and $\bar{B} = \bar{A}^{[m]}$ then conditions (iii) and (iv) are satisfied.

Proof of Proposition 1.1.

Let $S = k[f_1,\dots,f_r]$ and note that $A = S[x] = S^{[1]}$. We have $S \cap xB = 0$ and

$$B_x = A_x[v_1,\dots,v_m] = A_x^{[m]} \quad \text{and} \quad \bar{B} = \bar{A}[z_1,\dots,z_m] = \bar{A}^{[m]},$$

so (S,B,x) satisfies the hypothesis of Lemma 3.1. Consequently B is an $\mathbb{A}^m$-prefibration over A. As A,B are polynomial rings over a field, Lemma 2.1 implies that B is an $\mathbb{A}^m$-fibration over A. This proves (i).

Part (ii) follows from Corollary 2.1; part (iii) follows from Corollary 2.3; and part (iv) follows from Corollary 2.2. □

Remark 3.2. The example of Vénéreau (details of which are discussed below) uses a subring $A = \mathbb{C}[x,f] \subset B = \mathbb{C}[x,y,z,u] = \mathbb{C}^{[4]}$ of the form hypothesized in the proposition. Vénéreau proved that for every $\gamma(x) \in \mathbb{C}[x]$, $B/(f-\gamma(x)) = \mathbb{C}[x]^{[2]}$. Item (iv) in the proposition is thus a generalization of Vénéreau's result.

Construction of examples.

As above, let $B = k[x,y_1,\dots,y_r,z_1,\dots,z_m] = k^{[r+m+1]}$. In addition, let $R = k[x,y_1,\dots,y_r]$. Here are two ways to choose $v_1,\dots,v_m \in B$ satisfying

$$R[v_1,\dots,v_m]_x = B_x\ .$$

1. Let M be an $m \times m$ matrix with entries in R such that $\det M = x^n$ for some non-negative integer n. In particular, $M \in GL_m(R_x)$. Define $v_1,\dots,v_m \in B$ by

$$\begin{bmatrix} v_1 & \vdots & v_m \end{bmatrix} = M \begin{bmatrix} z_1 & \vdots & z_m \end{bmatrix}\ .$$

   Then $B_x = R_x[z_1,\dots,z_m] = R_x[v_1,\dots,v_m]$.

2. For this construction, we need to assume that k has characteristic 0. Choose $D \in \mathrm{LND}_R(B_x)$ and consider the R_x-automorphism $\exp(D) : B_x \to B_x$, which we abbreviate $\alpha : B_x \to B_x$. Choose $i_1,\dots,i_m \in \mathbb{Z}$ such that $\alpha(x^{i_j} z_j) \in B$ for all $j = 1,\dots,m$ and define

$$v_j = \alpha(x^{i_j} z_j) \quad (1 \le j \le m).$$

   Then

$$B_x = R[x^{i_1} z_1,\dots,x^{i_m} z_m]_x = R[\alpha(x^{i_1} z_1),\dots,\alpha(x^{i_m} z_m)]_x = R[v_1,\dots,v_m]_x\ .$$

4 Dimension four

In this section, assume k is any field of characteristic zero. We consider a family of $\mathbb{A}^2$-fibrations $\mathbb{A}^4 \to \mathbb{A}^2$ which includes the examples of Bhatwdekar–Dutta and Vénéreau.

Let $B = k[x,y,z,u]$ be a polynomial ring in four variables. We consider the set of polynomials in B of the form $p = yu + \lambda(x,z)$, where $\lambda = z^2 + r(x)z + s(x)$ for some $r,s \in k[x]$. Given $p = yu + \lambda(x,z)$ of this form, define $\theta \in \mathrm{LND}(B_x)$ by $\theta x = \theta y = 0$, $\theta z = x^{-1}y$, and $\theta u = -x^{-1}\lambda_z$, noting that $\theta p = 0$. Set

$$v = \exp(p\theta)(xz) = xz + yp \quad \text{and} \quad w = \exp(p\theta)(x^2u) = x^2u - xp\lambda_z - yp^2 .$$

In addition, given $n \geq 1$, define $f_n \in B$ by $f_n = y + x^n v$. Note that $f_n, v, w \in B$, and these depend on our choice of p. Let $\varphi_n(p) : \mathbb{A}^4 \to \mathbb{A}^2$ denote the morphism defined by the inclusion $k[x, f_n] \subset B$. It is easy to see that the hypothesis of Proposition 1.1 is satisfied (with $r = 1, m = 2$), so $\varphi_n(p)$ is an $\mathbb{A}^2$-fibration over $\mathbb{A}^2$ and more precisely:

Corollary 4.1. *For any choice of p and n as above, if $A = k[x, f_n]$ then:*

(i) *B is an $\mathbb{A}^2$-fibration over A.*
(ii) *$B^{[s]} = A^{[s+2]}$ for some $s \in \mathbb{N}$.*
(iii) $\ker D = A^{[1]} = k^{[3]}$ *for every nonzero $D \in \mathrm{LND}_A(B)$.*
(iv) *If $a \in A$ is such that $A/aA = k^{[1]}$, then $B/aB = (A/aA)^{[2]} = k^{[3]}$.*

In addition, we have the following lemma.

Lemma 4.1. *$\varphi_n(p)$ is trivial for each $n \geq 3$.*

Proof. Assume $n \geq 3$, and define the derivation d of B by

$$d = \frac{\partial(x,\cdot,v,w)}{\partial(x,y,z,u)} .$$

Since $k(x)[y,v,w] = k(x)[y,z,u]$, it follows that d is locally nilpotent. And since $dx = dv = 0$, we have that $x^{n-3}vd$ is also locally nilpotent. In addition, we have by direct calculation that $dy = v_z w_u - v_u w_z = x^3$. Therefore, if $\beta = \exp(x^{n-3}vd)$, then β is an automorphism of B for which $\beta(x) = x$ and $\beta(y) = y + x^{n-3}vd(y) = y + x^n v = f_n$. Therefore, $B = k[x, f_n]^{[2]}$ when $n \geq 3$. □

Example 1. $p = yu + z^2 + z$. This choice of p yields the 1992 example of Bhatwadekar and Dutta [5] (Example 4.13). In particular, the authors work over a DVR (R,π) containing $\mathbb{Q}$, and they define $F \in R[X,Y,Z] = R^{[3]}$ by

$$F = (\pi Y^2)Z + Y + \pi Y(X + X^2) + \pi^2 X .$$

By the substitutions

$$\pi \to x\, ,\; X \to z\, ,\; Y \to y\, ,\; Z \to u$$

we see that F becomes exactly the polynomial f_1 for this p, namely,

$$f_1 = y + x(xz + y(yu + z^2 + z)) \ .$$

The authors also list rational cogenerators G and H, which under these substitutions become $G \to v$ and $H \to w$. Here, R should be viewed as the localization of $k[x]$ at the prime ideal defined by x. The authors ask (Question 4.14) if $R[X,Y,Z] = R[F]^{[2]}$, and this is equivalent to the question whether $\varphi_1(p)$ is trivial. This question is still open.

Example 2. $p = yu + z^2$. This choice of p yields the 2001 examples of Vénéreau [17]. The polynomials f_n defined using $p = yu + z^2$ are called the *Vénéreau polynomials*. Vénéreau showed that $\varphi_n(p)$ is trivial for $n \geq 3$, which is equivalent to the condition $B = k[x, f_n]^{[2]}$. Later, the second author showed that $\varphi_n(p)$ is stably trivial for all $n \geq 1$ [8]. It remains an open question whether $\varphi_1(p)$ or $\varphi_2(p)$ is trivial. However, Vénéreau proved for all $\gamma(x) \in k[x]$ that the quotient $B/(f_1 - \gamma(x))$ is x-isomorphic to $k[x]^{[2]}$. In particular, f_1 defines a hyperplane in $\mathbb{A}^4$, but the question as to whether f_1 (or f_2) is a variable of B remains open.

Question 1. Are the fibrations $\varphi_1(p) : \mathbb{A}^4 \to \mathbb{A}^2$ equivalent for $p = yu + z^2$ and $p = yu + z^2 + z$?

5 A remark on stable variables

Let $B = k^{[n]}$ for a field k. If $f \in B$ is a variable of $B^{[q]}$, where $q \in \mathbb{N}$, we say that f is a *q-stable variable* of B (or simply a *stable variable* of B). It is not known whether every stable variable is a variable.

Example 3. Let $f \in B = k^{[n]}$, where k is of characteristic zero. By Proposition 3.20 of [7], if there exists $D \in \mathrm{LND}(B)$ such that $D(f) = 1$ then f is a 1-stable variable of B. Using this fact, one can show that if every 1-stable variable is a variable then the cancellation problem for affine spaces has an affirmative answer.

Example 4. Consider the situation of Example 2: Let $B = k[x,y,z,u] = k^{[4]}$, $p = yu + z^2$ and $n \in \{1,2\}$. Define $f_n \in B$ as in Section 4 (Vénéreau polynomials) and let $A_n = k[x, f_n]$. By Corollary 4.1 we have $B^{[q]} = A_n^{[q+2]}$ for some q, but in fact the second author showed in [8] that $B^{[1]} = A_n^{[3]}$. It follows in particular that f_n is a variable of $B^{[1]} = k^{[5]}$, i.e., f_n is a 1-stable variable of B. It is not known whether f_n is a variable of B, or whether there exists $D \in \mathrm{LND}(B)$ satisfying $D(f_n) = 1$.

We now explain how stable variables can be used to construct rings which are of interest in relation to the cancellation problem.

We begin with a general observation. Let k be a field and suppose that $R \subset S$ are k-algebras satisfying $S^{[q]} = R^{[q+m]}$. If $R \to R'$ is any k-homomorphism and if we define $S' = R' \otimes_R S$, then $S'^{[q]} = R'^{[q+m]}$; so in the case where $R' = k^{[r]}$, we have $S'^{[q]} = k^{[q+m+r]}$ and hence S' is a potential counterexample to the cancellation problem.

Now suppose that f is a q-stable variable of $B = k^{[n]}$ and let $R = k[f]$. As $B^{[q]} = R^{[n+q-1]}$, it follows that if $R \to R'$ is any k-homomorphism such that $R' = k^{[r]}$ for some r, then the algebra $B' = R' \otimes_R B$ satisfies $B'^{[q]} = k^{[n+q+r-1]}$.

For instance let $d \geq 2$ be an integer, W an indeterminate, $R' = k[W] = k^{[1]}$ and let $R \to R'$ be the k-homomorphism which maps f to W^d. This gives $R' \otimes_R B = B\left[\sqrt[d]{f}\right]$, so we obtain the following observation.

For a field k, suppose that f is a q-stable variable of $B = k^{[n]}$ and define $B_d = B\left[\sqrt[d]{f}\right]$. Then $B_d^{[q]} = k^{[n+q]}$ for every positive integer d.

In particular let $f_n \in B = k^{[4]}$ be as in Example 2 and let $B_d = B\left[\sqrt[d]{f_n}\right]$ (with $n \in \{1,2\}$ and $d \geq 2$); then $B_d^{[1]} = k^{[5]}$ but we don't know whether B_d is $k^{[4]}$.

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On the Depth of Modular Invariant Rings for the Groups $C_p \times C_p$

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Summary. Let G be a finite group, k a field of characteristic p and V a finite dimensional kG-module. Let $R := \mathrm{Sym}(V^*)$, the symmetric algebra over the dual space V^*, with G acting by graded algebra automorphisms. Then it is known that the depth of the invariant ring R^G is at least $\min\{\dim(V), \dim(V^P) + cc_G(R) + 1\}$. A module V for which the depth of R^G attains this lower bound was called flat by Fleischmann, Kemper and Shank [13]. In this paper some of the ideas in [13] are further developed and applied to certain representations of $C_p \times C_p$, generating many new examples of flat modules. We introduce the useful notion of "strongly flat" modules, classifying them for the group $C_2 \times C_2$, as well as determining the depth of R^G for any indecomposable modular representation of $C_2 \times C_2$.

Key words: Modular invariant theory, modular representation theory, depth, transfer, cohomology

Mathematics Subject Classification (2000): 13A50

1 Introduction

Let G be a finite group acting linearly on a finite-dimensional vector space V over a field k. We assume that the characteristic of k is a prime number p which divides $|G|$, so that V is a modular representation of G. Define $R := S(V^*) := \mathrm{Sym}(V^*)$ to be the symmetric algebra over the dual space V^* with k-basis $x_1, \ldots, x_n$. Then R is isomorphic to the polynomial ring $k[x_1, \ldots, x_n]$, on which G acts naturally by graded

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algebra automorphisms, extending the linear action of G on V^*. We usually adopt the convention of writing V as a left kG–module and V^* as a right module with $\lambda g = \lambda \circ g$ for $\lambda \in V^*$ and $g \in G$. The *ring of invariants* or *invariant ring* R^G is defined as follows.

Definition 1.1.
$$R^G := \{r \in R : rg = r\, \forall g \in G\}.$$

If $\mathrm{char}(k)$ divides $|G|$, we call R^G a *modular* ring of invariants, if $\mathrm{char}(k)$ does not divide $|G|$, R^G will be called *nonmodular*.

For example, if $G = \Sigma_n$ and V is the standard permutation representation, then $R = k[e_1, \ldots, e_n]$ is a polynomial ring generated by the elementary symmetric polynomials $e_1, \ldots, e_n$. In the general case, R is always a finitely generated k-algebra, due to a classical result of Emmy Noether. However, it will not necessarily be a polynomial ring. It is well known that in the nonmodular case, R^G is a polynomial ring if and only if the image of G in $\mathrm{GL}(V)$ is generated by reflections. In the modular case, a classification of representations with R^G being a polynomial ring is not known, except for p-groups and representations over the prime field $\mathbb{F}_p$. It is known however, that R^G being a polynomial ring is a rare condition. For these statements and more general background on invariant theory we refer the reader to the excellent monographs [2], [6] and [18].

From "Noether normalisation", another classical result of Emmy Noether, we know that there is a subset of homogenous elements $\{y_1, \ldots, y_n\} \subseteq R^G$, generating a polynomial subring $\mathscr{P} := k[y_1, \ldots, y_n] \leq R^G$, such that R^G is a finitely generated $\mathscr{P}$-module. Such a subset $\{y_1, \ldots, y_n\}$ is called a homogeneous system of parameters (hsop). If R^G is a free module over $\mathscr{P}$, then R^G is a Cohen–Macaulay ring. The technical definition of Cohen–Macaulay rings is in terms of *regular sequences* which we will now give for graded connected k-algebras, i.e. finitely generated $\mathbb{N}_0$ graded k-algebras whose degree zero component is k. Let $A = \oplus_{i \in \mathbb{N}_0} A_i$ be such a graded connected k-algebra with $A^+ := \oplus_{i>0} A_i$; furthermore let $J \subseteq A^+$ be a homogeneous ideal and M be a graded A-module. A sequence of homogeneous elements $(a_1, \ldots, a_k)$ with $a_i \in J$ is called M-regular, if for every $i = 1, \ldots, k$ the multiplication by a_i induces an injective map on the quotient ring $M/(a_1, \ldots, a_{i-1})M$. It is known that all maximal M-regular sequences in J have the same length $\mathrm{grade}(J, M)$, called the grade of J on M and one defines

$$\mathrm{depth}(A) := \mathrm{grade}(A^+, A).$$

It is a theorem in commutative algebra that A is Cohen–Macaulay if and only if $\mathrm{depth}(A) = \mathrm{Dim}(A)$, where $\mathrm{Dim}(A)$ denotes the Krull dimension of the ring A.

Eagon and Hochster [7] showed that every nonmodular invariant ring of a finite group is Cohen–Macaulay, but this does not remain true in the modular case. Determining the depth of modular rings of invariants remains an important problem but often is a very difficult challenge. The well known Auslander–Buchsbaum formula tells us that the difference between the depth and Krull dimension of R^G is equal to its projective dimension as a module over an hsop, so the depth can be viewed as a measure of structural complexity of R^G, telling us "how close" it is to being a free module over a polynomial ring.

In this chapter we are interested in calculating the depth of certain types of rings of invariants. For more details and results regarding depth, the reader is referred to [5]. Thanks to Ellingsrud and Skjelbred [9], we do have a lower bound for the depth of an invariant ring. Their result was strengthened in [13] to the following.

Theorem 1.1. *Let G be a finite group and k a field of characteristic p. Let P be a Sylow-p-subgroup and let V be a left kG-module. Let R denote the symmetric algebra $S(V^*)$ which has a natural right-module structure. Then*

$$\operatorname{depth}(R^G) \geq \min\{\dim(V), \dim(V^P) + cc_G(R) + 1\},$$

where V^P denotes the fixed-point space of P on V and $cc_G(R)$ is called the cohomological connectivity *and defined as* $\min\{i > 0 : H^i(G,R) \neq 0\}$.

A representation V for which this inequality is in fact an equality is called *flat*. Since for $R = S(V^*)$ the Krull–dimensions $\operatorname{Dim}(R^G)$ and $\operatorname{Dim}(R)$ coincide, both being equal to $\dim V$, it is clear that if $\dim(V^P) + cc_G(R) + 1 \geq \dim(V)$, then V is flat and R^G is Cohen–Macaulay. Such a representation is called *trivially flat* in this chapter.

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2 Flatness and strong flatness

In [13], the authors not only proved the inequality in Theorem 1.1, but also obtained necessary and sufficient geometric conditions for it actually to be an equality. We briefly sketch the results there.

Theorem 2.1. *Let V be a kG-module, $R := S(V^*)$, and $m := cc_G(R)$ where we assume $m <$ $\operatorname{codim}(V^P) - 1$. Then the following are equivalent:*

1. $\mathfrak{i}_G = \sqrt{\operatorname{Ann}_{R^G}(\alpha)}$ *for some* $0 \neq \alpha \in H^m(G,R)$*;*
2. $\operatorname{grade}(\mathfrak{i}_G, H^m(G,R)) = 0$*;*
3. $\operatorname{grade}(\mathfrak{i}_G, R^G) = m+1$*;*
4. *V is flat, i.e* $\operatorname{depth}(R^G) := \operatorname{grade}(R^G_+, R^G) = \dim(V^P) + m + 1$.

In the above, $\mathfrak{i}_G$ is the prime ideal of R^G defined as $\mathfrak{i}_G := R^G \cap \mathfrak{i}_P$, where $\mathfrak{i}_P :=$ $\mathfrak{I}(V^P)$ is the prime ideal of R generated by all linear forms in V^* which vanish on the P-fixed-point space V^P. The equivalence *3.* $\Leftrightarrow$ *4.* follows from the formula

$$\operatorname{depth}(R^G) = \dim(R/\mathfrak{i}) + \operatorname{grade}(\mathfrak{i}_G, R^G)$$

proved in [14]. The implication (2) $\Leftrightarrow$ (3) is more subtle, and follows from studying a certain spectral sequence. For details, see [13], Chapter 7.

We want to use condition (1). to identify flat representations. In our attempt to do this, we make a further simplification. Consider the following ideal of R^G,

$$I := \sum_{N<P} \mathrm{Tr}_N^G(R^N),$$

where the sum runs over all maximal subgroups of P. Here Tr_N^G denotes the transfer homomorphism $\mathrm{Tr}_N^G : R^N \to R^G$

$$\mathrm{Tr}_N^G r = \sum_{g \in S} rg,$$

where S is a set of right coset representatives for N in G.

The ideal I above is called the *relative transfer ideal*, and has been studied widely in connection with modular invariant theory. For example, it is known that the quotient ring $R^G/\sqrt{I}$ is always Cohen–Macaulay ([12]). So there is a sense in which I contains the extra complexity in a modular ring of invariants. It turns out (see [13], Chapter 3) that $\mathfrak{i}_G = \sqrt{I}$. Since for any ideal J and ring A one has $\mathrm{grade}(J,A) = \mathrm{grade}(\sqrt{J},A)$ (see [5]), we now have necessary and sufficient conditions for flatness which we can hope to exploit.

Theorem 2.2. *Let V be a kG-module and $R := S(V^*)$. Let $m := cc_G(R)$. Then the following are equivalent.*

1. *V is flat.*
2. *Either $m+1 \geq \mathrm{codim}(V^P)$ or there exists $0 \neq \alpha \in H^m(G,R)$ which is annihilated by every element of the relative transfer ideal.*

In practice it is difficult to find such an α, unless one knows explicitly the action of R^G on $H^m(G,R)$. Since finding the indecomposable summands of R as a kG-module is an unsolved problem, it is clear that this method is not really tractable. For this reason, we introduce a narrower class of representations.

Theorem 2.3. *Let V be a finite kG-module, and suppose that $0 \neq \tau \in H^m(G,R)$, where $m = cc_G(R)$ is a cohomology class such that*

$$\mathrm{res}_N^P(\tau) = 0$$

for each maximal subgroup $N < P$. Then V is flat.

Remark. This result is based on [13], Theorem 2.6.

Proof. Note that $H^m(G,R)$ is a direct summand of $H^m(P,R)$, with the restriction map res_P^G an injective map $H^m(G,R) \to H^m(P,R)$. So we may identify $H^m(G,R)$ with the image of res_P^G, and τ satisfying the conditions of the theorem satisfies $\mathrm{res}_N^G(\tau) = 0$ for all $N < P$. By [17], Lemma 1.3, elements in the image of the relative transfer Tr_N^G annihilate those in the kernel of the restriction $\ker \mathrm{res}_N^G$. Thus elements $\tau \in H^m(G,R)$ satisfying the conditions of the theorem are annihilated by every element of I. The result now follows from Theorem 2.2.

If G is a p-group with $p = \mathrm{char}(k)$, then $H^1(G,k) \neq 0$ and, since $k \cong R_0$, we get $H^1(G,R) \neq 0$, so $cc_G(R) = 1$. We call a representation V *strongly flat*, if $cc_G(R) = 1$ and if there exists a cohomology class $0 \neq \tau \in H^1(G,R)$ with $\mathrm{res}^P_N(\tau) = 0$ for every maximal subgroup $N < P$. Note that $H^1(G,R)$ has a direct summand isomorphic to $H^1(G,V^*)$; if we can find a suitable $0 \neq \tau$ satisfying the above condition in this direct summand, then the representation is called *linearly flat*.

As the name suggests, not every representation which is flat is strongly so; we show this by example in section 5. However, strong flatness is a sufficiently general notion to give us plenty of new examples of flat representations. It also has the following two desirable properties not shared by mere flatness.

Lemma 2.1. *Strong flatness has the following properties.*

1. *A representation V is strongly flat if and only if there is a direct summand W^* (as a G-module) of $S(V^*)$ for which W is linearly flat.*
2. *If U, V are kG-modules and U is strongly flat, then so is $U \oplus V$.*

Proof. The first property is a consequence of the splitting

$$H^m(G,R) = \bigoplus_{i \geq 0} (H^m(G,S^i(V^*)))$$

To prove the second property, use the formula $S(U \oplus V) = S(U) \otimes S(V)$. Now $k = S^0(V)$ is a direct summand of $S(V)$, and so $S(U) \otimes k \cong S(U)$ is a direct summand of $S(U \oplus V)$. The result now follows from the first property.

If the trivial kG-module k is strongly flat, then every kG-module is strongly flat, because for each kG-module V we have that $R_0 = k$. From this we obtain immediately the result of Ellingsrud and Skjelbred that every modular representation of a cyclic group of prime order is flat; in this case, there are no nontrivial subgroups to which to restrict, and k is strongly flat provided $H^1(G,k) \neq 0$, which is always true.

3 A sufficient condition for strong flatness

Now that we have a condition on modules which implies flatness, we aim to apply this to some representations. We need the following basic facts from group cohomology.

Theorem 3.1. *Let $C = \langle \sigma \rangle$ be a cyclic group of order p and X a kC-module, then*

$$H^i(C,X) = \begin{cases} X^C / \mathrm{Tr}_1^C(X) & \text{if } i > 0 \text{ is even,} \\ \ker(\mathrm{Tr}_1^C|_X)/(\sigma - 1)X & \text{if } i \text{ is odd.} \end{cases}$$

Proof. See [10], page 6.

Let P be a (noncyclic) p-group and W a right kP-module over a field k of characteristic p. We need to identify nonzero elements in $H^1(P,W)$ which restrict to zero for all maximal subgroups. Let $N \neq M$ denote two maximal subgroups of P. Note that N and M are normal in P. The inflation map gives an injection from $H^1(P/N, W^N)$ to $H^1(P,W)$ (see, for example, [10], Corollary 7.2.3.) Note that P/N is isomorphic to the cyclic group of order p and W^N is a vector space over a field of characteristic p. Thus, unless W^N is a projective P/N-module, $H^1(P/N, W^N)$ is nonzero. Elements in $H^1(P/N, W^N)$ can be represented by vectors in W^N which are in the kernel of the transfer. Assume W^N is not projective and choose $u \in W^N$ so that the equivalence class, $\{u\}$, is nonzero in $H^1(P/N, W^N)$. Clearly the image of $\{u\}$ under the inflation map restricts to zero in $H^1(N,W)$. Note that $M/(M \cap N)$ is also isomorphic to the cyclic group of order p so that elements of $H^1(M/(M \cap N), W^{M \cap N})$ can be represented by vectors in $W^{M \cap N}$. In [13] Lemma 6.2 it has been shown that the image of $\{u\}$ in $H^1(M,W)$, under the composition of inflation followed by restriction, is zero if and only if u represents zero in $H^1\left(M/(M \cap N), W^{M \cap N}\right)$. This can be used to derive a criterion of strong flatness for noncyclic p-groups.

Let $g, g' \in P \backslash M$, then $g' = hg^i$ for some $h \in M$ and $1 \leq i < p$. Since M is normal in P, P acts on W^M and

$$\begin{aligned} W^M(g'-1) &= W^M(hg^i - 1) \\ &= W^M(g^i - 1) = W^M(1 + g + \cdots + g^{i-1})(g-1) \subseteq W^M(g-1), \end{aligned}$$

hence $W^M(g'-1) = W^M(g-1)$ by symmetry. So we can define unambiguously

$$\mathscr{X}_M := W^M(g-1).$$

Define $X := N \cap M$, which is a maximal subgroup of M and of N. Let $u, u' \in N \backslash M$, then $N/X = \langle \overline{u} \rangle = \langle \overline{u}' \rangle$ and $u' = xu^i$ for some $x \in X$ and $1 \leq i < p$. In the same way as above we see $W^X(u'-1) = W^X(u-1)$. So for any maximal subgroup $N \neq M$ we can pick a $u_N \in N \backslash M$ and define

$$\mathscr{Y}_M := \bigcap_{\substack{N <_{\max} P \\ N \neq M}} W^{N \cap M}(u_N - 1).$$

Theorem 3.2. *For a noncyclic p-group P the following are equivalent.*

1. $\bigcap_{M <_{\max} P} \ker(\mathrm{res}^P_M|_{H^1(P,W)}) \neq 0$.
2. *For some $M \lhd P$ maximal, $\mathscr{X}_M < \mathscr{Y}_M \cap W^M$.*
3. *For all $M \lhd P$ maximal, $\mathscr{X}_M < \mathscr{Y}_M \cap W^M$.*

Proof. (3) $\Rightarrow$ (2) is clear.

Suppose (2) holds, and let $v \in \mathscr{Y}_M \cap W^M \backslash \mathscr{X}_M$. Pick $g \in P \backslash M$, so that $\overline{g} \in P/M$ generates $P/M \cong C_p$. Since P is noncyclic there exists a maximal subgroup $N \trianglelefteq P$ with $g \in N$ and we define $X := N \cap M$. Then clearly $N \neq M$ and, by assumption,

$v = w'(u_N - 1)$ with $w' \in W^X$. As before we see that $W^X(u_N - 1) = W^X(g - 1)$ so $v = w(g - 1)$ with some $w \in W^X$. Hence

$$\mathrm{Tr}_1^{P/M}(v) = v(\overline{g} - 1)^{p-1} = w(g - 1)^p = w(g^p - 1) = 0$$

since $g^p \in X$. So $v \in \ker(\mathrm{Tr}_1^{P/M})$. Since v is not in $W^M(g - 1)$, and P/M is a cyclic group generated by $\overline{g}$,

$$0 \neq [v] \in H^1(P/M, W^M).$$

Set $\tau := \mathrm{inf}_M^P([v]) \in H^1(P,W)$. Then $0 \neq \tau \in \ker(\mathrm{res}_M^P)$ by [10], Corollary 7.2.3, and by [13], Theorem 6.2, if we can show that

$$0 = [v] \in H^1(M'/M \cap M', W^{M \cap M'})$$

for any maximal subgroup $M' < P$, $M \neq M'$, then

$$\tau \in \bigcap_{N<P} \ker(\mathrm{res}_N^P|_{H^1(P,W)}).$$

Let $M' \neq M$ be such a maximal subgroup of P and set $Y := M' \cap M$, then there is $u_{M'} \in M' \backslash M$ and, by assumption $v \in W^Y(u_{M'} - 1)$. Since $\overline{u_{M'}}$ generates M'/Y, we have $0 = [v] \in H^1(M'/Y, W^Y)$. Thus we have proved (2) $\Rightarrow$ (1).

Finally suppose that $0 \neq \tau \in \bigcap_{N<_{\max}P} \ker(\mathrm{res}_N^P|_{H^1(P,W)})$, and let $M \triangleleft P$ be a maximal subgroup. Then by [10], Corollary 7.2.3,

$$\tau = \mathrm{inf}_M^P([v])$$

for some $0 \neq [v] \in H^1(P/M, W^M)$. This implies that

$$v \in W^M \backslash W^M(g - 1)$$

for each $g \in P \backslash M$. Let $N < P$ be a maximal subgroup with $N \neq M$, set $X := N \cap M$, and let $u_N \in N \backslash M$, then $N/X = \langle \overline{u_N} \rangle$. By assumption $\mathrm{res}_N^P(\tau) = 0$, so by [13], Theorem 6.2,

$$0 = [v] \in H^1(N/X, W^X);$$

i.e., $v \in W^X(u_N - 1)$. So we have shown that for each maximal subgroup M of P we can find an element v with

$$v \in \left(\bigcap_{M \neq N <_{\max} P} (W^{N \cap M}(u_N - 1)) \right) \cap W^M \backslash W^M(g - 1)$$

i.e., there exists $v \in \mathscr{Y}_M \cap W^M \backslash \mathscr{X}_M$.

4 Groups of the form $C_p \times C_p$

We now apply our results in the simplest case in which the depth of invariant rings is unknown. Let $P \cong C_p \times C_p$ be generated by X and Y, and define maximal subgroups $L := \langle XY \rangle$, $M := \langle X \rangle$, $N := \langle Y \rangle$. Let $W \cong V^*$ be a right P-module over a field k of characteristic p. Every maximal subgroup $H \neq N$ of P is of the form $\langle XY^i \rangle$ for $0 \leq i < p$. Hence, by Condition (2) of Theorem 3.2, V is linearly flat if and only if

$$W^N(X-1) < \bigcap_{l=0}^{p-1} W(XY^l - 1) \cap W^N.$$

When $p = 2$, the condition takes the particularly simple form

$$W^N(X-1) < W(X-1) \cap W(XY-1) \cap W^N.$$

Also by Part (3) of Theorem 3.2, we may interchange the roles of maximal subgroups, and when $p = 2$ we frequently use the alternative condition

$$W^L(X-1) < W(X-1) \cap W(Y-1) \cap W^L.$$

There are four infinite families of finite dimensional kP-Modules which are described below. Heller and Reiner [15] have shown that if $p = 2$ then these together with the regular kP-module form a complete set of isomorphism classes of indecomposable kP-modules. In the following, I_n denotes the $n \times n$-identity matrix and for $\lambda \in k$, J_λ denotes an $n \times n$ matrix in indecomposable (upper triangular) rational canonical form with eigenvalue λ, i.e. an upper triangular matrix with λ on the diagonal, 1 on the super - diagonal and zero elsewhere.

1. For every even dimension $2n$ there are representations $V_{2n,\lambda}$,

$$X \sim \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$$

$$Y \sim \begin{pmatrix} I_n & J_\lambda \\ 0 & I_n \end{pmatrix}.$$

   The dimension of the fixed-point space of the corresponding leftmodule, denoted $\dim(V^P)$ is n.

2. For every even dimension $2n$ there is a representation $V_{2n,\infty}$,

$$X \sim \begin{pmatrix} I_n & J_0 \\ 0 & I_n \end{pmatrix}$$

$$Y \sim \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}.$$

   In this case again $\dim(V^P) = n$.

3. For every odd dimension $2n+1$ there is a representation $V_{-(2n+1)}$,

$$X \sim \begin{pmatrix} & 0 & \\ I_n & \vdots & I_n \\ & 0 & \\ & & \\ 0 & & I_{n+1} \end{pmatrix}$$

$$Y \sim \begin{pmatrix} & & 0 \\ I_n & I_n & \vdots \\ & & 0 \\ & & \\ 0 & I_{n+1} & \end{pmatrix}.$$

These have $\dim(V^P) = n$.

4. For every odd dimension $2n+1$ there is a representation V_{2n+1},

$$X \sim \begin{pmatrix} I_{n+1} & 0\dots0 \\ & I_n \\ & \\ 0 & I_n \end{pmatrix}$$

$$Y \sim \begin{pmatrix} I_{n+1} & I_n \\ & 0\dots0 \\ & \\ 0 & I_n \end{pmatrix}.$$

For these modules $\dim(V^P) = n+1$.

Theorem 4.1. *Each even-dimensional module $V_{2n,\lambda}$ with $n \geq 2$ and $\lambda \in k$ or $\lambda = \infty$ is linearly flat. Each odd-dimensional module of the form V_{2n+1} with $n \geq 2$ is linearly flat, whereas the modules of the form $V_{-(2n+1)}$ are linearly flat if $n \geq p$.*

Corollary 4.1. *Every nonprojective indecomposable modular representation of the group $P := C_2 \times C_2$ is flat.*

Proof. To prove the corollary observe that when $p = 2$, the theorem states that every nonprojective indecomposable kP-module of dimension ≥ 4 is linearly flat. But it is also easily seen that each indecomposable kP-module of dimension < 4 in our classification above is trivially flat. Note that there is only one projective indecomposable module, namely the regular module.

Proof. We now show in each of the three cases that there is an element

$$v \in \bigcap_{l=0}^{p-1} W(XY^l - 1) \cap W^N \backslash W^N(X-1).$$

This proves that the left-module V with $V^* \cong W$ is linearly flat, by Theorem 3.2.

Consider the representation $V := V_{2n,\lambda}$, where for the time being $\lambda \neq \infty$. Then we have

$$W^N = \begin{cases} \langle x_{n+1}, \ldots, x_{2n} \rangle & \lambda \neq 0 \\ \langle x_n, \ldots, x_{2n} \rangle & \lambda = 0 \end{cases}$$

so that

$$W^N(X-1) = \begin{cases} 0 & \lambda \neq 0 \\ \langle x_{2n} \rangle & \lambda = 0. \end{cases}$$

Also for each $0 \leq l \leq p-1$,

$$W(XY^l - 1) = \langle (1+l\lambda)x_{n+1} + lx_{n+2}, (1+l\lambda)x_{n+2} + lx_{n+3}, \ldots, (1+l\lambda)x_{2n} \rangle$$

$$W(XY^l - 1) = \begin{cases} \langle x_{n+1}, \ldots, x_{2n} \rangle & \lambda l \neq -1 \\ \langle x_{n+2}, \ldots, x_{2n} \rangle & \lambda l = -1. \end{cases}$$

Now k is a field of characteristic p, so it contains a copy of the field $\mathbb{F}_p$ of order p. If $\lambda \in \mathbb{F}_p \backslash \{0\}$, there is precisely one $l \in \{1, \ldots, p-1\}$ for which $\lambda l = -1$. If $\lambda = 0$ or if $\lambda \in k \backslash \mathbb{F}_p$, then $\lambda l \neq -1$ for every l. It follows that

$$\bigcap_{l=0}^{p-1} W(XY^l - 1) \cap W^N = \begin{cases} \langle x_{n+1}, \ldots, x_{2n} \rangle & \lambda = 0 \text{ or } \lambda \in k \backslash \mathbb{F}_p \\ \langle x_{n+2}, \ldots, x_{2n} \rangle & \lambda \in \mathbb{F}_p \backslash \{0\}. \end{cases}$$

So we have linear flatness for all λ if $n \geq 2$, and for $n = 1$ when $\lambda \notin \mathbb{F}_p$. $V_{2n,\infty}$ is obtained from $V_{2n,0}$ under the automorphism of P which swaps X and Y, and so the same results hold here.

Now consider the representation $V := V_{2n+1}$. This time, we have

$$W^N = \langle x_{n+1}, \ldots, x_{2n+1} \rangle$$

so that $W^N(X-1) = \langle x_{2n+1} \rangle$. Furthermore we have

$$W(XY^l - 1) = \langle lx_{n+2}, x_{n+2} + lx_{n+3}, x_{n+3} + lx_{n+4}, \ldots, x_{2n} + lx_{2n+1}, x_{2n+1} \rangle$$
$$W(XY^l - 1) = \langle x_{n+2}, x_{n+3}, \ldots, x_{2n+1} \rangle$$

for all $l \in \{0, \ldots, p-1\}$. Therefore

$$\bigcap_{l=0}^{p-1} W(XY^l - 1) \cap W^N \backslash W^N(X-1)$$

is nonempty, and hence we have linear flatness, provided that $n \geq 2$.

Finally consider the representation $V := V_{-(2n+1)}$. In this case

$$W^N = \langle x_{n+1}, \ldots, x_{2n+1} \rangle$$

so $W^N(X-1)=0$, and

$$W(XY^l-1)=\langle lx_{n+1}+x_{n+2}, lx_{n+2}+x_{n+3}, \ldots, lx_{2n}+x_{2n+1}\rangle.$$

Let us denote each space $W(XY^l-1)$ by the shorthand W_l. Since $W_l \subset W^N$ for each $l=1,\ldots,p-1$, it is enough to show that $\bigcap_{l=0}^{p-1} W_l \neq 0$. It is easy to see that each W_l is a hyperplane in the $n+1$-dimensional space W^N. It then follows that

$$\dim \bigcap_{l=0}^{p-1} W_l \geq n+1-p.$$

And so $V_{-(2n+1)}$ is linearly flat when $n \geq p$.

5 Decomposable representations

We have proved that all nonprojective indecomposable representations of $C_2 \times C_2$ are flat, and that provided $\dim(V^P)+2<\dim(V)$, any representation containing the indecomposable V as a direct summand will also be flat. We now go on to classify the modular representations of $C_2 \times C_2$ that are strongly flat, by investigating which direct summands may appear in their symmetric algebras. Before we do so, we restate and extend Theorem 5.2 specifically for the case $p=2$.

Theorem 5.1. *Let V be an indecomposable representation of $P := C_2 \times C_2$ over a field of characteristic 2. Then:*

1. *If $V \cong V_{2,\lambda}$ for $(\lambda \neq 0,1,\infty)$, $V_{4,\lambda}$, or V indecomposable of dimension ≥ 5, then V is linearly flat.*
2. *If $\dim(V) \leq 4$ and V is not projective, then V is trivially flat.*
3. *If $V \cong$ the projective indecomposable of dimension 4, then V is not flat.*
4. *The representations $V_{2,0}, V_{2,1}, V_{2,\infty}, V_3$ and V_{-3}, though flat, are not strongly so.*

Note that since this is a p-group in characteristic p, there is only one projective indecomposable, namely the regular representation. From now on this is abbreviated to $\overline{V}_4$.

Proof. (1) and (2) are clear. The remaining statements are proven in the following lemmas.

$\overline{V}_4$ is equivalent to a permutation representation. We can think of X and Y acting on V^* as the permutations $(12)(34)$ and $(13)(24)$ respectively, by which we mean that

$$x_1X=x_2, \;\; x_2X=x_1, \;\; x_3X=x_4, \;\; x_4X=x_3$$

and the action of Y calculated similarly. It is clear that $\dim(V^P)=1$, therefore, V is either flat or R^P is Cohen–Macaulay. In this case we can find explicit generators for R^P in order to show that the latter holds, and V is not flat.

Remark. The ring of invariants for $C_2 \times C_2$ and $\overline{V}_4$ has been determined in [1], where k is assumed to be the field of order 2. The version we present here for convenience and completeness, assumes k to be any field of characteristic 2. We use the following lemma.

Lemma 5.1. *Let $R := S(V^*) = k[x_1, x_2, x_3, x_4]$ with $V \cong \overline{V}_4$. Then the ring of invariants R^M may be described as*

$$S := k[a,b,c,d](1, \beta_1)$$

where $a := x_1 + x_2, b := x_3 + x_4, c := x_1x_2, d := x_3x_4$ form an hsop and $\beta_1 := x_1x_3 + x_2x_4$, with $\beta_1^2 = \beta_1 ab + a^2d + b^2c$.

Proof. This is a special case of [19], Proposition 11.

To find R^P, we apply the lemma twice and obtain the following.

Lemma 5.2. *We keep the notation from Lemma 5.1. Then the ring of invariants R^P is Cohen–Macaulay and can be described as*

$$\begin{aligned} R^P &= k[a+b, c+d, ab, cd](1, \beta_1)(1, ac+bd) \\ &= k[x_1+x_2+x_3+x_4, x_1x_2+x_3x_4, (x_1+x_2)(x_3+x_4), x_1x_2x_3x_4](1, \beta_1, \beta_2, \beta_1\beta_2) \end{aligned}$$

where $\beta_2 = x_1^2x_2 + x_1x_2^2 + x_3^2x_4 + x_3x_4^2$. In particular R^P is not flat.

Proof. The quotient P/M acts on $R^M = k[a,b,c,d](1, \beta_1)$ by swapping a with b, swapping c with d, and by fixing β_1. Hence Lemma 5.1 gives the first equality, with primary invariants of degrees 1, 2, 2 and 4. Since the product of these degrees is 16 and the minimal set of secondary invariants has $4 = 16/|P|$ elements, we conclude by [6], Theorem 3.7.1, that R^P is a Cohen–Macaulay ring. Since $\mathrm{Dim}(R^P) = 4$ and $\dim(\overline{V}_4^P) = 1$, the ring R^P cannot be flat.

The proof of (4) is again broken down into two lemmas. Recall that in order to prove strong flatness we would need to find an element of

$$R(X+1) \cap R(Y+1) \cap R^L \backslash R^L(X+1).$$

If $V \cong V_{2,0}$ then $R(Y+1) = 0$, if $V \cong V_{2,\infty}$ then $R(X+1) = 0$, and if $V \cong V_{2,1}$ then $R^L(X+1) = R(X+1)$. So none of these modules are strongly flat as P-modules.

Lemma 5.3. *Let $R := S(V^*) = k[x_1, x_2, x_3]$ with $V \cong V_3$. Then the ring of invariants R^P is polynomial of the form $B := k[u, v, w]$ $u := x_3$, $v := x_1(x_1 + x_3)$, and $w := x_2(x_2 + x_3)$. The representation V_3 is flat but not strongly flat.*

Proof. Note that P acts as follows,

$$\begin{aligned} X &: \; x_1 \mapsto x_1, \; x_2 \mapsto x_2 + x_3, \; x_3 \mapsto x_3 \\ Y &: \; x_1 \mapsto x_1 + x_3, \; x_2 \mapsto x_2, \; x_3 \mapsto x_3 \\ XY &: \; x_1 \mapsto x_1 + x_3, \; x_2 \mapsto x_2 + x_3, \; x_3 \mapsto x_3, \end{aligned}$$

hence $u,v,w \in R^P$ and R is integral over B (e.g. $x_1^2 + x_1u - v = 0$); moreover the product of their degrees is $4 = |P|$, hence by [6], Theorem 3.7.5, $R^P = k[u,v,w]$ as required. Since $R = B(1,x_1,x_2,x_1x_2)$ as a B-module and $[\mathrm{Frac}(R):\mathrm{Frac}(B)] = 4$, we see that $R = B \oplus Bx_1 \oplus Bx_2 \oplus Bx_1x_2$. Using this decomposition it is easily seen that

$$R^M = B(1,x_1), \quad R^N = B(1,x_2), \quad R^L = B(1,x_1+x_2).$$

Now we obtain immediately

$$\begin{aligned} R^L(X+1) &= Bx_3 \\ R(X+1) &= R^M(x_3) = B(x_3,x_1x_3) \\ R(Y+1) &= R^N(x_3) \ = B(x_3,x_2x_3). \end{aligned}$$

Therefore $R(X+1) \cap R(Y+1) = Bx_3 = R^L(X+1)$, and so there cannot be an element in $R(X+1) \cap R(Y+1) \cap R^L \backslash R^L(X+1)$.

Lemma 5.4. *Let $R := S(V^*) = k[x_1,x_2,x_3]$ with $V \cong V_{-3}$. Then the ring of invariants R^P is polynomial of the form $B := k[x_2,x_3,z]$ with $z := N_P(x_1) := \prod_{g\in P}(x_1g) = x_1^4 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1x_2x_3^2$. The representation V_{-3} is flat but not strongly flat.*

Proof. Clearly $B \subseteq R^P$ and the equation

$$x_1^4 = x_1^2(x_2^2 + x_3^2 + x_2x_3) + x_1(x_2^2x_3 + x_2x_3^2) + z$$

shows that R is integral over B. Since the degree product of x_2, x_3, and z is $4 = |P|$, again by [6], Theorem 3.7.5, we have $B = R^P$ with a decomposition

$$R = B[x_1] = B \oplus Bx_1 \oplus Bx_1^2 \oplus Bx_1^3.$$

Let $u := N_M(x_1) := x_1(x_1+x_3)$, $v := N_N(x_1) := x_1(x_1+x_2)$ and $w := N_L(x_1) := x_1(x_1+x_2+x_3)$. Since $z = N_P(x_1) = N_N(u) = u\cdot(u)Y = u(u+x_2^2+x_2x_3)$, it follows that $u^2 \in B + Bu = B \oplus Bu$. Similarly $v^2 \in B \oplus Bv$ and $w^2 \in B \oplus Bw$ and we find

$$\begin{aligned} R^M &= B[u] \ = B \oplus Bu \\ R^N &= B[v] \ = B \oplus Bv \\ R^L &= B[w] = B \oplus Bw. \end{aligned}$$

From this we see

$$\begin{aligned} R(X+1) &= R^M(x_3) = B(x_3,ux_3) \\ R(Y+1) &= R^N(x_2) = B(x_2,vx_2) \\ R(XY+1) &= R^L(x_1) = B(x_1,wx_1) \\ R^L(X+1) &= B((w)(X+1)) = B(x_2x_3). \end{aligned}$$

Suppose $s \in R(X+1) \cap R(Y+1) \subseteq B$, then $s = x_3 r = x_2 r'$ with $r \in R^M$ and $r' \in R^N$, hence $r'/x_3 \in R$, so $s = x_2 x_3 s'$ with P-fixed $s' \in R$ and we conclude $s' \in B$ and $s \in Bx_2x_3 = R^L(X+1)$. This shows that V_{-3} cannot be strongly flat.

This last lemma also terminates the proof of 5.1.

Corollary 5.1. *A complete list of indecomposable kP-modules for $P := C_2 \times C_2$ whose polynomial invariants are Cohen–Macaulay is given by*

$$V_1, V_{2,\lambda}, V_{-3}, V_3, V_{4,\lambda}, \overline{V}_4, V_5.$$

Proof. We have stated already that the modules $V_1, V_{2,\lambda}, V_{-3}, V_3$ are trivially flat. Observe that $\dim(V_{4,\lambda}^P) = 2$ and $\dim(V_5^P) = 3$, so both of these are trivially flat. That $\overline{V}_4$ gives rise to a Cohen–Macaulay invariant ring is proved in Theorem 5.1. Finally we check the statement of Theorem 4.1, to see that every other indecomposable representation is flat, but not trivially so.

Definition 5.1. We denote by NSF the following set of isomorphism types of $k(C_2 \times C_2)$-modules:

$$\{V_1, V_{2,1}, V_{2,0}, V_{2,\infty}, V_3, V_{-3}, \overline{V}_4\}.$$

Corollary 5.2. *The symmetric algebras of the representations V_3,V_{-3} and $\overline{V}_4$ contain only indecomposable direct summands of the form $V_1, V_{2,1}, V_{2,0}, V_{2,\infty}, V_3, V_{-3}$, and $\overline{V}_4$.*[1]

Proof. Let $V \in NSF$. All indecomposable modules not in NSF are linearly flat. If $S(V^*)$ contains a direct summand W^* for which W is linearly flat, then V is strongly flat. Observing that NSF is closed under the operation of taking duals (where $V_3^* \cong V_{-3}$, and the rest are self-dual) completes the proof.

Now if we can determine precisely which indecomposable summands appear in the symmetric algebras of modules in NSF, and how the tensor products decompose into indecomposable summands, we will then be able to list all possible kP-modules which are strongly flat.

Theorem 5.2. *Let* $\mathrm{add}(NSF)$ *denote the set of modules which are direct sums of modules in NSF. Let W be any kP-module. Then W is strongly flat unless $W \in$* $\mathrm{add}(NSF)$ *and neither $V_3 \oplus V_3$ nor $V_{-3} \oplus V_{-3}$ is a direct summand of W.*

Proof. If $W \notin \mathrm{add}(NSF)$, then W must contain a direct summand which is strongly flat. Consequently, W is strongly flat. So we assume $W \in \mathrm{add}(NSF)$. We write

$$W = aV_1 \oplus b_0 V_{2,0} \oplus b_1 V_{2,1} \oplus b_\infty V_{2,\infty} \oplus c_+ V_3 \oplus c_- V_{-3} \oplus d\overline{V}_4$$

[1] V_1 denotes the trivial module.

So that

$$W^* \cong aV_1 \oplus b_0V_{2,0} \oplus b_1V_{2,1} \oplus b_\infty V_{2,\infty} \oplus c_+V_{-3} \oplus c_-V_3 \oplus d\overline{V}_4$$

and $S(W^*) \cong$

$$S(V_1)^{\otimes a} \otimes S(V_{2,0})^{\otimes b_0} \otimes S(V_{2,1})^{\otimes b_1} \otimes S(V_{2,\infty})^{b_\infty} \otimes S(V_{-3})^{\otimes c_+} \otimes S(V_3)^{\otimes c_-} \otimes S(\overline{V}_4)^{\otimes d}.$$

In order to determine which isomorphism classes of modules may appear as summands in the above expression, we need to know which summands appear in the symmetric algebras of the modules in NSF, and also how their tensor products decompose. Clearly the symmetric algebras of the nonfaithful modules $V_{2,0}, V_{2,1}$ and $V_{2,\infty}$ contain as direct summands only themselves and the trivial module, as any direct summands in the symmetric algebra must also be nonfaithful. Still more obviously, the symmetric algebra of V_1 contains only copies of the trivial module. In order to ascertain which modules appear in the symmetric algebras of V_3, V_{-3}, and $\overline{V}_4$, we use [16], Theorem 1.2. This tells us that all summands which do appear as summands of the symmetric algebra can be found in degrees $\leq 2^n - n - 1$, where n is the dimension of the representation. This allows for computation, using the meataxe in MAGMA [4]. The result is:

- $S(V_{-3})$ contains summands isomorphic to $V_1, V_{2,1}, V_{2,\infty}, V_{2,0}, V_{-3}$, and $\overline{V}_4$;
- $S(V_3)$ contains summands isomorphic to V_1, V_3, and $\overline{V}_4$;
- $S(\overline{V}_4)$ contains summands isomorphic to $V_1, V_{2,1}, V_{2,\infty}, V_{2,0}$, and $\overline{V}_4$.

The first two results require a decomposition of the first four symmetric powers, and the third requires a decomposition of the first eleven. This takes considerable computation time. However, there is a far easier way to obtain this result: $\overline{V}_4$ is a permutation module of dimension 4, so each indecomposable summand in its symmetric algebra has dimension dividing 4. This tells us that $S(\overline{V}_4)$ cannot contain summands isomorphic to V_3 or V_{-3} and the result now follows from Corollary 5.2. The following table lists the decompositions of tensor products in NSF. We omit from the table results concerning V_1, as $W \otimes V_1 = W$ for any kP-module W.

Table 1. Decomposing Tensor Products of kP-modules.

$\otimes$	$V_{2,1}$	$V_{2,\infty}$	$V_{2,0}$	V_3	V_{-3}	$\overline{V}_4$
$V_{2,1}$	$V_{2,1}^{\oplus 2}$	$\overline{V}_4$	$\overline{V}_4$	$V_{2,1} \oplus \overline{V}_4$	$V_{2,1} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
$V_{2,\infty}$		$V_{2,\infty}^{\oplus 2}$	$\overline{V}_4$	$V_{2,\infty} \oplus \overline{V}_4$	$V_{2,\infty} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
$V_{2,0}$			$V_{2,0}^{\oplus 2}$	$V_{2,0} \oplus \overline{V}_4$	$V_{2,0} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
V_3				$\overline{V}_4 \oplus V_5$	$V_1 \oplus \overline{V}_4^{\oplus 2}$	$\overline{V}_4^{\oplus 3}$
V_{-3}					$\overline{V}_4 \oplus V_{-5}$	$\overline{V}_4^{\oplus 3}$
$\overline{V}_4$						$\overline{V}_4^{\oplus 4}$

Note that the last column of that list is explained by the fact that the tensor product of any module with a projective one is projective and $\overline{V}_4$ is the only indecomposable projective module for $k(C_2 \times C_2)$. We conclude that the modules in the set $\mathrm{add}\{V_1, V_{2,1}, V_{2,\infty}, V_{2,0}, \overline{V}_4\}$ form a closed system under taking symmetric algebras. Moreover all the modules in the decomposition of $S(W^*)$ belong to NSF unless either c_+ or c_- is at least two. If $c_+ \geq 2$, then $S(W^*)$ contains a direct summand isomorphic to $V_{-5} \cong V_5^*$ and W is therefore strongly flat. Similarly if $c_- \geq 2$, then $S(W^*)$ contains a direct summand isomorphic to $V_5 \cong V_{-5}^*$ and again W is strongly flat.

It must be stressed that in this chapter we have determined the depth of R^G for any indecomposable representation of $P := C_2 \times C_2$ (and so classified its flat indecomposable representations), and also classified all the strongly flat representations of this group. We have not, however, classified all the flat representations of this group. Consider, for example, the kP-module $W := V_3 \oplus V_{2,1} \oplus V_{2,1}$. Then $W \in \mathrm{add}(NSF)$ and neither $V_3 \oplus V_3$ nor $V_{-3} \oplus V_{-3}$ is a direct summand of W, so by Theorem 5.2, W is not strongly flat. A calculation in MAGMA establishes that the depth of the corresponding invariant ring R^G is 6, and so W is in fact flat. We may easily verify that W is not trivially flat. So then, it must be the case (by Theorem 2.2) that there is a cohomology class $0 \neq \alpha \in H^1(P,R)$ which is annihilated by each element of the relative transfer ideal. At the same time, the intersection of the kernels of the restrictions to each maximal subgroup is zero. Finding such an α directly is, at least for the moment, beyond the scope of the methods presented in this chapter.

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Kählerian Reduction in Steps

Daniel Greb[1] and Peter Heinzner[2]

Summary. We study Hamiltonian actions of compact Lie groups K on Kähler manifolds which extend to a holomorphic action of the complexified group $K^{\mathbb{C}}$. For a closed normal subgroup L of K we show that the Kählerian reduction with respect to L is a stratified Hamiltonian Kähler $K^{\mathbb{C}}/L^{\mathbb{C}}$-space whose Kählerian reduction with respect to K/L is naturally isomorphic to the Kählerian reduction of the original manifold with respect to K.

Key words: Hamiltonian actions, symplectic reduction, group actions on Kähler manifolds

Mathematics Subject Classification (2000): 53D20, 32M05, 53C55

This note is dedicated to Prof. Gerry Schwarz on the occasion of his 60*th birthday*

1 Introduction

Reduction of variables for physical systems with symmetries is a fundamental concept in classical Hamiltonian dynamics. It is based on Noether's principle that every 1-parameter group of symmetries of a physical system corresponds to a constant of motion. The mathematical formalisation is known as the Marsden–Weinstein-reduction or *symplectic reduction*. Consider a symplectic manifold X with a smooth symplectic form ω and a smooth action of a Lie group K on X by ω-isometries.

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Assume that there exists a smooth map μ from X into the dual space $\mathfrak{k}^*$ of the Lie algebra $\mathfrak{k}$ of K such that

a) for every $\xi \in \mathfrak{k}$, the function $\mu^\xi : X \to \mathbb{R}$, $\mu^\xi(x) = \mu(x)(\xi)$, fulfills $\mathrm{d}\mu^\xi = \iota_{\xi_X}\omega$, where ξ_X denotes the vector field on X induced by the action of K on X and ι_{ξ_X} denotes contraction with respect to ξ_X,
b) the map $\mu : X \to \mathfrak{k}^*$ is equivariant with respect to the action of K on X and the coadjoint representation Ad^* of K on $\mathfrak{k}^*$, i.e., for all $x \in X$ and for all $\mathfrak{k} \in K$ we have $\mu(k \bullet x) = \mathrm{Ad}^*(k)(\mu(x))$.

The map $\mu : X \to \mathfrak{k}^*$ is called a (K-equivariant) *momentum map* and the action of K on X is called *Hamiltonian*.

The motion of a classical particle is given by a Hamiltonian function H. More precisely, let H be a smooth function on X. The time evolution of the underlying physical system is described by the flow of the vector field V_H associated with the Hamiltonian H via the equation $\mathrm{d}H = \iota_{V_H}\omega$. A Hamiltonian system with symmetries is a system (X, ω, K, μ, H) where H is invariant with respect to the K-action. In this case it follows from (a) that

$$\mathrm{d}\mu^\xi(V_H) = \omega(\xi_X, V_H) = -\mathrm{d}H(\xi_X) = 0$$

holds for all $\xi \in \mathfrak{k}$. This implies Noether's principle in the following geometric formulation. Every component μ^ξ of the momentum map is a constant of motion for the system described by H.

The previous considerations imply that level sets of μ are invariant under the flow of V_H. In many cases questions can be organised so that $\mu^{-1}(0)$ is the momentum fibre of interest. By (b), the group K acts on the level set $\mu^{-1}(0)$. Let us first consider this action on an infinitesimal level. If we fix $x_0 \in \mathscr{M} := \mu^{-1}(0)$, it follows from a) that $\ker(d\mu(x_0)) = (\mathfrak{k} \bullet x_0)^{\perp_\omega}$, where $\mathfrak{k} \bullet x_0 = \{\xi_X(x_0) \mid \xi \in \mathfrak{k}\}$ and $^{\perp_\omega}$ denotes the perpendicular with respect to ω. The optimal situation appears if μ has maximal rank at x_0. In this case, $\mathscr{M}$ is smooth at x_0 and $T_{x_0}\mathscr{M} = \ker(\mathrm{d}\mu(x_0)) = (\mathfrak{k} \bullet x_0)^{\perp_\omega}$ holds. It follows that $\mathscr{M}$ is a coisotropic K-stable submanifold of X and the symplectic form ω_{x_0} on $T_{x_0}\mathscr{M}$ induces a symplectic form $\widetilde{\omega}_{x_0}$ on $T_{x_0}\mathscr{M}/T_{x_0}\mathscr{M}^{\perp_\omega} = T_{x_0}\mathscr{M}/\mathfrak{k} \bullet x_0$. These observations imply that once the space $\mathscr{M}/K$ is smooth, it will be a symplectic manifold. This is the content of the *Marsden–Weinstein–Theorem* (see [11]):

If K acts freely and properly on $\mathscr{M}$, the quotient $\mathscr{M}/K$ is a symplectic manifold whose symplectic form $\widetilde{\omega}$ is characterised via the equation

$$\pi^*\widetilde{\omega} = i^*_{\mathscr{M}}\omega\,.$$

Here, $\pi : \mathscr{M} \to \mathscr{M}/K$ denotes the quotient map and $i_{\mathscr{M}} : \mathscr{M} \to X$ is the inclusion. Furthermore, the restriction of the K-invariant Hamiltonian H to $\mathscr{M}$ induces a smooth function $\widetilde{H}$ on $\mathscr{M}/K$. The Hamiltonian system on $(\mathscr{M}/K, \widetilde{\omega})$ associated with $\widetilde{H}$ captures the essential (symmetry-independent) properties of the original K-invariant system that was given by H.

The Marsden–Weinstein construction is natural in the sense that it can be done in steps. This means that for a normal closed subgroup L of K, the restricted momentum map $\mu_L : X \to \mathfrak{l}^*$ is K-equivariant, the induced K-action on $\mu_L^{-1}(0)/L$ is Hamiltonian with momentum map $\widetilde{\mu}$ induced by μ, and the symplectic reduction $\widetilde{\mu}^{-1}(0)/K$ is symplectomorphic to $\mathscr{M}/K$.

Removing the restrictive regularity assumptions of [11], it is proven in [17] that symplectic reduction can be carried out for general group actions of compact Lie groups yielding stratified symplectic quotient spaces $\mathscr{M}/K$, i.e., stratified spaces where all strata are symplectic manifolds. The paper [5] proposed an approach to this singular symplectic reduction based on embedding symplectic manifolds into Kähler manifolds and a Kähler reduction theory for Kähler manifolds. Roughly speaking, $\mathscr{M}/K$ is realised as a locally semi-algebraic subspace of a Kähler space Q and the symplectic structure on $\mathscr{M}/K$ is given by restriction. Inspecting the proofs one sees that the construction of [5] is compatible with reduction in steps.

Our interest here is to study the problem of reduction in steps in a Kählerian context using techniques related to the complex geometry and invariant theory for the complexification $K^{\mathbb{C}}$ of a compact Lie group K.

Holomorphic actions of compact groups K on complex spaces X very often extend to holomorphic actions of the complexification $K^{\mathbb{C}}$, at least in the sense that there exists an open K-equivariant embedding $X \hookrightarrow X^c$ of X into a holomorphic $K^{\mathbb{C}}$-space X^c (see [6]). In this note, we consider the case where $K^{\mathbb{C}}$ already acts on X. More precisely, we consider Kähler manifolds X with a Hamiltonian action of a compact Lie group K that extends to a holomorphic action of the complexification $K^{\mathbb{C}}$. In this setup, an extensive quotient theory has been developed. See Section 1.1 for a survey of the results used in this note. Due to the action of the complex group $K^{\mathbb{C}}$, it is possible to introduce a notion of semistability. The reduced space $\mathscr{M}/K$ carries a complex structure given by holomorphic invariant theory for the action of $K^{\mathbb{C}}$ on the set of semistable points. The reduced symplectic structure is compatible with this complex structure, i.e., $\mathscr{M}/K$ is a Kähler space. Since the quotient $\mathscr{M}/K$ in general will be singular, the Kähler structure is locally given by continuous strictly plurisubharmonic functions that are smooth along the natural stratification of $\mathscr{M}/K$ given by the orbits of the action of K on $\mathscr{M}$. We emphasize at this point that both the complex structure and the Kähler structure are defined globally, i.e., across the strata. We call $\mathscr{M}/K$ the *Kählerian reduction* of X by K. This reduction theory works more generally for stratified Kählerian complex spaces, however, in order to reduce technical difficulties and not to obscure the main principles at work, we restrict our attention to actions on complex manifolds. Nevertheless, we also have to consider complex spaces which appear as quotients by normal subgroups of $K^{\mathbb{C}}$.

Using the approach of [5] and the quotient theory for complex-reductive group actions, we show that Kählerian reduction can be done in steps. More precisely, we prove that the Kählerian reduction of X by a normal subgroup L of K is a stratified Hamiltonian Kähler $K^{\mathbb{C}}$-space and that its Kählerian reduction is isomorphic to the reduction of X by K (see Theorem 2.1). This shows that reduction in steps respects the Kähler geometry. It also exhibits the class of Kählerian stratified spaces as natural for a Kählerian reduction theory. Furthermore, we look carefully at the stratifications obtained on the various quotient spaces.

1.1 Reductive group actions on Kähler spaces

As we have noted above, symplectic reduction of Kähler manifolds yields spaces that are endowed with a complex structure. This is due to a close relation between the quotient theory of a compact Lie group K and the quotient theory for its complexification $K^{\mathbb{C}}$ which we now explain.

In this chapter a complex space refers to a reduced complex space with countable topology. If G is a Lie group, then a *complex G-space* Z is a complex space with a real-analytic action $G \times Z \to Z$ which for fixed $g \in G$ is holomorphic. For a complex Lie group G a *holomorphic G-space* Z is a complex G-space such that the action $G \times Z \to Z$ is holomorphic.

We note that given a compact real Lie group K, there exists a complex Lie group $K^{\mathbb{C}}$ containing K as a closed subgroup with the following universal property. Given a Lie homomorphism $\varphi : K \to H$ into a complex Lie group H, there exists a holomorphic Lie homomorphism $\varphi^{\mathbb{C}} : K^{\mathbb{C}} \to H$ extending φ.

If not only K but also $K^{\mathbb{C}}$ acts holomorphically on a manifold X, it is natural to relate the quotient theory of K to the quotient theory of $K^{\mathbb{C}}$. Due to the existence of non-closed $K^{\mathbb{C}}$-orbits, in contrast to actions of K, actions of $K^{\mathbb{C}}$ will in general not give rise to reasonable orbit spaces. However, as we show, it is often possible to find an open subset U of X and a complex space Y that parametrises closed $K^{\mathbb{C}}$-orbits in U. More precisely, we call a complex space Y an *analytic Hilbert quotient* of U by the action of $K^{\mathbb{C}}$ if there exists a $K^{\mathbb{C}}$-invariant Stein holomorphic map $\pi : U \to Y$ with $(\pi_* \mathscr{O}_U)^{K^{\mathbb{C}}} = \mathscr{O}_Y$. Here, a *Stein map* is a map such that inverse images of Stein subsets are again Stein. It can be shown (see [7], for example) that Y is the quotient of U by the equivalence relation

$$x \sim y \quad \text{if and only if} \quad \overline{K^{\mathbb{C}} \bullet x} \cap \overline{K^{\mathbb{C}} \bullet y} \neq \emptyset\, .$$

Analytic Hilbert quotients are universal with respect to $K^{\mathbb{C}}$-invariant holomorphic maps, i.e., given a $K^{\mathbb{C}}$-invariant holomorphic map $\varphi : U \to Z$ into a complex space Z, there exists a uniquely determined holomorphic map $\bar{\varphi} : Y \to Z$ such that $\varphi = \bar{\varphi} \circ \pi$. It follows that an analytic Hilbert quotient of U by $K^{\mathbb{C}}$ is unique up to biholomorphism once it exists. We denote it by $U /\!\!/ K^{\mathbb{C}}$.

The theory of analytic Hilbert quotients is interwoven with the Kählerian quotient theory as follows (see [7]). Let K be a compact Lie group with Lie algebra $\mathfrak{k}$, and X a holomorphic $K^{\mathbb{C}}$-manifold. Assume that the action of K is Hamiltonian with respect to a K-invariant Kähler form on X with K-equivariant momentum map $\mu : X \to \mathfrak{k}^*$. In this situation, we call X a *Hamiltonian Kähler $K^{\mathbb{C}}$-manifold*. Let $\mathscr{M} = \mu^{-1}(0)$ and define

$$X(\mathscr{M}) := \{x \in X \mid \overline{K^{\mathbb{C}} \bullet x} \cap \mathscr{M} \neq \emptyset\}\, .$$

A point $x \in X(\mathscr{M})$ is called *semistable*. The set $X(\mathscr{M})$ is open in X and an analytic Hilbert quotient Q of $X(\mathscr{M})$ by $K^{\mathbb{C}}$ exists. Each fibre of the quotient map $\pi : X(\mathscr{M}) \to Q$ contains a unique closed $K^{\mathbb{C}}$-orbit which is the unique orbit of minimal dimension in that fibre. This closed $K^{\mathbb{C}}$-orbit intersects $\mathscr{M}$ in a unique K-orbit, $K \bullet x$. The isotropy group $(K^{\mathbb{C}})_x$ of points $x \in \mathscr{M}$ is complex-reductive and equal to the complexification of K_x. The analytic Hilbert quotient $X(\mathscr{M})/\!\!/K^{\mathbb{C}}$ is related to the Kählerian reduction $\mathscr{M}/K$ by the following fundamental commutative diagram

$$\begin{tikzcd} \mathscr{M} \arrow[r, hook, "\iota"] \arrow[d, "\pi_K"'] & X(\mathscr{M}) \arrow[d, "\pi"] \\ \mathscr{M}/K \arrow[r, "\bar{\iota}", "\simeq"'] & Q\,. \end{tikzcd} \tag{1}$$

Here, the induced map $\bar{\iota}$ is a homeomorphism. Hence, the symplectic reduction $\mathscr{M}/K$ has a complex structure induced via the homeomorphism $\bar{\iota}$. The inverse of $\bar{\iota}$ is induced via a retraction $\psi : X(\mathscr{M}) \to \mathscr{M}$ that is related to the stratification of X via the gradient flow of the norm square of the momentum map (see [13], [9], and [15]).

Using this two-sided picture of the quotient it can be shown using the techniques of [5] that $\mathscr{M}/K$ carries a natural Kählerian structure that is smooth along the orbit type stratification.

1.2 Stratifying holomorphic G-spaces

Invariant stratifications are a powerful tool in the study of group actions and their quotient spaces (see [10] and [14]). We recall the definitions and basic properties of these stratifications.

Definition 1.1. A *complex stratification* of a complex space X is a countable, locally finite covering of X by disjoint subspaces (the so called *strata*) $\mathbb{S} = (S_\gamma)_{\gamma \in \Gamma}$ with the following properties.

1. Each stratum S_γ is a locally closed submanifold of X that is Zariski-open in its closure.
2. The boundary $\partial(S_\gamma) = \overline{S_\gamma} \setminus S_\gamma$ of each stratum S_γ is a union of strata of lower dimension.

Example 1.1. The singular set X_{sing} of a complex space X is a closed complex subspace of smaller dimension. Iterating this procedure, i.e., considering the singular set of X_{sing}, we obtain a natural stratification on X. If a Lie group G acts holomorphically on X, this stratification is G-invariant.

We now consider stratifications related to group actions. Let G be a complex-reductive Lie group. Let X be a holomorphic G-manifold such that the analytic

Hilbert quotient $\pi_G : X \to X/\!\!/G$ exists. Let p be a point in $X/\!\!/G$. The fibre $\pi_G^{-1}(p)$ over p contains a unique closed orbit. We denote this orbit by $C(p)$. We say that p is of *G-orbit-type* (G_1) if the stabilizer of one (and hence any point) in $C(p)$ is conjugate to G_1 in G.

The next result follows from the holomorphic slice theorem (see [4]).

Proposition and Definition 1.1. *Let X be a holomorphic G-manifold such that the analytic Hilbert quotient $\pi_G : X \to X/\!\!/G$ exists. Then, each connected component of the set of points of orbit type (G_1) is a locally closed manifold and the corresponding decomposition of $X/\!\!/G$ is a stratification of X which we call the* orbit type stratification of $X/\!\!/G$.

We obtain a related G-invariant complex stratification of X by stratifying the preimage $\pi_G^{-1}(S_\gamma)$ of each orbit type stratum $S_\gamma \subset X/\!\!/G$ as a complex space.

Over a stratum S_γ, the structure of X and of the quotient $\pi_G : X \to X/\!\!/G$ is particularly simple. More precisely, let S_γ be a stratum of $X/\!\!/G$ and let (G_0) be the conjugacy class of isotropy groups corresponding to S_γ. Let us assume for the moment that $X/\!\!/G = S_\gamma$. Then, the slice theorem implies that each point $q \in X/\!\!/G$ has an open neighbourhood U such that $\pi_G^{-1}(U)$ is G-equivariantly biholomorphic to $G \times_{G_0} Z$, where Z is a locally closed G_0-stable Stein submanifold of X. The union of the closed orbits in $\pi_G^{-1}(U)$ is equal to $G \times_{G_0} Z^{G_0} \cong G/G_0 \times Z^{G_0}$. Noticing that $U \cong (G \times_{G_0} Z)/\!\!/G \cong Z/\!\!/G_0 \cong Z^{G_0}$, we see that the set of closed orbits in $\pi_G^{-1}(S_\gamma)$ is a smooth fibre bundle over S_γ with typical fibre G/G_0 and structure group G.

If $X/\!\!/G$ is irreducible, there exists a Zariski-open and dense stratum S_{princ} in $X/\!\!/G$, called the *principal stratum*. It corresponds to the minimal conjugacy class (G_0) of isotropy groups of closed orbits, i.e., if x is any point X with closed G-orbit, G_0 is conjugate in G to a subgroup of G_x.

In the Hamiltonian setup there is a stratification related to the action of K on the momentum zero fibre: let X be a Hamiltonian Kähler $K^{\mathbb{C}}$-space. Assume that $X = X(\mathscr{M})$. Decompose the quotient $\mathscr{M}/K$ by orbit types for the action of K on $\mathscr{M}$. It has been shown in a purely symplectic setup in [17] that this defines a stratification of $\mathscr{M}/K$. Since $\mathscr{M}/K$ is homeomorphic to $X/\!\!/K^{\mathbb{C}}$ (see Diagram (1)) this defines a second decomposition of $X/\!\!/K^{\mathbb{C}}$. However, it can be shown that the two constructions yield the same result: given a stratum S corresponding to K-orbit type (K_1), it coincides with one of the connected components of the set of points in $X/\!\!/K^{\mathbb{C}}$ with orbit type $(K_1^{\mathbb{C}})$ (see [16]).

1.3 Kählerian reduction

In this section we recall the basic definitions necessary for Kählerian reduction theory. Due to the presence of singularities in the spaces under consideration, Kähler structures are defined in terms of strictly plurisubharmonic functions.

Definition 1.2. Let Z be a complex space. A continuous function $\rho : Z \to \mathbb{R}$ is called *plurisubharmonic*, if for every holomorphic map φ from the unit disc D in $\mathbb{C}$ to Z, the pullback $\varphi^*(\rho)$ is subharmonic on D, i.e., for each $0 < r < 1$ the *mean value inequality*

$$\varphi^*(\rho)(0) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi^*(\rho)(re^{i\theta})\,\mathrm{d}\theta$$

holds.

A *perturbation* of a continuous function $\rho : Z \to \mathbb{R}$ at a point $x \in Z$ is a function $\rho + f$, where f is smooth and defined in some neighbourhood U of x. The function ρ is said to be *strictly plurisubharmonic* if for every perturbation $\rho + f$ there exist $\varepsilon > 0$ and a perhaps smaller neighbourhood V of x such that $\rho + \varepsilon f$ is plurisubharmonic on V.

Remark 1.1. If Z is a complex manifold and $\rho : Z \to \mathbb{R}$ is smooth, then ρ is strictly plurisubharmonic if and only if its Levi form $(\mathrm{i}/2)\,\partial\bar{\partial}\rho$ is positive definite, i.e., if it defines a Kähler form.

Definition 1.3. A *Kähler structure* on a complex space Z is given by an open cover (U_j) of Z and a family of strictly plurisubharmonic functions $\rho_j : U_j \to \mathbb{R}$ such that the differences $\rho_j - \rho_k$ are *pluriharmonic* on $U_{jk} := U_j \cap U_k$ in the sense that there exists a holomorphic function $f_{jk} \in \mathscr{O}_Z(U_{jk})$ with $\rho_j - \rho_k = \mathfrak{Re}(f_{jk})$. Two Kähler structures (U_j, ρ_j) and $(\widetilde{U}_k, \widetilde{\rho}_k)$ are considered equal if there exists a common refinement (V_l) of (U_j) and $(\widetilde{U}_k)$ such that $(\rho_l - \widetilde{\rho}_l)|_{V_l}$ is pluriharmonic for every l.

Again, we remark that the definition made above coincides with the usual definition of a Kähler form on a complex manifold if all the ρ_j's are assumed to be smooth. For more information on strictly plurisubharmonic functions we refer to [18] and [1].

We have seen that analytic Hilbert quotients of manifolds have a natural stratification by orbit types. Taking into account this additional structure, we make the following definition.

Definition 1.4. A complex space Z is called a *stratified Kähler space* if there exists a complex stratification $\mathbb{S} = (S_\gamma)_{\gamma \in \Gamma}$ on Z which is finer than the stratification of Z as a complex space and there exists a Kähler structure $\omega = (U_\alpha, \rho_\alpha)_{\alpha \in I}$ on Z such that $\rho_\alpha|_{S_\gamma \cap U_\alpha}$ is smooth.

The following theorem is a special case of results proven in [5] and [7].

Theorem 1.2. *Let K be a compact Lie group and $G = K^{\mathbb{C}}$ its complexification. Let X be a Hamiltonian Kähler $K^{\mathbb{C}}$-manifold with $X = X(\mathscr{M})$. We denote the quotient map by $\pi_G : X \to X/\!\!/G$. Let $\mathbb{S}_G^{X/\!/G}$ be the orbit type stratification of $X/\!\!/G$ defined above. Then there exists a Kähler structure $\widetilde{\omega} = (U_\alpha, \widetilde{\rho}_\alpha)_{\alpha \in I}$ on $X/\!\!/G$ with the following properties.*

1. *The triple $(X, \mathbb{S}_G^{X/\!/G}, \widetilde{\omega})$ is a stratified Kähler space.*
2. *There exist smooth functions $\rho_\alpha : \pi_H^{-1}(U_\alpha) \to \mathbb{R}$ with $\omega|_{\pi_H^{-1}(U_\alpha)} = 2i\partial\bar{\partial}(\rho_\alpha)$ such that the following equality holds: $\pi_G^*(\widetilde{\rho}_\alpha)|_{\mathscr{M} \cap \pi_G^{-1}(U_\alpha)} = \rho_\alpha|_{\mathscr{M} \cap \pi_G^{-1}(U_\alpha)}$.*

We recall the construction of the reduced Kähler structure. With the notation of the previous theorem, the key technical result for the construction can be stated as follows:

Proposition 1.1. *Let $x \in \mathscr{M}$. Then, there exists a π_G-saturated neighbourhood U of x in X such that the Kähler structure ω is given by a smooth strictly plurisubharmonic function $\rho : U \to \mathbb{R}$. The function ρ is an exhaustion along every fibre of π_G that is contained in U. Furthermore, the restriction of the momentum map μ to U fulfills*

$$\mu^{\xi}(x) = \mu^{\xi}_{\rho}(x) := \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \rho(\exp(\mathrm{i}t\xi)\cdot x) \qquad \forall \xi \in \mathfrak{k}, \forall x \in U\ . \tag{2}$$

The set $\mathscr{M} \cap U$ coincides with the set of critical points for the restriction of ρ to fibres of the quotient map π_G.

Remark 1.2. If a momentum map μ fulfills Equation (2) for some function ρ, we say that μ *is associated with* ρ and we write $\mu = \mu_\rho$.

The Kähler structure on the quotient $X(\mathscr{M})/\!\!/G$ is constructed as follows: every point $y \in X/\!\!/G$ has a neighbourhood of the form $U/\!\!/G$ which has the properties of Proposition 1.1. The restriction of ρ to $\mathscr{M} \cap U$ induces a continuous function $\widetilde{\rho}$ on $U/\!\!/G$. We show that it is strictly plurisubharmonic and smooth along the strata of $\mathbb{S}_G^{X/\!\!/G}$. Let S be a stratum. Let Y be the set of closed G-orbits in $\pi_G^{-1}(S)$. This is smooth and contains $\mathscr{M} \cap \pi_G^{-1}(S)$ as a smooth submanifold. We know that ω and hence ρ are smooth along Y. The quotient map $\pi_K : \mathscr{M} \cap \pi_G^{-1}(S) \to S$ is a smooth submersion, hence $\widetilde{\rho}$ is smooth along each stratum S. Over S, we can assume that $X = Y = G/\Sigma \times X/\!\!/G$. For $z_0 \in \mathscr{M}$, we have

$$T_{z_0}(\mathscr{M}) = T_{z_0}(K\cdot z_0) \oplus T_{z_0}(K^{\mathbb{C}}\cdot z_0)^{\perp}\ ,$$

where $\perp$ denotes the perpendicular with respect to the Riemannian metric associated with ω. This allows us to construct a smooth section $\sigma : X/\!\!/G \to G/\Sigma \times X/\!\!/G$ for π with image in $\mathscr{M}$ in such a way that the differential of σ at $\pi(z_0)$ is a complex linear isomorphism from $T_{\pi(z_0)}(X/\!\!/G)$ to $T_{z_0}(K^{\mathbb{C}}\cdot z_0)^{\perp}$. Let $\sigma(w) = (\eta(w), w)$ and let $\widetilde{f}$ be a smooth function near $\pi(z_0)$. Since ρ is strictly plurisubharmonic there exists an $\varepsilon > 0$ such that $\rho_\varepsilon := \rho + \varepsilon(\widetilde{f}\circ\pi)$ is plurisubharmonic on X. By construction, $\widetilde{\rho}_\varepsilon := \widetilde{\rho} + \varepsilon\widetilde{f}$ is equal to $\rho_\varepsilon \circ \sigma$. Now use the chain rule, the fact that $\rho_\varepsilon|_{K^{\mathbb{C}}\cdot z}$ is critical at $K^{\mathbb{C}}\cdot z \cap \mathscr{M}$ and $\mathrm{d}\eta(\pi(z_0)) = 0$ to show that

$$\frac{\partial^2\widetilde{\rho}_\varepsilon}{\partial w_i \partial \bar{w}_j}(\pi(z_0)) = \frac{\partial^2\rho_\varepsilon}{\partial w_i \partial \bar{w}_j}(z_0)\ .$$

It follows from the considerations above that $\widetilde{\rho}_\varepsilon$ is plurisubharmonic on each stratum $S \subset X/\!\!/G$. Since $\widetilde{\rho}_\varepsilon$ extends continuously to $X/\!\!/G$, the results of [2] imply that $\widetilde{\rho}_\varepsilon$ is plurisubharmonic on $X/\!\!/G$. Hence, $\widetilde{\rho}$ is strictly plurisubharmonic.

Covering $X /\!\!/ G$ with sets $U_\alpha /\!\!/ G$, the corresponding strictly plurisubharmonic functions $\widetilde{\rho}_\alpha$ fit together to define a Kähler structure on $X /\!\!/ G$ with the desired smoothness properties. We emphasize at this point that the induced Kähler structure does not depend on the choice of the pairs (U_α, ρ_α).

Remark 1.3. Inspecting the proof that we have outlined above, one sees that Kählerian reduction works under the following weaker regularity assumptions: as before, let K be a compact Lie group and $G = K^{\mathbb{C}}$ its complexification. Let X be a holomorphic Kähler G-space with analytic Hilbert quotient $\pi_G : X \to X /\!\!/ G$. Let $\mathbb{S}$ be a stratification of $X /\!\!/ G$ which is finer than the stratification of $X /\!\!/ G$ as a complex space and finer than the decomposition of $X /\!\!/ G$ by G-orbit types. Assume that the Kähler structure of X is smooth along every G-orbit in X, on a Zariski-open smooth subset X_{reg} of X and along the set of closed orbits in $\pi_G^{-1}(S)$ for each stratum S of $X /\!\!/ G$. Assume that the Kähler structure is K-invariant and that there exists a continuous map $\mu : X \to \mathfrak{k}^*$ which is a smooth momentum map for the K-action on X_{reg} as well as on every G-orbit. In this situation, we call X a *stratified Hamiltonian Kähler $K^{\mathbb{C}}$-space*. Assume that $X = X(\mathscr{M})$. Then, the fundamental diagram (1) holds (in particular, $X /\!\!/ G$ is homeomorphic to $\mu^{-1}(0)/K$) and the construction outlined above yields a Kähler structure on $X /\!\!/ G$ which is smooth along the stratification $\mathbb{S}$.

The following example shows that even if the quotient is a smooth manifold, we cannot expect the reduced Kähler structure to be smooth. This illustrates that the reduction procedure is also sensitive to singularities of the map $\mathscr{M} \to \mathscr{M}/K$.

Example 1.2. Consider $X = \mathbb{C}^2$ with the action of $\mathbb{C}^* = (S^1)^{\mathbb{C}}$ given by $t \bullet (z,w) = (tz, t^{-1}w)$. The standard Kähler form on $\mathbb{C}^2$ can be written as $(i/2)\partial\bar{\partial}\rho$, where $\rho : v \mapsto \|v\|^2$ denotes the square of the norm function associated with the standard Hermitian product on $\mathbb{C}^2$. After identification of $\mathrm{Lie}(S^1)^*$ with $\mathbb{R}$, the momentum map associated with ρ is given by $\mu(z,w) = |z|^2 - |w|^2$. It follows that the momentum zero fibre is singular at the origin. Every point in $\mathbb{C}^2$ is semistable and the analytic Hilbert quotient is realised by $\pi : \mathbb{C}^2 \to \mathbb{C}, (z,w) \mapsto zw$. Restriction of ρ to $\mu^{-1}(0)$ induces the function $\widetilde{\rho}(\widetilde{z}) = |\widetilde{z}|$ on $\mathbb{C}^2 /\!\!/ \mathbb{C}^* = \mathbb{C}$. It is continuous strictly plurisubharmonic and smooth along the orbit type stratification of $\mathbb{C}$.

2 Reduction in steps

From now on we consider the following situation. Let K be a connected compact Lie group, $G = K^{\mathbb{C}}$ its complexification, and let X be a connected Hamiltonian Kähler $K^{\mathbb{C}}$-manifold with K-equivariant momentum map $\mu : X \to \mathfrak{k}^*$. Let L be a closed normal subgroup of K. Then, $L^{\mathbb{C}} =: H$ is contained in $K^{\mathbb{C}}$ as a closed normal complex subgroup. The inclusion $\iota : \mathfrak{l} \to \mathfrak{k}$ of $\mathfrak{l} = \mathrm{Lie}(L)$ into $\mathfrak{k}$ induces an adjoint map $\iota^* : \mathfrak{k}^* \to \mathfrak{l}^*$. The composition $\mu_L : X \to \mathfrak{l}^*$, $\mu_L := \iota^* \circ \mu$ is a momentum map for the action of L on X. Set $\mathscr{M}_L := \mu_L^{-1}(0)$. There is a corresponding set of semistable points $X(\mathscr{M}_L) := \{x \in X \mid \overline{H \bullet x} \cap \mathscr{M}_L \neq \emptyset\}$, and an analytic Hilbert quotient

$\pi_H : X(\mathscr{M}_L) \to X(\mathscr{M}_L)/\!\!/H$. In the following we investigate the relations between this quotient and the quotient $\pi_G : X(\mathscr{M}) \to X(\mathscr{M})/\!\!/G$.

The main result we show here is the following.

Theorem 2.1 (Kählerian reduction in steps). *With the notation introduced above, the following holds.*

1. *The analytic Hilbert quotient $Q_L := X(\mathscr{M})/\!\!/H$ exists and is realised as an open subset of $X(\mathscr{M}_L)/\!\!/H$. There is a holomorphic G-action on Q_L such that the quotient map $\pi_H : X(\mathscr{M}) \to Q_L$ is G-equivariant. The analytic Hilbert quotient $\bar{\pi} : Q_l \to Q_L/\!\!/G$ exists. It is naturally biholomorphic to $X(\mathscr{M})/\!\!/G$ and the following diagram commutes.*

$$\begin{tikzcd}
X(\mathscr{M}_L) \arrow[d, "\pi_H"'] & X(\mathscr{M}) \arrow[l, hook'] \arrow[r, "\pi_G"] \arrow[d, "\pi_H"'] & X(\mathscr{M})/\!\!/G \\
X(\mathscr{M}_L)/\!\!/H & Q_L \arrow[l, hook'] \arrow[ur, "\bar{\pi}"'] &
\end{tikzcd}$$

2. *The restriction of the momentum map μ to $\mathscr{M}_L$ is L-invariant and induces a momentum map for the K-action on Q_L. This makes Q_L into a stratified Hamiltonian Kähler $K^{\mathbb{C}}$-space. The analytic Hilbert quotient $Q_L/\!\!/G$ carries a Kählerian structure induced by the Kählerian structure of Q_L. This Kählerian structure coincides with the Kählerian structure obtained by reduction for the quotient $\pi_K : \mathscr{M} \to \mathscr{M}/K \simeq X(\mathscr{M})/\!\!/G$.*
3. *The G-orbit type stratification of $X(\mathscr{M})/\!\!/G$ coincides with the 2-step stratification which is defined in Section 2.2.*

2.1 Analytic reduction in steps

In this section we prove part (1) of Theorem 2.1.

Theorem 2.2 (Analytic reduction in steps). *Let X be a holomorphic G-space for the complex-reductive Lie group G and let $H \lhd G$ be a reductive normal subgroup. Assume that the analytic Hilbert quotient $\pi_G : X \to X/\!\!/G$ exists. Then, the analytic Hilbert quotient $\pi_H : X \to X/\!\!/H$ exists and admits a holomorphic G-action such that π_H is G-equivariant. Furthermore, the analytic Hilbert quotient of $X/\!\!/H$ by G exists and is naturally biholomorphic to $X/\!\!/G$. If $\bar{\pi} : X/\!\!/H \to X/\!\!/G$ denotes the quotient map, the diagram*

$$\begin{tikzcd}
X \arrow[r, "\pi_G"] \arrow[d, "\pi_H"'] & X/\!\!/G \\
X/\!\!/H \arrow[ur, "\bar{\pi}"'] &
\end{tikzcd} \tag{3}$$

commutes.

Proof. The quotient map $\pi_G : X \to X/\!\!/G$ is an H-invariant Stein map. Hence, the quotient $X/\!\!/H$ exists (see [8]). Let $\pi_H : X \to X/\!\!/H$ be the quotient map. The map

$$\mathrm{id}_G \times \pi_H : G \times X \to G \times X/\!\!/H$$

is an analytic Hilbert quotient for the H-action on $G \times X$ which is given by the action of H on the second factor. Since H is a normal subgroup of G, the map that is obtained by composition of the action map $G \times X \to X$ with the quotient map π_H is an H-invariant holomorphic map from $G \times X$ to $X/\!\!/H$. By the universal property of analytic Hilbert quotients we obtain a holomorphic map $G \times X/\!\!/H \to X/\!\!/H$ such that the following diagram commutes,

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\mathrm{id}_G \times \pi_H"'] & X \arrow[d, "\pi_H"'] \\ G \times X/\!\!/H \arrow[r] & X/\!\!/H\,. \end{tikzcd}$$

This defines a holomorphic action of G on $X/\!\!/H$ such that π_H is G-equivariant.

The G-invariant map $\pi_G : X \to X/\!\!/G$ descends to a G-invariant map $\bar{\pi} : X/\!\!/H \to X/\!\!/G$. We claim that $\bar{\pi}$ is Stein. Indeed, let $U \subset X/\!\!/G$ be Stein. Since π_G is an analytic Hilbert quotient map, the inverse image $\pi_G^{-1}(U)$ is a π_H-saturated Stein subset of X. Since analytic Hilbert quotients of Stein spaces are Stein (see [4]), $\pi_H(\pi_G^{-1}(U)) = \bar{\pi}^{-1}(U)$ is a Stein open set in $X/\!\!/H$. Furthermore, using the fact that $\bar{\pi}$ is induced by π_G and that π_H is G-equivariant, we see that $\mathscr{O}_{X/\!\!/G} = (\bar{\pi}_* \mathscr{O}_{X/\!\!/H})^G$. This shows that the map $\bar{\pi} : X/\!\!/H \to X/\!\!/G$ is the analytic Hilbert quotient of $X/\!\!/H$ by the action of G. $\square$

In the situation of Theorem 2.1, it follows from the previous theorem that the analytic Hilbert quotient of $X(\mathscr{M})$ by the action of a normal complex-reductive subgroup H of G exists. For our purposes, it is important to relate this quotient to the quotient of $X(\mathscr{M}_L)$ by H and to the momentum geometry of μ_L. For later reference, we make the following.

Definition 2.1. Let X be a holomorphic G-space and $A \subset X$ a subset. We define $\mathscr{S}_G(A) := \{x \in X \mid \overline{G \cdot x} \cap A \neq \emptyset\}$ and call it the *saturation* of A with respect to G. If we explicitly want to indicate the space X on which the group G acts, we also write $\mathscr{S}_G^X(A)$.

Lemma 2.1. *Let L be a compact subgroup of K and $H := L^{\mathbb{C}} \subset G = K^{\mathbb{C}}$. Then, $X(\mathscr{M})$ is a π_H-saturated subset of $X(\mathscr{M}_L)$.*

Proof. Let $x_0 \in X(\mathscr{M})$. By Proposition 1.1 there exists a strictly plurisubharmonic exhaustion function ρ of the fibre $F := \pi_G^{-1}(\pi_G(x_0))$ with the property that $\mu|_F$ is associated with ρ. This implies that $\mathscr{M} \cap F$ is the set where $\rho|_F$ assumes its minimum. The restriction of ρ to $C := \overline{H \cdot x_0} \cap X(\mathscr{M}) \subset F$ also is an exhaustion. This implies that $\rho|_C$ attains its minimum at some point $y_0 \in \overline{H \cdot x_0}$. For all $\xi \in \mathfrak{l}$, we have $\mu_L^{\xi}(y_0) = \frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} \rho(\exp(\mathrm{i}t\xi) \cdot y_0) = 0$. Hence, $y_0 \in \mathscr{M}_L$ and $x_0 \in X(\mathscr{M}_L)$.

The first part of the proof shows that $X(\mathscr{M}) \subset \mathscr{S}_H(X(\mathscr{M}) \cap \mathscr{M}_L)$. Conversely, let $x \in \mathscr{S}_H(X(\mathscr{M}) \cap \mathscr{M}_L)$. Then, by definition, $\overline{H \bullet x} \cap (X(\mathscr{M}) \cap \mathscr{M}_L) \neq \emptyset$. Since $X(\mathscr{M})$ is an open H-invariant neighbourhood of $X(\mathscr{M}) \cap \mathscr{M}_L$, this implies $x \in X(\mathscr{M})$. Hence, we have shown that $X(\mathscr{M}) = \mathscr{S}_H(X(\mathscr{M}) \cap \mathscr{M}_L)$. This concludes the proof. □

As before, let $\pi_G : X(\mathscr{M}) \to X(\mathscr{M})/\!\!/G$ and $\pi_H : X(\mathscr{M}_L) \to X(\mathscr{M}_L)/\!\!/H$ denote the quotient maps. From Lemma 2.1, we obtain the following.

Corollary 2.1. *The analytic Hilbert quotient for the H-action on $X(\mathscr{M})$ is given by $\pi_H|_{X(\mathscr{M})} : X(\mathscr{M}) \to \pi_H(X(\mathscr{M})) \subset X(\mathscr{M}_L)/\!\!/H$. In particular, $X(\mathscr{M})/\!\!/H$ can be realised as an open subset of $X(\mathscr{M}_L)/\!\!/H$.*

Theorem 2.2 and Corollary 2.1 prove part (1) of Theorem 2.1.

As a preparation for the proof of part (2) of Theorem 2.1, we now relate the preceding discussion to the actions of the groups K and L.

Lemma 2.2. *The momentum zero fibre $\mathscr{M}_L$ of μ_L as well as the set of semistable points $X(\mathscr{M}_L)$ is K-invariant.*

Proof. Let $x \in \mathscr{M}_L$. Since L is normal in K, we have $\mathrm{Ad}(K)(\mathfrak{l}) \subset \mathfrak{l}$. This implies $\mu^{\xi}(k \bullet x) = \mathrm{Ad}^*(k)(\mu(x)) = 0$. Hence, we have $k \bullet x \in \mathscr{M}_L$.

Since $\mathscr{M}_L$ is K-invariant and $kH = Hk$ for all $k \in K$, we have $\overline{H \bullet (k \bullet x)} \cap \mathscr{M}_L = k \bullet (\overline{H \bullet x} \cap \mathscr{M}_L)$ for all $x \in X(\mathscr{M}_L)$. This shows the claim. □

Since $X(\mathscr{M}_L)$ is K-invariant, it follows by considerations analogous to those in the proof of Theorem 2.2 that $X(\mathscr{M}_L)/\!\!/H$ is a complex K-space. On $\pi_H(X(\mathscr{M}))$ the K-action coincides with the restriction of the $K^{\mathbb{C}}$-action to K. However, we do not know if the action defined in this way on $\pi_H(X(\mathscr{M})) \subset X(\mathscr{M}_L)/\!\!/H$ extends to a holomorphic action of $K^{\mathbb{C}}$ on $X(\mathscr{M}_L)/\!\!/H$ in general.

In two important special cases, there is a holomorphic $K^{\mathbb{C}}$-action on $X(\mathscr{M}_L)/\!\!/H$. If $X(\mathscr{M}_L)/\!\!/H$ is compact, its group of holomorphic automorphisms $\mathscr{A}$ is a complex Lie group. The action of K on $X(\mathscr{M}_L)/\!\!/H$ yields a homomorphism of K into $\mathscr{A}$. This extends to a holomorphic homomorphism of $K^{\mathbb{C}}$ into $\mathscr{A}$ by the universal property of $K^{\mathbb{C}}$, hence $K^{\mathbb{C}}$ acts holomorphically on $X(\mathscr{M}_L)/\!\!/H$. The second case is the following. As we have seen in Lemma 2.2, the complement of $X(\mathscr{M}_L)$ in X is K-invariant. If it is an analytic subset of X, its K-invariance implies its $K^{\mathbb{C}}$-invariance. In this case, $K^{\mathbb{C}}$ acts on $X(\mathscr{M}_L)$ and hence on $X(\mathscr{M}_L)/\!\!/H$.

Lemma 2.3. *We have $\mathscr{S}_G^{\pi_H(X(\mathscr{M}))}(\pi_H(\mathscr{M})) = \pi_H(X(\mathscr{M}))$.*

Proof. Let $x \in X(\mathscr{M})$. Then, by definition, $\overline{G \bullet x} \cap \mathscr{M} \neq \emptyset$. Since $\overline{G \bullet \pi_H(x)} \supset \pi_H(\overline{G \bullet x})$, this implies that $\overline{G \bullet \pi_H(x)}$ intersects $\pi_H(\mathscr{M})$ nontrivially. □

2.2 Kählerian reduction in steps

In this section we prove part (2) of Theorem 2.1. The results of Section 2.1 show that it is sensible to restrict to the situation where $X = X(\mathscr{M}) = X(\mathscr{M}_L)$ for the discussion of Kählerian reduction in this section and we do this from now on.

First we take a closer look at the compatibility of the G-action on $X/\!\!/H = X(\mathscr{M})/\!\!/H$ with the orbit type stratification $\mathbb{S}_H^{X/\!\!/H}$ of $X/\!\!/H$. For later reference, we note the following basic observation.

Lemma 2.4. *Let M be a Lie group with finitely many connected components and let M_0 be a closed subgroup of M. Then, the following are equivalent.*

1. *M and M_0 are isomorphic as topological groups.*
2. *$M_0 = M$.*

As a first application, we get the following.

Lemma 2.5. *The stratification $\mathbb{S}_H^{X/\!\!/H}$ is G-invariant.*

Proof. Let $z \in \mathscr{M}_L$ and consider the H-action on $G \bullet z$. The orbit $H \bullet z$ is closed. Since $H \bullet g \bullet z = g \bullet H \bullet z$ holds for all $g \in G$, all H-orbits in $G \bullet z$ are closed (and have the same dimension). Since $G \bullet z$ is connected, there is a principal H-stratum S in $G \bullet z$ and we may assume $z \in S$. For any $g \in G$ this yields

$$hH_zh^{-1} \subset H_{g \bullet z} = g(H \cap G_z)g^{-1} = gH_zg^{-1} \text{ for some } h \in H\,.$$

Lemma 2.4 implies $hH_zh^{-1} = gH_zg^{-1} = H_{g \bullet z}$ and therefore $G \bullet z = S$. □

We now investigate the compatibility of the $K^{\mathbb{C}}$-action on $X/\!\!/H$ with the induced Kähler structure.

Proposition 2.1. *In the setup of Theorem 2.1 assume additionally that $X = X(\mathscr{M}) = X(\mathscr{M}_L)$ holds. Then, we have:*

1. *The reduced Kähler structure of $X/\!\!/H$ is smooth along each G-orbit in $X/\!\!/H$.*
2. *The reduced Kähler structure is K-invariant. The restriction of μ to $\mathscr{M}_L$ is L-invariant and induces a well-defined continuous map $\widetilde{\mu} : X/\!\!/H \to \mathfrak{k}^*$ which is a smooth momentum map on each stratum of $X/\!\!/H$ as well as on every G-orbit.*

Proof. The induced Kähler structure $\widetilde{\omega}$ is smooth along the stratification $\mathbb{S}_H^{X/\!\!/H}$ by construction. Furthermore, Lemma 2.5 shows that this stratification is G-invariant. Hence, $\widetilde{\omega}$ is smooth along every G-orbit in $X/\!\!/H$.

Let $x \in \mathscr{M}_L \subset X$. Applying Propostion 1.1 to X, to the momentum map $\mu : X \to \mathfrak{k}^*$ and to the quotient $\pi_G : X \to X/\!\!/G$, we get a π_G-saturated neighbourhood U of x in X on which the Kähler structure and the momentum maps μ and μ_L are induced by a K-invariant strictly plurisubharmonic function $\rho : U \to \mathbb{R}$. Since the Kähler structure on $\pi_H(U) \subset X/\!\!/H$ is induced by the restriction of ρ to $\mathscr{M}_L \cap U$ it is K-invariant.

For the L-invariance of $\mu|_{\mathscr{M}_L}$ we follow [17]: first, we equivariantly identify $\mathfrak{k}$ with $\mathfrak{k}^*$. The image of $\mu : \mathscr{M}_L \to \mathfrak{k}$ is contained in $\mathfrak{l}^\perp \subset \mathfrak{k}$. Here, $^\perp$ denotes the perpendicular with respect to a chosen K-invariant inner product on $\mathfrak{k}$. Hence, the equation $\mu(l \cdot x) = \mathrm{Ad}(l)(\mu(x))$ implies that it is sufficient to show that L acts trivially on $\mathfrak{l}^\perp$. As $\mathfrak{l}^\perp$ is L-invariant, $[\mathfrak{l}, \mathfrak{l}^\perp]$ is contained in $\mathfrak{l}^\perp$. But $\mathfrak{l}$ is an ideal in $\mathfrak{k}$ and hence, $[\mathfrak{l}, \mathfrak{l}^\perp]$ is also contained in $\mathfrak{l}$. This implies that the component of the identity L_0 of L acts trivially on $\mathfrak{l}^\perp$. Since K is connected, the image of the map $\phi : K \times L_0 \to K, (k,l) \mapsto klk^{-1}$ lies in some connected component of L, which has to be L_0. Hence, L_0 is normal in K. The finite group L/L_0 is normal in the connected group K/L_0, hence central. It follows that L/L_0 acts trivially on $\mathrm{Lie}(K/L_0) = \mathfrak{l}^\perp$. This shows that L acts trivially on $\mathfrak{l}^\perp$.

Hence, $\mu|_{\mathscr{M}_L}$ defines a continuous map $\widetilde{\mu}$ on $X/\!\!/H$. This map is a smooth momentum map along each stratum S of $X/\!\!/H$, since $\pi_L : \mathscr{M}_L \cap \pi_H^{-1}(S) \to \widetilde{S}$ is a smooth K-equivariant submersion. Since the stratification is G-invariant, $\widetilde{\mu}$ is a smooth momentum map along each G-orbit in $X/\!\!/H$. $\square$

Remark 2.1. By construction, we see that $\widetilde{\mathscr{M}} := \widetilde{\mu}^{-1}(0)$ is equal to $\pi_H(\mathscr{M})$. Furthermore, Lemma 2.3 shows that $\mathscr{S}_G(\widetilde{\mathscr{M}}) = X/\!\!/H$.

We now investigate the possibility of carrying out Kähler reduction with respect to the quotient $\bar{\pi} : X/\!\!/H \to X/\!\!/G$.

First we define a second stratification of $X/\!\!/G$ as follows: let S be a stratum of the H-orbit type stratification of $X/\!\!/H$ such that $S \cap \widetilde{\mathscr{M}} \neq \emptyset$. Stratify $\bar{\pi}(\overline{S}) \setminus \bar{\pi}(\partial S)$ by G-orbit types with respect to the G-action on $X/\!\!/H$. Repeating this procedure for all strata yields a stratification of $X/\!\!/G$ which we call the 2-*step-stratification*. We see later on that it coincides with the stratification of $X/\!\!/G$ by G-orbit types (see Section 2.3).

We notice that by Proposition 2.1 and by the construction of the 2-step stratification $X/\!\!/H$ is a stratified Hamiltonian Kähler $K^{\mathbb{C}}$-space. We also note that by the fundamental diagram (1), $X/\!\!/G$ is homeomorphic to $\widetilde{\mathscr{M}}/K$. Applying the procedure described in Section 1.3, we have the following.

Proposition 2.2. *The analytic Hilbert quotient $X/\!\!/G$ carries a Kähler structure induced from $X/\!\!/H$ by Kählerian reduction for the quotient $\widetilde{\pi}_K : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{M}}/K$. This Kähler structure is smooth along the* 2*-step stratification.*

We now have two Kählerian structures on the complex space $X/\!\!/G$: the structure ω_{red} that we get by reducing the Kähler form ω on X to $X/\!\!/G$ and the Kähler structure $\widehat{\omega}$ that we get by first reducing ω to the Kähler structure $\widetilde{\omega}$ on $X/\!\!/H$ and then reducing $\widetilde{\omega}$ to $\widehat{\omega}$ on $X/\!\!/G$. In order to complete the proof of Theorem 2.1(b), we have to prove that ω_{red} and $\widehat{\omega}$ coincide.

Let $y_0 \in X/\!\!/G$. Proposition 1.1 yields a open neighbourhood U of y_0 in $X/\!\!/G$ such that on $\pi_G^{-1}(U)$ the Kähler structure and the momentum maps μ and μ_L are given by ρ. It follows that the Kähler structure on $\bar{\pi}^{-1}(U)$ is given by $\widetilde{\rho}$ which is induced via $\rho|_{\mathscr{M}_L \cap \pi_G^{-1}(U)}$.

Lemma 2.6. *$\widetilde{\rho}$ is an exhaustion along every fibre of $\bar{\pi}$. On $\bar{\pi}^{-1}(U)$, we have $\widetilde{\mu} = \mu_{\widetilde{\rho}}$.*

Proof. Let $y \in U$ and let $F := \pi_G^{-1}(y)$. We have $\mathscr{M} \cap F = K \cdot x$ for some $x \in F$. The restriction $\rho|_F$ is an exhaustion. It is minimal along $\mathscr{M} \cap F = K \cdot x$. Since $\widetilde{F} := \bar{\pi}^{-1}(y) = \pi_H(F)$ and since $\widetilde{\rho}|_{\widetilde{F}}$ is induced via the restriction of ρ to $\mathscr{M}_L \cap F$, $\widetilde{\rho}|_{\widetilde{F}}$ is minimal along $\widetilde{\mathscr{M}} \cap \widetilde{F} = K \cdot \pi_H(x)$. It follows that $\widetilde{\rho}|_{\widetilde{F}}$ is an exhaustion.

Since both $\widetilde{\mu}$ and $\mu_{\widetilde{\rho}}$ are momentum maps for the K-action on $\bar{\pi}^{-1}(U)$, they differ by a constant $c \in \mathfrak{k}^*$. Hence, it suffices to show that there exists a $z \in \bar{\pi}^{-1}(U)$ such that $\widetilde{\mu}(z) = \mu_{\widetilde{\rho}}(z)$ holds. Let $z_0 \in \widetilde{\mathscr{M}} \cap \bar{\pi}^{-1}(U)$. As we have seen above, $\widetilde{\rho}|_{\bar{\pi}^{-1}(\bar{\pi}(z_0))}$ is critical along $K \cdot z_0$. Therefore, $\mu_{\widetilde{\rho}}(z_0) = \widetilde{\mu}(z_0) = 0$ and hence $c = 0$. □

The equality of the two Kähler structures on U now follows from the construction of reduced structures: the Kähler structure ω_{red} is formed by a single function ρ_{red} which is induced by the restriction of ρ to $\mathscr{M} \cap \pi_G^{-1}(U)$. The Kähler structure $\widetilde{\omega}$ on $\bar{\pi}^{-1}(U)$ is given by $\widetilde{\rho}$. Lemma 2.6 shows that the Kähler structure $\widehat{\omega}$ is given by a single function $\widehat{\rho}$ that is induced by the restriction of $\widetilde{\rho}$ to $\widetilde{\mathscr{M}} \cap \bar{\pi}^{-1}(U)$. However, $\widetilde{\mathscr{M}} = \mathscr{M}/L$ and therefore, $\widehat{\rho}$ coincides with ρ_{red} after applying the homeomorphism $(\mathscr{M}/L)/K \simeq \mathscr{M}/K$. This concludes the proof of part (2) of Theorem 2.1.

2.3 Stratifications in steps

Here we prove part (3) of Theorem 2.1. We formulate it as the following.

Proposition 2.3. *The stratification of $X /\!\!/ G$ by G-orbit types coincides with the 2-step-stratification of $X /\!\!/ G$.*

Proof. We do this in two steps: first, we claim that the G-stratification of $X /\!\!/ G$ is finer than the 2-step-stratification. Let S be a stratum of the G-stratification of $X /\!\!/ G$ and let Y be the set of closed orbits in $\pi_G^{-1}(S)$. Then, since S and G are connected, Y is a connected holomorphic H-manifold.

Every H-orbit in Y is closed. Stratify Y by H-orbit types. Since Y is connected, there exists a principal orbit type (H_0) for the H-action on Y. Then, $H_0 = G_0 \cap H$, where (G_0) is the orbit type of the G-stratum S. Let $y \in Y$. We have $H_y = gH_0g^{-1}$ for some $g \in G$. Since (H_0) is the principal isotropy type, there exists an $h \in H$ such that $hH_0h^{-1} < H_y$. Lemma 2.4 implies that H_y is conjugate to H_0 in H. Hence, $\pi_H(Y)$ is a union of closed G-orbits that is contained in a single stratum $\mathbb{S}_H^{X/\!\!/H}$. Furthermore, the G-isotropy group of each element in $\pi_H(Y)$ is conjugate in G to G_0H. This implies that the G-stratification of $X /\!\!/ G$ is finer than the 2-step-stratification.

In the second step, we prove that the 2-step-stratification of $X /\!\!/ G$ is finer than the G-stratification of $X /\!\!/ G$. Let S be a stratum of the 2-step stratification of $X /\!\!/ G$. Let $w, w' \in \pi_G^{-1}(S) \cap \mathscr{M}$. Then, by construction of S, we have

1. H_w is conjugate to $H_{w'}$ in H and
2. HG_w is conjugate to $HG_{w'}$ in G.

We claim that $\dim G_w = \dim G_{w'}$. Indeed, (1) and (2) imply the following dimension equalities on the Lie algebra level.

$$\begin{aligned}\dim \mathfrak{g}_w &= \dim(\mathfrak{g}_w + \mathfrak{h}) - \dim \mathfrak{h} + \dim(\mathfrak{g}_w \cap \mathfrak{h}) \\ &= \dim(\mathfrak{g}_{w'} + \mathfrak{h}) - \dim \mathfrak{h} + \dim(\mathfrak{g}_{w'} \cap \mathfrak{h}) \\ &= \dim \mathfrak{g}_{w'} \ .\end{aligned}$$

Hence, $\dim G \bullet w = \dim G \bullet w' =: m_0$. The set

$$Y := \{x \in \pi_G^{-1}(S) \mid G \bullet x \ \text{ closed in } X\} = \pi_G^{-1}(S) \cap \{x \in X \mid \dim G \bullet x \leq m_0\}$$

is analytic in $\pi_G^{-1}(S)$. There exists a principal orbit type (G_0) for the action of G on Y. Let $w \in Y$ such that $(G_w) = (G_0)$. Let w' be any other point in Y. Since (G_w) is the principal orbit type for the action on Y, there exists a $g_0 \in G$ such that $g_o G_w g_0^{-1} \subset G_{w'}$. Hence, we can assume that $G_w \subset G_{w'}$. Let $\iota : G_w \to G_{w'}$ be the inclusion. Since $G_w \subset G_{w'}$, we also have $H_w \subset H_{w'}$. Together with assumption (1) and Lemma 2.4, this implies that $H_w = H_{w'}$. If we let $H_w = H_{w'}$ act on G_w and $G_{w'}$ as a normal subgroup, ι is clearly equivariant. This shows that ι induces an injective group homomorphism $\widetilde{\iota} : G_w/H_w \to G_{w'}/H_w$. For every $y \in Y$ we have the following exact sequence

$$1 \to H_y \to G_y \to G_y H/H \to 1 \ .$$

Together with assumption (2), this implies that G_w/H_w and $G_{w'}/H_w$ are isomorphic as topological groups. Since both G_w/H_w and $G_{w'}/H_w$ have a finite number of connected components, Lemma 2.4 implies that $\widetilde{\iota}$ is surjective. It follows that ι is surjective and that $G_{w'}$ is equal to G_w. This shows that the 2-step stratification is finer than the G-stratification of $X /\!\!/ G$ and we are done. □

3 Projectivity and Kählerian reduction

One source of examples of quotients for actions of complex-reductive Lie groups is Geometric Invariant Theory (GIT). In this section we explain the construction of GIT-quotients in detail, noting that it is an example of the Kählerian reduction in steps procedure yielding projective algebraic quotient spaces. In addition we discuss projectivity results for general Hamiltonian actions on complex algebraic varieties and their compatibility with reduction in steps.

A complex-reductive group $G = K^{\mathbb{C}}$ carries a uniquely determined algebraic structure making it into a linear algebraic group. In this section we study algebraic actions of complex-reductive groups G with respect to this algebraic structure. For this, we define a *complex algebraic G-variety X* to be a complex algebraic variety X together with an action of G such that the action map $G \times X \to X$ is algebraic.

Let X be a projective complex algebraic G-variety. Let L be a very ample G-linearised line bundle on X, i.e., a very ample line bundle L on X and an action of G on L by bundle automorphisms making the bundle projection G-equivariant. The action of G on X and on L induces a natural representation on $V := \Gamma(X,L)^*$ and there exists a G-equivariant embedding of X into $\mathbb{P}(V)$. Let

$$\mathcal{N}(V) = \{v \in V \mid f(v) = 0 \text{ for all nonconstant homogen } f \in \mathbb{C}[V]^G\}$$

be the nullcone of V. Here $\mathbb{C}[V]^G$ denotes the algebra of polynomials on V that are invariant under the action of G. Let $X(V) := X \setminus p(\mathcal{N}(V))$, where $p : V \setminus \{0\} \to \mathbb{P}(V)$ denotes the projection. Then, it is proven in geometric invariant theory that the analytic Hilbert quotient $\pi_G : X(V) \to X(V)/\!\!/G$ exists, that $X(V)/\!\!/G$ is a projective algebraic variety, that the quotient map is algebraic and affine, and that the algebraic structure sheaf on the quotient is the sheaf of invariant polynomials on $X(V)$ (see [12]). In this situation, we call π an *algebraic Hilbert quotient*. We will explain the relation of this algebraic quotient theory to reduction in steps. Consider the action of the complex-reductive group $G \times \mathbb{C}^*$ on V that is given by the G-representation and the action of $\mathbb{C}^*$ by multiplication. By a theorem of Hilbert, the analytic Hilbert quotient $V/\!\!/G$ exists as an affine algebraic variety. It can be embedded into a $\mathbb{C}^*$-representation space W with $\mathbb{C}[W]^{\mathbb{C}^*} = \mathbb{C}$ as a $\mathbb{C}^*$-invariant algebraic subvariety. The orbit space $W \setminus \{0\}/\mathbb{C}^*$ is a weighted projective space. It follows that the analytic Hilbert quotient of $V \setminus \mathcal{N}(V)$ by the action of $G \times \mathbb{C}^*$ exists as a projective algebraic variety. Let $C(X) := \overline{\pi^{-1}(X)} \subset V$ be the cone over X. The considerations above imply that the analytic Hilbert quotient of $C(X) \setminus \mathcal{N}(V)$ by the action of $G \times \mathbb{C}^*$ exists as a projective algebraic variety. Now we note that $X(V) = (C(X) \setminus \mathcal{N}(V))/\mathbb{C}^*$, i.e., $X(V)$ is the quotient of $C(X) \setminus \mathcal{N}(V)$ by the normal subgroup $\mathbb{C}^*$ of $G \times \mathbb{C}^*$. By Theorem 2.2, the analytic Hilbert quotient $\pi_G : X(V) \to X(V)/\!\!/G$ exists and the following diagram commutes.

$$\begin{tikzcd} C(X) \setminus \mathcal{N}(V) \arrow[r] \arrow[d, "p"'] & X(V)/\!\!/G \\ X(V) \arrow[ur, "\pi_G"'] & \end{tikzcd}$$

The variety $C(X)$ is Kähler with Kähler structure given by the square of the norm function associated with a K-invariant Hermitian product $\langle \cdot, \cdot \rangle$ on V . Furthermore, the action of $K \times S^1$ is Hamiltonian. A momentum map is given by

$$\mu^{(\xi,\sqrt{-1})}(v) = 2\mathrm{i}\langle \xi v, v\rangle + \|v\|^2 - 1 \qquad \forall \xi \in \mathfrak{k}\,.$$

Here, $\sqrt{-1}$ is considered as an element of $\mathrm{Lie}(S^1)$. The set of semistable points coincides with $C(X) \setminus \mathcal{N}(V)$. The set of semistable points for the $\mathbb{C}^*$-action and the restricted momentum map is $V \setminus \{0\}$. The Fubini–Study-metric on X is obtained by Kählerian reduction for the quotient $p : C(X) \setminus \{0\} \to X$ and the K-action on X is Hamiltonian with respect to the Fubini–Study-metric with momentum map $\widetilde{\mu}^{\xi}([v]) = 2\mathrm{i}\langle \xi v, v\rangle/\|v\|^2$. The set of semistable points with respect to $\widetilde{\mu}$ coincides with $X(V)$.

We now take a closer look at the Kählerian structure that we get by Kählerian reduction of the Fubini–Study-metric to $X(V)/\!\!/G \simeq \widetilde{\mu}^{-1}(0)/K$. We have already noted that $X(V)/\!\!/G$ is projective algebraic. It carries an ample line bundle L_{red} such that there exists an $n_0 \in \mathbb{N}$ with $\pi_G^*(L_{\mathrm{red}}) = H^{n_0}$, where H denotes the restriction of the hyperplane bundle to $X(V)$. With this notation, the reduced Kähler structure ω_{red} on $X(V)/\!\!/G$ fulfills

$$n_0 \cdot c_1(\omega_{\mathrm{red}}) = c_1(L_{\mathrm{red}}) \in H^2(X,\mathbb{R}) ,$$

where c_1 denotes the first Chern class of ω_{red} and of L_{red}, respectively. Hence, the cohomology class of the Kähler structure obtained by Kählerian reduction of the Fubini–Study-metric lies in the real span of the ample cone of $X(V)/\!\!/G$.

The discussion above shows that analytic Hilbert quotients obtained by Kählerian reduction of projective algebraic varieties are again projective algebraic if the K-actions under consideration are Hamiltonian with respect to the Fubini–Study-metric. For arbitrary Kähler forms on an algebraic variety (i.e., forms that are not the curvature form of a very ample line bundle) and associated momentum maps there is no obvious relation to ample G-line bundles and the corresponding analytic Hilbert quotients of sets of semistable points are not a priori algebraic.

The best algebraicity result for momentum map quotients known so far is proven in the first author's thesis [3].

Theorem 3.1. *Let K be a compact Lie group and $G = K^{\mathbb{C}}$. Let X be a smooth complex algebraic G-variety such that the K-action on X is Hamiltonian with respect to a K-invariant Kähler form on X with momentum map $\mu : X \to \mathfrak{k}^*$. Let $\mathscr{M} := \mu^{-1}(0)$, and let $\pi_G : X(\mathscr{M}) \to X(\mathscr{M})/\!\!/G$ denote the quotient map. If $\mathscr{M}$ is compact, the following holds.*

1. *The analytic Hilbert quotient $X(\mathscr{M})/\!\!/G$ is (the complex space associated with) a projective algebraic variety.*
2. *There exists a G-equivariant biholomorphic map Φ from $X(\mathscr{M})$ to an algebraic G-variety Y with algebraic Hilbert quotient $p_G : Y \to Y/\!\!/G$, and the induced map $\bar{\Phi} : X(\mathscr{M})/\!\!/G \to Y/\!\!/G$ is an isomorphism of algebraic varieties.*
3. *The algebraic G-variety Y is uniquely determined up to G-equivariant isomorphism of algebraic varieties.*

As before, let $L \lhd K$ be a normal closed subgroup of K and $H = L^{\mathbb{C}}$. Then, reduction in steps is compatible with the algebraicity results obtained above in the following way.

We have already seen that the analytic Hilbert quotient $\pi_H : X(\mathscr{M}) \to X(\mathscr{M})/\!\!/H$ exists. Furthermore, there exists an algebraic Hilbert quotient $p_H : Y \to Y/\!\!/H$ for the action of H on Y and the map Φ induces a G-equivariant biholomorphic map $\Phi_H : X(\mathscr{M})/\!\!/H \to Y/\!\!/H$. In addition, the algebraic Hilbert quotient $(Y/\!\!/H)/\!\!/G$ exists and it is biregular to $Y/\!\!/G$. If $\bar{p} : Y/\!\!/H \to Y/\!\!/G$ denotes the quotient map, we obtain the following commutative diagram.

$$
\begin{tikzcd}
X(\mathscr{M})/\!\!/H \arrow[dr, "\bar{\pi}"] \arrow[ddd, bend right=60, "\Phi_H"'] & \\
X(\mathscr{M}) \arrow[u, "\pi_H"'] \arrow[r, "\pi_G"] \arrow[d, "\Phi"] & X(\mathscr{M})/\!\!/G \arrow[d, "\bar{\Phi}"] \\
Y \arrow[r, "p_G"] \arrow[d, "p_H"] & Y/\!\!/G \\
Y/\!\!/H \arrow[ur, "\bar{p}"'] &
\end{tikzcd}
$$

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On Orbit Decompositions for Symmetric *k*-Varieties

A. G. Helminck[1]

Summary. Orbit decompositions play a fundamental role in the study of symmetric k-varieties and their applications to representation theory and many other areas of mathematics, such as geometry, the study of automorphic forms and character sheaves. Symmetric k-varieties generalize symmetric varieties and are defined as the homogeneous spaces G_k/H_k, where G is a connected reductive algebraic group defined over a field k of characteristic not 2, H the fixed point group of an involution σ and G_k (resp., H_k) the set of k-rational points of G (resp., H).
In this contribution we give a survey of results on the various orbit decompositions which are of importance in the study of these symmetric k-varieties and their applications with an emphasis on orbits of parabolic k-subgroups acting on symmetric k-varieties. We will also discuss a number of open problems.

Key words: Symmetric spaces, symmetric varieties, involutions of algebraic groups

Mathematics Subject Classification (2000): 20G20, 22E15, 22E46, 53C35

Dedicated to Gerry Schwarz on the occasion of his 60th birthday

1 Introduction

Symmetric k-varieties were introduced in the late 1980s as a generalization of both real reductive symmetric spaces and symmetric varieties to homogeneous spaces defined over general fields of characteristic not 2. The *real reductive symmetric spaces*, are the homogeneous spaces $G_{\mathbb{R}}/H_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is a reductive Lie group of Harish-Chandra class and $H_{\mathbb{R}}$ is an open subgroup of the fixed point group

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of an involution of $G_{\mathbb{R}}$. The representations associated with these real reductive symmetric spaces (i.e., a decomposition of $L^2(G_{\mathbb{R}}/H_{\mathbb{R}})$ into irreducible components) had been studied intensively by many prominent mathematicians starting with a study of compact groups and their representations by Cartan [16], to a study of Riemannian symmetric spaces and real Lie groups by Harish-Chandra [27] to a more recent study of the non Riemannian symmetric spaces starting in the 1970s by work of Faraut [22], Flensted Jensen [23] and Oshima and Sekiguchi [49]. These were soon studied by many mathematicians, including Brylinski, Carmona, Delorme, Matsuki, Oshima, Schlichtkrull, van der Ban and many others (see, e.g., [48, 14, 3, 15, 5, 4, 20]). In the mid 1980s a Plancherel formula for the general real reductive symmetric spaces was announced by Oshima, although a full proof was not published until 1996 by Delorme [20]. See also van der Ban and Schlichtkrull for a different approach [5, 4]. In the late 1980s it seemed natural to generalize the concept of these real reductive symmetric spaces to similar spaces over the p-adic numbers and study the representations associated with these spaces. At that same time generalizations of these real reductive symmetric spaces to other base fields started to play a role in other areas, for example in the study of arithmetic subgroups (see [59]), the study of character sheaves (see for example [45, 24]), geometry (see [18, 19] and [1]), singularity theory (see [46] and [43]), and the study of Harish-Chandra modules (see [7] and [61, 60]). This prompted Helminck and Wang to commence a study of rationality properties of these homogeneous spaces over general base fields; see [39] for some first results. For any field k of characteristic not 2 they defined a *symmetric k-variety* as the homogeneous space $X_k := G_k/H_k$, where G is a reductive algebraic group defined over k and $H = G^{\sigma}$ the fixed point group of a k-involution $\sigma : G \to G$ of G. Here we have used the notation H_k for the set of k-rational points of an affine algebraic group H defined over k. As in the real case the p-adic symmetric k-varieties are also called reductive p-adic symmetric spaces or simply p-adic symmetric spaces.

For the p-adic numbers k it is natural to study the harmonic analysis of these p-adic symmetric spaces, similar to the real case. Namely, let dx be a G_k-invariant measure on the symmetric k-variety $X_k = G_k/H_k$. Given a complex vector bundle over these spaces we get a natural representation π_ρ of G_k on its space of global sections, where ρ is the representation of H_k on the fibers. If the representation ρ of H_k is unitary, then the G_k-action on the space of global sections that are square-integrable with respect to dx is a unitary representation $(\pi_\rho, L^2(G_k/H_k, \rho))$ of G_k. In particular for the trivial line bundle over G_k/H_k, this leads to the regular representation π_1 of G_k on the Hilbert space $L^2(G_k/H_k)$ of square integrable functions on G_k/H_k. Since the group G_k is of type I, any unitary representation R of G_k on a separable Hilbert space $\mathscr{H}_R$ has an abstract direct integral decomposition

$$R \simeq \int_{\hat{G}_k}^{\oplus} R^{\pi} d\mu_R(\pi), \tag{1}$$

where $\hat{G}_k$ is the unitary dual of $G_k, d\mu_R$ a Borel measure on $\hat{G}_k, (\pi, \mathscr{H}_\pi)$ is a representative of a class in $\hat{G}_k$, and R^{π} is a multiple of π, see [21]. This holds in particular for $(\pi_\rho, L^2(G_k/H_k, \rho))$. The main aim of harmonic analysis is to decompose this representation as explicitly as possible into irreducible components, what is also called finding the “Plancherel decomposition”.

Most of the representations occurring in this decomposition are representations induced from a parabolic k-subgroup. So in order to study these representations it is essential to first have a thorough understanding of the orbits of parabolic k-subgroups acting on these symmetric k-varieties. Other decompositions which play an important role in the study of these symmetric k-varieties are orbits of symmetric subgroups, orbits of maximal k-anisotropic (compact) subgroups, and in the p-adic case also orbits of parahoric subgroups.

Although there are descriptions for some of these orbit decompositions, many properties of these symmetric k-varieties remain open. In this chapter we give a survey of both known and open results about these orbit decompositions for symmetric k-varieties, and illustrate these with a number of examples.

A brief overview of this chapter follows. In Section 2 we set the notation and review a few properties of symmetric k-varieties. In Section 3 we discuss orbits of various parabolic subgroups acting on symmetric varieties and symmetric k-varieties. These orbit decompositions involve conjugacy classes of σ-stable tori. These are discussed in Section 4. In Section 5 we discuss the combinatorics of the orbit decomposition as well as orbit closures and a Bruhat type order on these orbits. Finally in Section 6 we briefly discuss some results about other orbit decompositions and indicate some open problems.

2 Preliminaries about symmetric *k*-varieties

2.1 Notations

Given an algebraic group G, the identity component is denoted by G^0. We use $L(G)$ (or $\mathfrak{g}$, the corresponding lowercase German letter) for the Lie algebra of G. If H is a subset of G, then we write $N_G(H)$ (resp., $Z_G(H)$) for the normalizer (resp., centralizer) of H in G. We write $Z(G)$ for the center of G. The commutator subgroup of G is denoted by $D(G)$ or $[G,G]$. The group of automorphisms of G leaving H invariant is denoted by $\mathrm{Aut}(G,H)$. For $H = G$ we abbreviate this by $\mathrm{Aut}(G)$. For $g \in G$ let $\mathrm{Int}(g) \in \mathrm{Aut}(G)$ denote the inner automorphism defined by $\mathrm{Int}(g)(x) = g^{-1}xg$, $x \in G$.

An algebraic group defined over k is also called an algebraic k-group. For an extension K of k, the set of K-rational points of G is denoted by G_K or $G(K)$.

If G is a reductive k-group and A a torus of G then we denote by $X^*(A)$ (resp., $X_*(A)$) the group of characters of A (resp., one-parameter subgroups of A) and by $\Phi(A) = \Phi(G,A)$ the set of the roots of A in G. The group $X^*(A)$ can be put in duality with $X_*(A)$ by a pairing $\langle \cdot\,,\,\cdot \rangle$ defined as follows. If $\chi \in X^*(A)$, $\lambda \in X_*(A)$, then $\chi(\lambda(t)) = t^{\langle \chi,\lambda \rangle}$ for all $t \in k^*$. Let $W(A) = W(G,A) = N_G(A)/Z_G(A)$ denote the Weyl group of G relative to A. If $\alpha \in \Phi(G,A)$, then let U_α denote the unipotent subgroup of G corresponding to α. If A is a maximal torus, then U_α is one-dimensional.

2.1.1 Symmetric *k*-varieties

Throughout the chapter G denotes a connected reductive algebraic k-group, σ an involution of G defined over k, $G_\sigma = \{g \in G \mid \sigma(g) = g\}$ the set of fixed points of σ, and H a k-open subgroup of G_σ. The involution σ is also called a *k-involution of G*. The variety G/H is called a *symmetric variety* and the variety G_k/H_k is called a *symmetric k-variety*.

Given $g, x \in G$, the *twisted action* associated with σ is given by $(g,x) \mapsto g * x = gx\sigma(g)^{-1}$. This is also called *$\sigma$-twisted conjugation*. Let $Q = \{g\sigma(g)^{-1} \mid g \in G\}$ and $Q' = \{g \in G \mid \sigma(g) = g^{-1}\}$. The set Q is contained in Q'. Both Q and Q' are invariant under the twisted action associated with σ. There are only a finite number of twisted G-orbits in Q' and each such orbit is closed (see [52]). In particular, Q is a connected closed k-subvariety of G. Define a morphism $\tau : G \to G$ by

$$\tau(x) = x\sigma(x)^{-1}, \quad (x \in G). \tag{2}$$

The image $\tau(G) = Q$ is a closed k-subvariety of G and τ induces an isomorphism of the coset space G/G_σ onto $\tau(G)$. Note that $\tau(x) = \tau(y)$ if and only if $y^{-1}x \in G_\sigma$ and $\sigma(\tau(x)) = \tau(x)^{-1}$ for $x \in G$. This map is essential in the study of involutions of reductive algebraic groups and their symmetric varieties. If one considers $G_\sigma \backslash G$ instead of G/G_σ and lets the fixed point group G_σ act from the left, then one defines τ as $\tau(g) = g^{-1}\sigma(g)$. This leads to the same results for the corresponding symmetric varieties, only using a different characterization. In some computations it is easier to let G_σ act from the left.

2.1.2 σ-split tori

If $T \subset G$ is a torus and $\phi \in \mathrm{Aut}(G,T)$ an involution, then we write $T_\phi^+ = (T \cap G_\phi)^0$ and $T_\phi^- = \{x \in T \mid \phi(x) = x^{-1}\}^0$. It is easy to verify that the product map

$$\mu : T_\phi^+ \times T_\phi^- \to T, \quad \mu(t_1,t_2) = t_1t_2$$

is a separable isogeny. In particular $T = T_\phi^+ T_\phi^-$ and $T_\phi^+ \cap T_\phi^-$ is a finite group. (In fact it is an elementary abelian 2-group.) The automorphisms of $\Phi(G,T)$ and $W(G,T)$ induced by ϕ are also denoted by ϕ. If $\phi = \sigma$ we reserve the notation T^+ and T^- for T_σ^+ and T_σ^- respectively. For other involutions of T, we keep the subscript.

Recall from [28] that a torus A is called *σ-split* if $\sigma(a) = a^{-1}$ for every $a \in A$. If A is a maximal σ-split torus of G, then $\Phi(G,A)$ is a root system with Weyl group $W(A) = N_G(A)/Z_G(A)$ (see [52]). This is the root system associated with the symmetric variety G/H. With the symmetric k-variety G_k/H_k one can also associate a natural root system. To see this we consider the following tori.

Definition 2.1. A k-torus A of G is called (σ,k)-split if it is both σ-split and k-split.

Consider a maximal (σ,k)-split torus A in G. In [39, 5.9] it was shown that $\Phi(G,A)$ is a root system and $N_{G_k}(A)/Z_{G_k}(A)$ is the Weyl group of this root system. We can also obtain this root system by restricting the root system of G_k. Namely let $A_0 \supset A$ be a σ-stable maximal k-split torus of G. Then $A = (A_0)^-_\sigma$ and $\Phi(G,A)$ can be identified with $\overline{\Phi_\sigma} = \{\alpha|A \neq 0 \mid \alpha \in \Phi(G,A_0)\}$.

2.1.3 σ-split parabolic k-subgroups

There exists a natural class of parabolic k-subgroups related to these (σ,k)-split tori and the symmetric k-varieties. These are defined as follows. A parabolic k-subgroup P is called σ-split if $P \cap \sigma(P)$ is a σ-stable Levi factor of P. The minimal σ-split parabolic k-subgroups can be characterized as follows.

Proposition 2.1 ([39]). *Let P be a σ-split parabolic k-subgroup of G and A a σ-stable maximal k-split torus of P. Then the following conditions are equivalent:*

1. *P is a minimal σ-split parabolic k-subgroup of G.*
2. *$P \cap \sigma(P)$ has no proper σ-split parabolic k-subgroups.*
3. *σ is trivial on the isotropic factor of $P \cap \sigma(P)$ over k.*
4. *A^- is a maximal (σ,k)-split torus of G and $Z_G(A^-) = P \cap \sigma(P)$.*

The minimal σ-split parabolic k-subgroups correspond to the open orbit of minimal parabolic k-subgroups acting on symmetric varieties.

Proposition 2.2 ([39]). *Let P be a minimal σ-split parabolic k-subgroup of G and P_0 a minimal parabolic k-subgroup of G contained in P. Then we have the following conditions.*

1. *$H^0P = H^0P_0$.*
2. *H^0P_0 is open in G.*

For $k = \mathbb{R}$ and $\mathbb{Q}_p$ these σ-split parabolic k-subgroups play an important role in the study of symmetric k-varieties and their representations; see for example [2, 5, 20, 34, 35, 36].

2.2 Classification of symmetric k-varieties

In order to classify the symmetric k-varieties, one needs to classify the isomorphism classes of k-involutions. A first question that arises is when an involution of an algebraic k-group (and hence the corresponding symmetric variety) is defined over k. By a result of Helminck and Wang [39] this is completely determined by the k-structure of the fixed point group.

Proposition 2.3. *Let G be a connected semi-simple algebraic k-group and σ an involution of G. Then σ is defined over k if and only if H^0 is defined over k.*

2.2.1 Characterization of the isomorphism classes

A characterization of the isomorphism classes of the k-involutions was given in [31] essentially using the following three invariants.

1. Classification of admissible (Γ,σ)-indices.
2. Classification of the G_k-isomorphism classes of k-involutions of the k-anisotropic kernel of G.
3. Classification of the G_k-isomorphism classes of k-inner elements of G.

For more details, see [31]. The admissible (Γ,σ)-indices determine most of the fine structure of the symmetric k-varieties and a classification of these was included in [31] as well. For k algebraically closed or k the real numbers the full classification can be found in [28]. For other fields a classification of the remaining two invariants is still lacking. In particular the case of symmetric k-varieties over the p-adic numbers is of interest. In some special cases, such as $G = \mathrm{SL}(n,k)$ or $G = \mathrm{SO}(n,k)$ detailed classifications of these k-involutions exist (see [40, 42, 41]). Throughout this chapter we illustrate results with examples for $G = \mathrm{SL}(2,k)$. The classification of involutions in this case can be summarized as follows.

Theorem 2.1 ([40]). *Let $G = \mathrm{SL}(2,k)$. Then we have the following.*

1. *Every k-isomorphism class of involutions over G contains an inner automorphism $\mathrm{Int}(A)$ with $A = \left(\begin{smallmatrix} 0 & 1 \\ a & 0 \end{smallmatrix}\right) \in \mathrm{GL}(2,k)$.*
2. *If $\sigma = \mathrm{Int}(A)$ and $\phi = \mathrm{Int}(B) \in \mathrm{Aut}(G)$ involutions with $A = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ q & 0 \end{smallmatrix}\right)$, then σ is conjugate to ϕ if and only if q/p is a square in k^*.*

For $k = \mathbb{Q}_p$, $p \neq 2$ this gives four isomorphism classes, of which three have a compact fixed-point group.

2.3 *k-Anisotropic symmetric subgroups*

In the real case the symmetric spaces with H_k compact are called Riemannian symmetric spaces and these were studied first, since they have many additional properties. For general fields k the equivalent would be the case that the fixed-point group H is k-anisotropic. In this case these symmetric k-varieties have a lot of additional properties as well. In this subsection we review some of the known results from [39]. Many of them are similar to the well-known results in the case of real Riemannian symmetric spaces. The condition of k-anisotropic fixed-point group can be described as follows.

Proposition 2.4. *Let G be a connected reductive k-group and σ an involution of G defined over k. The following conditions are equivalent.*

1. *The group $[G,G]\cap H$ is anistropic over k;*
2. *Any parabolic k-subgroup of G is σ-split;*
3. *Any minimal parabolic k-subgroup of G is σ-split.*

The condition of k-anisotropic fixed point group implies an Iwasawa type decomposition of G_k.

Corollary 2.1. *Let G be a connected reductive k-group and σ an involution of G defined over k. Then we have the following.*

1. *If $[G,G]\cap H$ is anistropic over k, then $G_k = (PH^0)_k$ for any minimal parabolic k-subgroup P of G.*
2. *Conversely if $G_k = (PH^0)_k$ for any minimal parabolic k-subgroup P of G, then there exists an almost direct product $G = G_1 \cdot G_2$ of k-groups such that $\sigma|G_2$ is trivial and $\sigma|G_1$ satisfies the equivalent conditions in Proposition 2.4.*

It also implies that any σ-stable maximal k-split torus of G has representatives in H.

Corollary 2.2. *Let A be a σ-stable maximal k-split torus of G. If H is anisotropic over k, then $N_G(A) = N_{H^0}(A)Z_G(A)$.*

But most important is that the corresponding symmetric varieties consist of semi-simple elements, when the characteristic of the field k is zero.

Proposition 2.5. *Let G be a connected reductive algebraic k-group with $\mathrm{ch}(k) = 0$, $Q = \{g\sigma(g)^{-1} \mid g \in G\}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ the decomposition of $\mathfrak{g} = L(G)$ into eigensubspaces of σ where $\mathfrak{h} = L(H)$. Suppose that $H \cap [G,G]$ is anisotropic over k. Then we have the following conditions.*

1. *Q_k consists of semi-simple elements.*
2. *$\mathfrak{q}_k$ consists of semi-simple elements.*
3. *If M is a connected k-subgroup of G containing H^0, then there exists a connected normal k subgroup G' of G such that $M = H^0G'$.*

Remark 2.1. (1) If G has no anisotropic factors over k, (3) yields that H^0 is a maximal connected anisotropic k-subgroup of G. The results in Proposition 2.4, Corollary 2.1, and Proposition 2.5 strikingly resemble the usual properties of Cartan involutions for real groups. One can also easily derive the Iwasawa decomposition for real groups from these results.
(2) It is an open question whether this result or a variation of it holds in characteristic not zero.

Example 2.1. (1) $k = \mathbb{R}$ and $G = \mathrm{SL}(2,\mathbb{R})$, $\sigma(x) = {}^t x^{-1} = \mathrm{Int}(A)(x)$ with $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $H = \mathrm{SO}(2)$. Then H is anisotropic over $\mathbb{R}$ and Q is the set of symmetric matrices with positive real eigenvalues.

(2) $k=\mathbb{R}$ and $G=\mathrm{SL}(2,\mathbb{R})$, $\sigma(x)=\mathrm{Int}(A)(x)$ with $A=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, and

$$H=\left\{\begin{pmatrix}x&y\\y&x\end{pmatrix}\;\middle|\;x^2-y^2=1\right\}=\left\{\begin{pmatrix}\cosh\phi&\sinh\phi\\\sinh\phi&\cosh\phi\end{pmatrix}\;\middle|\;\phi\in\mathbb{R}\right\}$$

which is noncompact. In this case

$$Q=\left\{\begin{pmatrix}a^2-b^2&-ca+bd\\ca-bd&d^2-c^2\end{pmatrix}\;\middle|\;a,b,c,d\in\mathbb{R},ad-bc=1\right\}$$

consists of nonsymmetric matrices and contains nonsemi-simple elements. For example, take $a=1,b=2,c=0,d=1$; then $\left(\begin{smallmatrix}-3&1\\-1&1\end{smallmatrix}\right)$ has Jordan normal form $\left(\begin{smallmatrix}-1&1\\0&-1\end{smallmatrix}\right)$, which is not semi-simple.

(3) $G=\mathrm{SL}(2,k)$ with $k=\mathbb{Q}_p$, the p-adic numbers, $\sigma_a(x)=\mathrm{Int}(A)(x)$ with $A=\left(\begin{smallmatrix}0&1\\a&0\end{smallmatrix}\right)$ and $a\in k^*/(k^*)^2$. If H^a is the fixed point group of σ_a, then

$$H^a=\left\{\begin{pmatrix}x&y\\ay&x\end{pmatrix}\;\middle|\;x^2-ay^2=1\right\}.$$

It was shown in [40] that H^a is compact if and only if $a\neq 1$ (see also Proposition 4.4). Since $|k^*/(k^*)^2|=4$ for $p\neq 2$ and $|k^*/(k^*)^2|=8$ for $p=2$, there are three non-isomorphic symmetric k-varieties consisting of semi-simple elements if $p\neq 2$ and 7 if $p=2$.

3 Orbits of parabolic subgroups acting on symmetric varieties

The orbits of a parabolic k-subgroup P acting on the symmetric k-variety G_k/H_k play a fundamental role in the study of representations associated with these symmetric k-varieties. These orbits were studied for many fields and can be characterized in several equivalent ways. They can be characterized as the P_k-orbits acting on the symmetric k-variety G_k/H_k by σ-twisted conjugation, or as the H_k-orbits acting on the flag variety G_k/P_k by conjugation or also as the set $P_k\backslash G_k/H_k$ of (P_k,H_k)-double cosets in G_k. The last is the same as the set of $P_k\times H_k$-orbits on G_k. For k algebraically closed and $P=B$ a Borel subgroup these orbits were characterized by Springer [56] and a characterization of these orbits for general parabolic subgroups was given by Brion and Helminck in [12, 32]. For $k=\mathbb{R}$ and P a minimal parabolic k-subgroup characterizations were given by Matsuki [47] and Rossmann [55] and for general fields these orbits were characterized by Helminck and Wang [39]. For general fields one can first consider the set of (P,H)-double cosets in G. Then the (P_k,H_k)-double cosets in G_k can be characterized by the (P,H)-double cosets in G defined over k plus an additional invariant describing the decomposition of a (P,H)-double coset into (P_k,H_k)-double cosets. In this section we first recall the characterization of orbits of parabolic subgroups acting on symmetric varieties from [56, 12] and then discuss the decomposition of a (P,H)-double coset into (P_k,H_k)-double cosets from [39]. We also illustrate the additional complications with examples and discuss some recent results.

3.1 k Algebraically closed and P = B a Borel subgroup

This case was studied extensively in [56]. Springer proved several characterizations of the double cosets $B\backslash G/H$, depending on whether one considers orbits of H, B or $B\times H$. We outline these characterizations. First we consider H-orbits. For a torus T of G, let $N_G(T)$ denote its normalizer, $Z_G(T)$ its centralizer, $W_G(T) = N_G(T)/Z_G(T)$ the Weyl group of T, and $W_H(T) = N_H(T)/Z_H(T) = \{w \in W_G(T) \mid w \text{ has a representative in } N_H(T)\}$. All of these are σ-stable. Let $\mathscr{C}$ denote the set of pairs (B',T') where T' is a σ-stable maximal torus of a Borel subgroup B' of G and let $\mathscr{B}$ denote the variety of all Borel subgroups of G. The group G acts on $\mathscr{B}$ and $\mathscr{C}$ on the right by conjugation. Let $\mathscr{B}/H$ (resp., $\mathscr{C}/H$) denote the set of H-orbits in $\mathscr{B}$ (resp., $\mathscr{C}$). The H-orbits in $\mathscr{C}$ can be broken up into two parts. First one can consider the H-conjugacy classes of σ-stable maximal tori in G and then for each σ-stable maximal torus representing a H-conjugacy class the Borel subgroups containing it that are not H-conjugate. So, if $\{T_i \mid i\in I\}$ are representatives of the H-conjugacy classes of σ-stable maximal tori in G, then the H-orbits in $\mathscr{C}$ can be identified with $\cup_{i\in I} W_G(T_i)/W_H(T_i)$. This description is similar to the well know Bruhat decomposition of the group.

For the B-orbits we identify G/H with Q and note that B acts on Q by the σ-twisted action. In this case one shows that if T is a σ-stable maximal torus and U is the unipotent radical of B, then any σ-twisted U orbit on Q meets $N_G(T)$ (see [56, 39]). Denote the set of σ-twisted B orbits on Q by $B\backslash Q$.

For the characterization as $H\times B$-orbits in G we need a bit more notation. Set $\mathscr{V} = \{g\in G \mid g\sigma(g)^{-1} \in N_G(T)\}$. Then $\mathscr{V}$ is stable under right multiplication by H and left multiplication by $N_G(T)$. Denote the (T,H)-double cosets in $\mathscr{V}$ by V. Springer showed that all these characterizations are isomorphic.

Theorem 3.1 ([56]). *Let B be a Borel subgroup of G and let $\{T_i \mid i\in I\}$ be representatives of the H-conjugacy classes of σ-stable maximal tori in G. Then*

$$B\backslash G/H \cong \mathscr{B}/H \cong \bigcup_{i\in I} W_G(T_i)/W_H(T_i) \cong \mathscr{C}/H \cong B\backslash Q \cong V.$$

Geometrically, the orbits of H, B, and $B\times H$ are completely different. For example, the B-orbits in G/H are always irreducible and connected, but the H-orbits in $\mathscr{B}$ are not necessarily connected since H need not be connected. All the H-orbits in $\mathscr{C}$ are closed; however, there is a bijection between the closures of the H-orbits in $\mathscr{B}$ and the closures of the $(B\times H)$-orbits in G. The closed $(B\times H)$-orbits are those corresponding to the H-conjugacy classes of pairs (B',T') where B' is σ-stable. If (B',T') is such a pair, then one can show that T' contains a maximal torus of H. Since all maximal tori containing a maximal torus of H are H-conjugate it follows that there is a bijection between the closed H-orbits in G/B and $W_G(T')^\sigma/W_H(T')$. See [53] for more about the $(B\times H)$-orbits and for a combinatorial description of the Bruhat order on orbit closures.

3.2 k Algebraically closed and P a parabolic subgroup

This case was studied extensively in [12] and is very similar to the case that $H=B$ is a Borel subgroup.

In terms of H-orbits we get the following. Let $\mathscr{P}(G)$ denote the variety of all parabolic subgroups of G and let $\mathscr{D}$ denote the set of triples (P,B,T) where P is a parabolic subgroup of G, B is a Borel subgroup of P such that the product $(P\cap H)B$ is open in P, and T is a σ-stable maximal torus of B. For a fixed parabolic subgroup P let $\mathscr{P}(G)^P$ denote the set of all parabolic subgroups of G, which are conjugate to P and $\mathscr{D}^P$ the set of all triples $(P',B',T')\in\mathscr{D}$ with $P'\in\mathscr{P}(G)^P$. The group G acts on $\mathscr{P}(G)$, $\mathscr{P}(G)^P$, $\mathscr{D}$, and $\mathscr{D}^P$ on the right by conjugation. Let $\mathscr{P}(G)/H$ (resp., $\mathscr{P}(G)^P/H$, $\mathscr{D}/H$, $\mathscr{D}^P/H$) denote the set of H-orbits in $\mathscr{P}(G)$ (resp., $\mathscr{P}(G)^P$, $\mathscr{D}$, $\mathscr{D}^P$). To get a Bruhat type decomposition we do the following. Fix a parabolic subgroup P, let $x_1,\dots,x_r\in G$ such that $T_1=x_1Tx_1^{-1},\dots,T_r=x_rTx_r^{-1}$ are representatives for the H-conjugacy classes of the σ-stable maximal tori occurring in the H-conjugacy classes of the triples (P,B,T) as above, $P=P_1=x_1Px_1^{-1},\dots,P_r=x_rPx_r^{-1}$ and $W_{P_i}(T_i)$, the Weyl group of P_i. Then for each T_i the H-conjugacy classes of the triples $(P',B',T_i)\in\mathscr{D}^P$ correspond to $W_{P_i}(T_i)\backslash W_G(T_i)/W_H(T_i)$.

To characterize the double cosets as $P\times H$-orbits set

$$\mathscr{V}^P:=\{g\in\mathscr{V}\mid BgH \text{ is open in } PgH\}.$$

Again there is a natural action of $T\times H$ on $\mathscr{V}^P$. Denote the set of (T,H)-double cosets in $\mathscr{V}^P$ by V^P. Then we have the following.

Proposition 3.1 ([12]). *There is a bijective map from the set of H-orbits in $\mathscr{P}(G)$ onto the set of H-conjugacy classes of triples $(P,B,T)\in\mathscr{D}$. Moreover for a fixed parabolic subgroup P, we have*

$$P\backslash G/H\cong\mathscr{P}(G)^P/H\cong\mathscr{D}^P/H\cong\bigcup_{i=1}^{r}W_{P_i}(T_i)\backslash W_G(T_i)/W_H(T_i)\cong V^P$$

where $T_1=x_1Tx_1^{-1},\dots,T_r=x_rTx_r^{-1}$ are representatives for the H-conjugacy classes of the σ-stable maximal tori occurring in the H-conjugacy classes of the triples (P,B,T) as above and $P=P_1=x_1Px_1^{-1},\dots,P_r=x_rPx_r^{-1}$.

The characterization of the closed double cosets is somewhat more complicated in this case, since there does not need to exist a σ-stable conjugate of P. The closed H-orbits in G/P are characterized by the H-conjugacy classes of triples (P',B',T') as above where B' is σ-stable and $P'\supset B'$ is a conjugate of P satisfying $P'\cap\sigma(P')$

is a parabolic subgroup of G. If (P',B',T') is such a triple, then similar to the case that $P=B$ one can show that there is a bijection between the closed H-orbits in G/P and $W_{P'}(T')^\sigma\backslash W_G(T')^\sigma/W_H(T')$. For more details we refer to [12].

The orbits $G^\sigma P/P\subseteq G/P$ that are affine are those for which P^σ is reductive. This condition holds if P is σ-split. Another example of an affine orbit occurs when the symmetric space G/G^σ is *Hermitian*; that is, there exists a parabolic subgroup $Q\subseteq G$ and a Levi subgroup $M\subseteq Q$ such that $(G^\sigma)^0=M$. Then $Q^\sigma=M$ is reductive; the corresponding orbit $G^\sigma Q/Q=G^\sigma/(G^\sigma)^0$ is finite. In the general case affine orbits arise from a combination of both examples. In particular open affine orbits can be characterized as follows.

Proposition 3.2 ([12]). *The parabolic subgroup P is σ-split if and only if the orbit $G^\sigma P/P$ is an open affine subset of G/P. Then this orbit consists of all σ-split G-conjugates of P.*

3.3 Parabolic subgroups with σ-stable Levi subgroups

To study rationality properties of these orbits we first need to look at orbits of minimal parabolic k-subgroups. In this case one easily shows that there exists a σ-stable Levi subgroup. This property enables one to refine the above results as follows. In this subsection, we assume that P contains a σ-stable Levi subgroup L. By [12, Lemma 5] any σ-stable Levi subgroup of P is conjugate to L in $R_u(P)^\sigma$. Let G^P be the set of all $g\in G$ such that gPg^{-1} contains a σ-stable Levi subgroup. Clearly, G^P is a union of (H,P)-double cosets, but it can happen that $G^P\subsetneq G$. In the case that $P=B$ we have $G^B=G$ (see [57]). Let $S=Z(L)^0$ denote the connected center of L, and $N_G(S)$ (resp., $Z_G(S)$) the normalizer, (resp., centralizer) of S in G. Then $L=Z_G(S)$, $N_G(L)=N_G(S)$, and these groups are σ-stable. Let $\mathscr{V}_S=\{g\in G\mid g\sigma(g)^{-1}\in N_G(S)\}$. Then $L\times H$ acts on $\mathscr{V}_S$ by $(x,z)\cdot y=xyz^{-1}$, $(x,z)\in L\times H$, $y\in\mathscr{V}_S$. So $\mathscr{V}_S$ is a union of $(N_G(S),H)$-double cosets contained in G^P. Denote the set of $L\times H$-orbits on $\mathscr{V}_S$ by V_S. Finally, let $\mathscr{V}_{S,P}=\mathscr{V}_S\cap\mathscr{V}^P$. Then by [12, Proposition 3] any (P,H)-double coset in G^P meets $\mathscr{V}_S$ along a unique (L,H)-double coset, which itself meets $\mathscr{V}_{S,P}$ along a unique (T,H)-double coset. Set $V_S:=L\backslash\mathscr{V}_S/H$; then we have the following.

Proposition 3.3. $P\backslash G^P/H=T\backslash\mathscr{V}_{S,P}/H=V_S$.

Remark 3.1. This characterization of these double cosets can be applied to the case where P is a minimal parabolic k-subgroup or a minimal σ-split parabolic k-subgroup. In both these cases P contains a σ-stable Levi subgroup, so all minimal (σ-split) parabolic k-subgroups are contained in G^P and it suffices to only consider the orbits above. In these cases we can also replace S by, respectively, a maximal k-split torus or a maximal (σ,k)-split torus contained in S. It was shown in [39] that the sets of roots of the latter tori are actually root systems with Weyl group $W(S)$. Using these root systems with their Weyl groups one can refine the above results.

3.3.1 The case where $\Phi(S)$ is a root system

In the remainder of this subsection assume that $\Phi(S)$ is a root system with Weyl group $W(S) = N_G(S)/Z_G(S)$. Since S is σ-stable its centralizer $Z_G(S)$ is σ-stable as well. By [57, 7.5] $Z_G(S)$ contains a σ-stable maximal torus $T \supset S$. Let $X_0(S) = \{\chi \in X(T) \mid \chi(S) = e\}$ and $\Phi_0(S) = \Phi(T) \cap X_0(S)$. This is a closed subsystem of $\Phi(T)$. Denote the Weyl group of $\Phi_0(S)$ by $W_0(S)$ and identify it with the subgroup of $W(T)$ generated by the reflections s_α, $\alpha \in \Phi_0(S)$. Let $W_1 = \{w \in W(T) \mid w(X_0(S)) = X_0(S)\}$. We note that $W_0(S)$ is a normal subgroup of W_1. Put $\bar{X} = X(T)/X_0(S)$, π the natural projection from $X(T)$ to $\bar{X}$ and let $\bar{\Phi} = \pi(\Phi(T) - \Phi_0(S))$ denote the set of restricted roots of Φ relative to $X_0(S)$. Using similar arguments as in [31, Proposition 4.11] one easily shows that $\bar{\Phi} = \Phi(S)$ and $W(S) \cong W_1/W_0(S)$. For $g \in G$, define an automorphism ζ_g of G by

$$\zeta_g := \mathrm{Int}(g^{-1}) \circ \sigma \tag{3}$$

and define the automorphism

$$\psi_g := \zeta_{g\sigma(g)^{-1}} = \mathrm{Int}(g^{-1}) \circ \sigma \circ \mathrm{Int}(g). \tag{4}$$

If L is a subgroup of G normalized by g, then $L^{\psi_g} = gL^\sigma g^{-1}$. We have now the following characterization of $\mathscr{V}_{S,P}$ in the case where $\Phi(S)$ is a root system.

Proposition 3.4 ([32]). *Let $g \in \mathscr{V}^P$. Then the following are equivalent.*

1. $g \in \mathscr{V}_{S,P}$.
2. *S is ψ_g-stable.*
3. $\psi_g(X_0(S)) = X_0(S)$.

3.4 Some examples

In this section we illustrate the above results with some examples. First we note that a parabolic subgroup P and the set $\mathscr{V}^P$ can also be characterized as follows. Let $B \subset P$ a Borel subgroup such that BgH is open in PgH and $T \subset B$ a σ-stable maximal torus. Let $\Phi = \Phi(T)$ denote the root system of T with respect to G and $\Delta \subset \Phi$ the basis of simple roots corresponding to the set of positive roots defined by (B,T). For each $\alpha \in \Phi$, let $U_\alpha \subset G$ be the corresponding root subgroup. Each simple root $\alpha \in \Delta$ defines a parabolic subgroup P_α of semi-simple rank one, generated by B and $U_{-\alpha}$. Any parabolic subgroup $P \supseteq B$ is generated by the P_αs that it contains. We write $P = P_\Pi$ where Π is the set of all $\alpha \in \Delta$ such that $P_\alpha \subseteq P$. We denote by Φ_Π the subroot system of Φ generated by Π, and by W_Π its Weyl group; we also denote $\mathscr{V}^P$ by $\mathscr{V}^\Pi$. In the examples below we let G^σ act from the left.

The first example shows that the above decompositions are a generalization of the Bruhat decomposition of the group. The Bruhat decomposition corresponds to the following case.

Example 3.1. Let $\mathbf{G}$ be a connected reductive group, $\mathbf{B} \subseteq \mathbf{G}$ a Borel subgroup, and $\mathbf{T} \subset \mathbf{B}$ a maximal torus. Consider $G = \mathbf{G} \times \mathbf{G}$ with involution σ defined by $\sigma(g_1, g_2) = (g_2, g_1)$. Then G^σ is the diagonal $\mathrm{diag}(\mathbf{G})$. The maximal torus $T = \mathbf{T} \times \mathbf{T}$ and the Borel subgroup $B = \mathbf{B} \times \mathbf{B}$ are σ-stable.

The map $(g_1, g_2) \mapsto g_1^{-1} g_2$ induces a bijection $G^\sigma \backslash G/B \to \mathbf{B}\backslash\mathbf{G}/\mathbf{B}$. If P is a parabolic subgroup of G containing B, then $P = \mathbf{P}_1 \times \mathbf{P}_2$ where $\mathbf{P}_1$ and $\mathbf{P}_2$ are parabolic subgroups of $\mathbf{G}$ containing $\mathbf{B}$, and we have a bijection $G^\sigma \backslash G/P \to \mathbf{P}_1\backslash\mathbf{G}/\mathbf{P}_2$ which is compatible with the partial orderings given by inclusion of closures. Thus, the results in this case can be derived from the Bruhat decomposition.

The root system of (G, T) is the disjoint union of two copies of the root system Φ of $(\mathbf{G}, \mathbf{T})$. Denote these copies by $\Phi \times 0$ and $0 \times \Phi$. Let $\mathbf{N}$ be the normalizer of $\mathbf{T}$ in $\mathbf{G}$; then

$$\mathscr{V} = \{(g_1, g_2) \mid g_1^{-1} g_2 \in \mathbf{N}\} = \mathrm{diag}(\mathbf{G})(1 \times \mathbf{N}).$$

For $g = (g_1, g_2) \in \mathscr{V}$, let w be the image of $n = g_1^{-1} g_2$ in $\mathbf{W} = \mathbf{N}/\mathbf{T}$. Then ψ_g acts on G by $\psi_g(x_1, x_2) = (n x_2 n^{-1}, n^{-1} x_1 n)$, and on roots by $\psi_g(\alpha, 0) = (0, w^{-1}(\alpha))$, $\psi_g(0, \alpha) = (w(\alpha), 0)$. For more details we refer to [53, 10.1] and [12, 2.4].

The following example illustrates some of the geometry associated with these double coset decompositions.

Example 3.2. Let $G = \mathrm{GL}_n$ with involution σ such that $\sigma(g) = zgz^{-1}$ where $z = \mathrm{diag}(1, \ldots, 1, -1)$; then $G^\sigma = \mathrm{GL}_{n-1} \times k^*$. Let B be the Borel subgroup of G consisting of upper triangular matrices, and let T be the maximal torus of diagonal matrices. Then T is σ-fixed, and B is σ-stable. One checks that a system of representatives of $G^\sigma \backslash \mathscr{V}/T$ consists of the

$$g_{i,j} : (e_1, \ldots, e_n) \mapsto (e_1, \ldots, e_{i-1}, e_i + e_n, e_{i+1}, \ldots, e_{j-1}, e_i - e_n, e_j, \ldots, e_{n-1}) \qquad (1 \le i < j \le n)$$

together with the

$$g_{i,i} : (e_1, \ldots, e_n) \mapsto (e_1, \ldots, e_{i-1}, e_n, e_i, e_{i+1}, \ldots, e_{n-1}) \qquad (1 \le i \le n).$$

Furthermore, for $i < j$, the corresponding involution $\psi_{g_{i,j}}$ is conjugation by the permutation matrix associated with the transposition (ij); and $\psi_{g_{i,i}}$ is conjugation by $\mathrm{diag}(1, \ldots, 1, -1, 1, \ldots, 1)$ where -1 occurs at the ith place. As a consequence, for a subset Π of Δ, we have: $g_{i,j} \in \mathscr{V}^\Pi$ if and only if α_{i-1} and α_j are not in Π.

We sketch a geometric interpretation of this result. Consider G/B as the variety of complete flags

$$\underline{V} = (V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = k^n),$$

where each V_i is a linear subspace of dimension i. Observe that G^σ is the isotropy group in G of the pair (ℓ, H_ℓ) where ℓ is the line spanned by e_n, and H_ℓ is the hyperplane spanned by $e_1, \ldots, e_{n-1}$. For $1 \le i \le j \le n$, set

$$X_{i,j} := \{\underline{V} \in G/B \mid \ell \subset V_j \text{ and } V_{i-1} \subset H_\ell\}.$$

Then one checks that the $X_{i,j}$ are the G^σ-orbit closures in G/B. More precisely, denoting by $\mathscr{O}_{i,j}$ the G^σ-orbit of $g_{i,j}B$ in G/B, we have

$$X_{i,j} = \overline{\mathscr{O}_{i,j}} = \mathscr{O}_{i,j} \cup X_{i+1,j} \cup X_{i,j-1},$$

where $X_{a,b}$ is empty if $a > b$. In particular, the closed orbits are the $X_{i,i} = \mathscr{O}_{i,i}$ $(1 \leq i \leq n)$.

The right stabilizer of $\overline{G^\sigma g_{i,j}B}$ is the largest parabolic subgroup $P^{i,j} = P \supseteq B$ such that $X_{i,j}$ is the pull-back of a subvariety of G/P under the projection $G/B \to G/P$. As a consequence, we see that $P^{i,j}$ is generated by the P_αs with $\alpha \notin \{\alpha_{i-1}, \alpha_j\}$.

If Π is the complement of $\{\alpha_{i-1}, \alpha_j\}$ in Δ, then there are three closed G^σ-orbits in $G/P_\Pi = G/P^{i,j}$, consisting of all pairs $(V_{i-1} \subset V_j)$ such that $V_j \subset H_\ell$ (resp., $\ell \subset V_{i-1}$; $V_{i-1} \subset H_\ell$ and $\ell \subset V_j$). For more details, see [53, 10.5] and [12, 2.4].

3.5 $k \neq \overline{k}$ and P a minimal parabolic k-subgroup

Let P be a minimal parabolic k-subgroup of G. As in the case of algebraically closed fields Helminck and Wang [39] characterized the double cosets $P_k \backslash G_k / H_k$ in several ways. First we consider H_k-orbits. Let $\mathscr{C}_k$ denote the set of pairs (P', A') where A' is a σ-stable maximal k-split torus of a minimal parabolic k-subgroup P' of G and let $\mathscr{P}_k$ denote the variety of all minimal parabolic k-subgroups of G. The group G_k acts on $\mathscr{P}_k$ and $\mathscr{C}_k$ on the right by conjugation. Let $\mathscr{P}_k/H_k$ (resp., $\mathscr{C}_k/H_k$) denote the set of H_k-orbits in $\mathscr{P}_k$ (resp., $\mathscr{C}_k$). The H_k-orbits in $\mathscr{C}_k$ can be broken up into two parts. First one can consider the H_k-conjugacy classes of σ-stable maximal k-split tori in G and then for each σ-stable maximal k-split torus representing an H_k-conjugacy class the minimal parabolic k-subgroups containing it that are not H_k-conjugate. So, if $\{A_i \mid i \in I\}$ are representatives of the H_k-conjugacy classes of σ-stable maximal k-split tori in G, then the H_k-orbits in $\mathscr{C}_k$ can be identified with $\cup_{i \in I} W_{G_k}(A_i)/W_{H_k}(A_i)$.

For the P_k-orbits we note that P_k acts on Q_k by the σ-twisted action. In this case one shows that if A is a σ-stable maximal k-split torus and U the unipotent radical of P, then any σ-twisted U_k orbit on Q_k meets $N_{G_k}(A)$ (see [39, Proposition 6.6]). Denote the set of σ-twisted P_k orbits on Q_k by $P_k \backslash Q_k$.

For the characterization as $P_k \times H_k$-orbits in G_k let A be a σ-stable maximal k-split torus of P and set $\mathscr{V}_k = \{x \in G_k \mid \tau(x) \in N_{G_k}(A)\}$. The group $Z_{G_k}(A) \times H_k$ acts on $\mathscr{V}_k$ by $(x,z) \cdot y = xyz^{-1}$, $(x,z) \in Z_{G_k}(A) \times H_k$, $y \in \mathscr{V}_k$. Let V_k be the set of $(Z_{G_k}(A) \times H_k)$-orbits on $\mathscr{V}_k$.

Theorem 3.2 ([39]). *Let P be a minimal parabolic k-subgroup of G and let $\{A_i \mid i \in I\}$ be representatives of the H_k-conjugacy classes of σ-stable maximal k-split tori in G. Then*

$$P_k \backslash G_k / H_k \cong \mathscr{P}_k / H_k \cong \bigcup_{i \in I} W_{G_k}(A_i)/W_{H_k}(A_i) \cong \mathscr{C}_k / H_k \cong V_k.$$

There is a natural map of the set of $P_k \times H_k$-orbits on G_k to the set of $P \times H$-orbits on G. Let $\mathcal{V}_A = \{x \in G \mid \tau(x) \in N_G(A)\}$. Similarly as in Section 3.3 $Z_G(A) \times H$ acts on $\mathcal{V}_A$ by $(x,z) \cdot y = xyz^{-1}$, $(x,z) \in Z_G(A) \times H$, $y \in \mathcal{V}_A$. Denote the set of $(Z_G(A) \times H)$-orbits on $\mathcal{V}_A$ by V_A. The set V_A is finite, but in general the set V_k is infinite. In a number of cases one can show that there are only finitely many $(P_k \times H_k)$-orbits on G_k. If k is algebraically closed, the finiteness of V_k was proved by Springer [56]. The finiteness of the orbit decomposition for $k = \mathbb{R}$ was discussed by Wolf [63], Rossmann [55] and Matsuki [47]. For general local fields this result can be found in [39]. The following example shows that in most cases the set V_k is infinite.

Example 3.3. Let $k = \mathbb{Q}$, $G = SL(2)$, $\sigma(x) = {}^t x^{-1}$, $B =$ the Borel subgroup of upper triangular matrices and A the group of diagonal matrices. In this case the computations work out nicer if we let $H = G^\sigma$ act from the left and define $\tau(g) = g^{-1}\sigma(g)$ and $\mathcal{V}_\mathbb{Q} = \{g \in G_\mathbb{Q} \mid \tau(g) \in N_{G_\mathbb{Q}}(A)\}$. If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2),$$

then

$$g^{-1}\sigma(g) = \begin{pmatrix} b^2+d^2 & -ab-cd \\ -ab-cd & a^2+c^2 \end{pmatrix} \in N_{G_\mathbb{Q}}(A)$$

if and only if

$$ab + cd = 0.$$

So

$$\tau(\mathcal{V}_\mathbb{Q}) \subset A_\mathbb{Q}$$

and it coincides with the set consisting of

$$\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \quad \text{with} \quad r = x^2 + y^2,\ (x,y) \in \mathbb{Q}^2 - \{(0,0)\}.$$

If $z = \begin{pmatrix} v & u \\ 0 & v^{-1} \end{pmatrix} \in B$ and $\tau(g) = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$, then

$$z^{-1}\tau(g)\sigma(z) = \begin{pmatrix} rv^{-2}+u^2r^{-1} & -uvr^{-1} \\ -uvr^{-1} & v^2r^{-1} \end{pmatrix} \in A_\mathbb{Q}$$

if and only if $u = 0$. It follows that $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ and $\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$ are in the same twisted $B_\mathbb{Q}$ orbit if and only if $r^{-1}s \in (\mathbb{Q}^*)^2$. Hence $V_\mathbb{Q} \cong \oplus_{\substack{p\equiv 1(4) \\ \text{prime}}} \mathbb{Z}/2\mathbb{Z}$ and the set $H_\mathbb{Q}\backslash G_\mathbb{Q}/B_\mathbb{Q}$ is infinite.

Remark 3.2. The classification of $P_k\backslash G_k/H_k$ for non-minimal parabolic k-subgroups P_k is more complicated than in the algebraically closed case. The main problem in this case is that there is no unique "open" orbit of a minimal parabolic k-subgroup, so one does not get a similar description as in the algebraically closed case. We will discuss this double coset decomposition in more detail in a forthcoming paper.

4 Conjugacy classes of σ-stable tori

As we have seen in the previous section, the conjugacy classes of σ-stable tori play a fundamental role in the description of the double cosets $P_k\backslash G_k/H_k$ and $P\backslash G/H$. In fact in many cases these conjugacy classes determine the double cosets modulo the action of the Weyl group. In this section we collect some results about these tori. First we consider the natural action of the Weyl group on the double cosets and the relation with the conjugacy classes of these σ-stable tori. Next we look at the natural map from V_A into W, induced by the map $\tau|\mathscr{V}_A : \mathscr{V}_A \to N_G(A)$. This map can not only be used to give another characterization of some of the conjugacy classes of σ-stable tori (or equivalently W-orbits in V_k and V_A), but also plays a fundamental role in the study of the geometry and combinatorics of these double cosets. As we show in the next section this map also enables us to port the natural combinatorial structure on the image (contained in the set of twisted involutions in W) to V_A and V_k. Finally, using all this, we discuss characterizations of the various conjugacy classes of σ-stable tori occurring in the characterization of the double cosets. We first consider the case of algebraically closed fields where the H-conjugacy classes of σ-stable maximal tori can be completely described by conjugacy classes of involutions in the Weyl group. After that we consider the H-conjugacy classes of σ-stable maximal k-split tori. We conclude with a discussion of some results about how these H-conjugacy classes split into H_k-conjugacy classes. For $k = \mathbb{R}$ a characterization and a classification of the H_k-conjugacy classes of σ-stable maximal k-split tori is given in [33]. For other fields this problem is still open, except in some specific cases.

4.1 W-Action on V_A and V_k

The Weyl group $W(A) = W_G(A)$ acts on both V_A and V_k. In this section we write W for $W(A)$. The action of W on V_A (resp., V_k) is defined as follows. For $v \in V_A$ (resp., V_k) let $x = x(v) \in \mathscr{V}_A$ (resp., $\mathscr{V}_k$) be a representative of the orbit $v \in V_A$ (resp., V_k). If $n \in N_G(A)$, then $nx \in \mathscr{V}_A$ and its image in V_A depends only on the image of n in W. We thus obtain a (left) action of W on V_A, denoted by $(w, v) \to w \cdot v$ $(w \in W,\ v \in V_A)$.

To get another description of $\mathscr{V}_k$ and $\mathscr{V}_A$ we first define the following.

Definition 4.1. A torus A of G is called a *quasi-k-split torus* if A is conjugate under G with a k-split torus of G.

Since all maximal k-split tori of G are conjugate, all maximal quasi-k-split tori of G are also conjugate. If A is a maximal quasi-k-split torus of G, then $\Phi(G,A)$ is a root system in $(X^*(A), X_0)$ in the sense of [10, Section 2.1] where X_0 is the set of characters of A which are trivial on $(A \cap [G,G])^0$; moreover the Weyl group of $\Phi(G,A)$ is $N_G(A)/Z_G(A)$.

4.1.1 Orbit maps

Let $\mathscr{A}$ be the variety of maximal quasi-k-split tori of G. This is an affine variety, isomorphic to $G/N_G(A)$, on which σ acts. Let $\mathscr{A}^\sigma$ be the fixed-point set of σ, i.e., the set of σ-stable maximal quasi-k-split tori. It is an affine variety on which H acts by conjugation.

Similarly let $\mathscr{A}_k$ denote the set of maximal k-split tori of G and let $\mathscr{A}_k^\sigma$ be the fixed point set of σ i.e., the set of σ-stable maximal k-split tori. The group H_k acts on $\mathscr{A}_k^\sigma$ by conjugation.

If $x \in \mathscr{V}_k$, then $x^{-1}Ax$ is again a maximal k-split torus and conversely any σ-stable maximal k-split torus in $\mathscr{A}_k^\sigma$ can be written as $x^{-1}Ax$ for some $x \in \mathscr{V}_k$. Similarly any σ-stable maximal quasi-k-split torus is of the form $x^{-1}Ax$ for some $x \in \mathscr{V}_A$.

If $v \in V_k$ (resp., $v \in V_A$), then $x(v)^{-1}Ax(v) \in \mathscr{A}_k^\sigma$ (resp., $\mathscr{A}^\sigma$). This determines maps of V_k, (resp., V_A) to the orbit sets $\mathscr{A}_k^\sigma/H_k$ (resp., $\mathscr{A}^\sigma/H$). It is easy to check that these maps are independent of the choice of the representative $x(v)$ for v and they are constant on W-orbits. So we also get maps of orbit sets: $\gamma_k : V_k/W \to \mathscr{A}_k^\sigma/H_k$ and $\gamma : V_A/W \to \mathscr{A}^\sigma/H$. In fact we have a bijection:

Proposition 4.1. *Let G, $\mathscr{A}^\sigma$, $\mathscr{A}_k^\sigma$, γ, and γ_k be as above. Then we have the following.*

(i) $\gamma_k : V_k/W \to \mathscr{A}_k^\sigma/H_k$ is bijective.
(ii) $\gamma : V_A/W \to \mathscr{A}^\sigma/H$ is bijective.

Proof. (i). Since any σ-stable maximal k-split torus is of the form $x^{-1}Ax$ for some $x \in \mathscr{V}_k$, the map γ_k is surjective.

Let $v_1, v_2 \in V_k$ with $\gamma_k(v_1) = \gamma_k(v_2)$. Let $x_1 = x(v_1), x_2 = x(v_2) \in \mathscr{V}_k$ be representatives of v_1, v_2 and let $A_1 = x_1^{-1}Ax_1$, $A_2 = x_2^{-1}Ax_2$. Since $\gamma_k(v_1) = \gamma_k(v_2)$ there exists $h \in H_k$ such that $hA_2h^{-1} = hx_2^{-1}Ax_2h^{-1} = x_1^{-1}Ax_1 = A_1$. So we may assume that $A_1 = A_2$. But then by [39, 6.10] there exists $n \in N_G(A_1)$ such that $P_knx_1H_k = P_kx_2H_k$. If $w \in W(A_1)$ is the image of n in $W(A_1)$, then $w \cdot v_1 = v_2$, which proves (i).

The proof of (ii) follows using a similar argument to that of (i).

4.1.2 H_k and H-conjugacy classes

Let $\mathscr{A}_0^\sigma$ denote the set of σ-stable quasi-k-split tori of G, which are H-conjugate to a σ-stable maximal k-split torus. Then $\mathscr{A}_0^\sigma/H \subset \mathscr{A}^\sigma/H$ can be identified with the set of H-conjugacy classes of σ-stable maximal k-split tori of G. There is a natural map

$$\zeta : \mathscr{A}_k^\sigma/H_k \to \mathscr{A}^\sigma/H, \tag{5}$$

sending the H_k-conjugacy class of a σ-stable maximal k-split torus onto its H-conjugacy class. Then $\mathscr{A}_0^\sigma/H$ is precisely the image of ζ. On the other hand the inclusion map $\mathscr{V}_k \to \mathscr{V}$ induces a map $\eta : V_k \to V_A$, where η maps the orbit $Z_{G_k}(A)gH_k$ onto $Z_G(A)gH$. This map is W-equivariant. Denote the corresponding orbit map by $\delta : V_k/W \to V_A/W$ and write V_0 for the image of η in V_A. Denote the restriction of γ to V_0/W by γ_0. Then γ_0 maps V_0/W onto $\mathscr{A}_0^\sigma/H$. This all leads to the following diagram:

$$\begin{array}{ccc} V_k/W & \xrightarrow{\gamma_k} & \mathscr{A}_k^\sigma/H_k \\ \Big\downarrow \delta & & \Big\downarrow \zeta \\ V_0/W & \xrightarrow{\gamma_0} & \mathscr{A}_0^\sigma/H. \end{array}$$

Since γ_0 and γ_k are bijections, it follows that there is a bijection between the fibers of δ and ζ. For a σ-stable maximal k-split torus A, the fiber $\zeta^{-1}(A)$ consists of all σ-stable maximal k-split tori, which are H-conjugate to A, but not H_k-conjugate. That these fibers can be infinite can be seen in Example 3.3.

4.2 *Action of the Weyl group on twisted involutions*

Throughout this subsection let A be a fixed σ-stable maximal k-split torus and let $W = W(A)$ be the Weyl group of A with respect to G.

Another way to characterize the W-orbits in V_k and V_A is by characterizing the image and fibers of the natural map from V_A into W, induced by the map $\tau|\mathscr{V}_A : \mathscr{V}_A \to N_G(A)$. This map also enables us to port the natural combinatorial structure on the image (contained in the set of twisted involutions in W) to V_A and V_k. Similar to the case of orbits of a Borel subgroup acting on a symmetric variety as in [53] this will enable us to describe most of the combinatorial structure involved.

4.2.1 Twisted involutions

Recall that an element $a \in W$ is a *twisted involution* if $\sigma(a) = a^{-1}$ (see [56, Section 3] or [39, Section 7]). Let

$$\mathscr{I} = \mathscr{I}_\sigma = \mathscr{I}(W,\sigma) = \{w \in W \mid \sigma(w) = w^{-1}\} \tag{6}$$

be the set of twisted involutions in W. If $v \in V_A$, then $\varphi(v) = \tau(x(v))Z_G(A) \in W$ is a twisted involution. The element $\varphi(v) \in \mathscr{I}$ is independent of the choice of representative $x(v) \in \mathscr{V}_A$ for v. So this defines a map $\varphi : V_A \to \mathscr{I}$.

We can define a map $\varphi_k : V_k \to \mathscr{I}$ in a similar manner. Namely if $v \in V_k$, then let $\varphi_k(v) = \tau(x(v))Z_G(A) \in W$. Again this is a twisted involution. From the above observations we get the following relation between φ_k and φ:

$$\varphi_k = \varphi \circ \eta. \tag{7}$$

The maps φ (resp., φ_k) play an important role in the study of the Bruhat order on V_A (resp., V_k). For more details, see Section 5.

4.2.2 W-Action on $\mathscr{I}$

The Weyl group W also acts on $\mathscr{I}$. This action comes from the *twisted action* of W on the set W, which is defined as follows. If $w, w_1 \in W$, then $w * w_1 = w w_1 \sigma(w)^{-1}$. If $w_1 \in W$, then $W * w_1 = \{w * w_1 \mid w \in W\}$ is the *twisted W-orbit* of w_1. Now $\mathscr{I}$ is stable under the twisted action, so that we get a twisted action of W on $\mathscr{I}$. The images of φ and φ_k in $\mathscr{I}$ are unions of twisted W-orbits, which leads to the following result.

Lemma 4.1. *Let $w \in W$ and $v \in V_A$ (resp., V_k). Then $\varphi(w \cdot v) = w * \varphi(v)$ (resp., $\varphi_k(w \cdot v) = w * \varphi_k(v)$).*

4.2.3 Orbit maps

From this result it follows now that the maps $\varphi : V_A \to \mathscr{I}$ (resp., $\varphi_k : V_k \to \mathscr{I}$) are equivariant with respect to the action of W on V_A (resp., V_k) and the twisted action of W on $\mathscr{I}$. So there are natural orbit maps $\phi : V_A/W \to \mathscr{I}/W$ and $\phi_k : V_k/W \to \mathscr{I}/W$. From (7) and Section 4.1.2 we get the following relation between ϕ and ϕ_k,

$$\phi_k = \phi \circ \delta.$$

Since γ and γ_k are one-to-one, we also get embeddings of $\mathscr{A}_0^{\sigma}/H$ and $\mathscr{A}_k^{\sigma}/H_k$ into $\mathscr{I}/W$. This indicates that the W-orbits of twisted involutions can be used as an invariant to characterize the conjugacy classes in $\mathscr{A}_0^{\sigma}/H$ and $\mathscr{A}_k^{\sigma}/H_k$. In fact in Section 4.3 we show that we can use conjugacy classes of involutions in the Weyl group W, instead of W-orbits of twisted involutions.

In the case of $P = B$ a Borel subgroup, Richardson and Springer [53] showed that the map ϕ_k is one-to-one. So in this case the classes in $\varphi_k(V_k)/W$ completely characterize the H_k-conjugacy classes of σ-stable maximal k-split tori. In this case one can then easily prove the following properties of the maps φ and ϕ (see [39] and [53]).

Proposition 4.2. *Let G, φ, and ϕ be as above and assume $A = T$ is a maximal torus. Then we have the following.*

1. *$\phi : V_T/W(T) \to \mathscr{I}_{\sigma}/W(T)$ is injective.*
2. *There is a bijection from $\varphi(V_T)/W(T)$ onto $\mathscr{A}^{\sigma}/H$.*

The map ϕ_k is usually not one-to-one. An example can be found in Section 3.3.

4.3 *Conjugacy classes of σ-stable maximal k-split tori*

As we have seen above, the σ-stable maximal k-split tori play a fundamental role in the description of the orbits of parabolic k-subgroups acting on symmetric k-varieties. To classify these orbits we first need a classification of the conjugacy classes of these σ-stable maximal k-split tori. For k-algebraically closed a classification of the H-conjugacy classes of σ-stable maximal tori was given by Helminck in [29]. For k not algebraically closed a characterization of $\mathscr{A}^\sigma/H$ and $\mathscr{A}_0^\sigma/H$ is considerably more difficult. This problem was studied in [30].

There is a natural conjugacy class of involutions associated with the H-conjugacy classes in $\mathscr{A}^\sigma$ and $\mathscr{A}_k^\sigma$. We can construct this involution in a natural manner, which gives more insight into the conjugacy class of the corresponding tori. We sketch the idea for $\mathscr{A}_0^\sigma$. The construction for $\mathscr{A}^\sigma$ uses similar arguments. For $v \in V_A$ let $x = x(v) \in \mathscr{V}_A$ be a representative, $n = x\sigma(x)^{-1} \in N_G(A)$, and denote the corresponding twisted involution in W by w_v or w_x. To show that we can choose a suitable representative $x(v) \in \mathscr{V}_A$, such that w_v becomes an involution, we conjugate all tori in a so-called standard position.

Definition 4.2. For A_1, $A_2 \in \mathscr{A}_0^\sigma$, the pair (A_1,A_2) is called *standard* if $A_1^- \subset A_2^-$ and $A_1^+ \supset A_2^+$. In this case, we also say that A_1 is *standard* with respect to A_2.

Let $A,S \in \mathscr{A}_0^\sigma$ be σ-stable maximal k-split tori of G, such that A^- is a maximal (σ,k)-split torus of G and S^+ is a maximal k-split torus of H. Moreover one can choose S to be standard with respect to A. In the following we fix such a standard pair (S,A). We can then show that we can reduce to the tori standard with respect to the standard pair (S,A).

Proposition 4.3 ([30]). *Let (S,A) be a standard pair as above and let $A_1 \in \mathscr{A}_0^\sigma$. Then A_1 is H-conjugate with a σ-stable maximal k-split torus, which is standard with respect to A and S.*

By the definition of $\mathscr{V}_A$ the σ-stable maximal quasi-k-split torus of G is of the form $A_1 = x_1^{-1}Ax_1$ with $x_1 = x_1(v) \in \mathscr{V}_A$. By the above result there exists $h \in H$ such that $A_2 = h^{-1}x_1^{-1}Ax_1h$ is standard with respect to (S,A). Then A and A_2 are maximal quasi-k-split tori in $Z_G(A_2^-A^+)$, so there exists $x \in Z_G(A_2^-A^+) \cap \mathscr{V}_A$ such that $A_2 = x^{-1}Ax$. Since A_1 and A_2 are H-conjugate, there exists $n \in N_G(A)$ such that $x = nx_1h$. If $w \in W(A)$ is the Weyl group element corresponding to n, then $w_x = w^{-1}w_{x_1}\sigma(w)$ and $w_x^2 = \mathrm{id}$. Hence any twisted W-orbit in $\mathscr{I}$ contains an involution. If $v \in V_A$ and $x = x(v) \in \mathscr{V}_A$ then we call the involution $w_v = w_x$ *σ-singular* (with respect to A_1) and if $v \in V_k$, then we also call this involution *(σ,k)-singular* (with respect to A_1).

4.3.1 Characterization of $\mathscr{A}_0^\sigma/H$ for k algebraically closed

In the case where k is an algebraically closed field the map ϕ is one-to-one and $\mathscr{A}^\sigma = \mathscr{A}_0^\sigma$. Together with the bijection γ_k in Proposition 4.1 we get a one-to-one correspondence between the H-conjugacy classes of σ-stable maximal tori and the W-conjugacy classes of σ-singular involutions.

Theorem 4.1 ([29]). *Let T be a σ-stable maximal torus of G with maximal T^- (resp., T^+). Then there is a one-to-one correspondence between the H^0-conjugacy classes of $\mathscr{A}^\sigma$ and the $W(T)$-conjugacy classes of σ-singular involutions in $W(T)$.*

It remains to classify the conjugacy classes of σ-singular involutions of $W(T)$. This appears to be easy.

Theorem 4.2 ([29]). *Let T be a σ-stable maximal torus of G with maximal T^- and $w \in W(T)$, $w^2 = e$. Then the following are equivalent.*

(i) *w is σ-singular.*
(ii) $T_w^- \subset T^-$.

A classification of the conjugacy classes of σ-singular involutions of $W(T)$, in the case where T^- is a maximal σ-split torus of G, can now be derived from the classification of involutorial automorphisms of G in Helminck [28].

4.3.2 Characterization of $\mathscr{A}_0^\sigma/H$ for k not algebraically closed

For k not algebraically closed a characterization of $\mathscr{A}^\sigma/H$ and $\mathscr{A}_0^\sigma/H$ is considerably more difficult. This problem was studied in [30]. We have a similar result to the algebraically closed case.

Theorem 4.3 ([30]). *Assume that A_1, $A_2 \in \mathscr{A}_0^\sigma$ such that they are standard with respect to A. Let w_1 and w_2 be the σ-singular involutions in $W(A)$ corresponding to A_1 and A_2, respectively. If A_1 and A_2 are H-conjugate then w_1 and w_2 are conjugate under $W(A,H)$.*

Similarly as in the algebraically closed case we have the following characterization of the σ-singular involutions.

Theorem 4.4 ([30]). *Let A be a σ-stable maximal k-split torus of G with A^- a maximal (σ,k)-split torus of G and $w \in W(A)$, $w^2 = e$. Then the following are equivalent.*

(i) *w is σ-singular.*
(ii) $A_w^- \subset A^-$.

A characterization of the (σ,k)-singular involutions is a bit more complicated and is field dependent. For more details we refer to [30].

Remark 4.1. Ideally one would like the converse of Theorem 4.3 to hold, but that is no longer true, since the map ϕ is not always one-to-one. So the W-conjugacy classes of the σ-singular involutions only provide a first invariant to characterize the conjugacy classes in $\mathscr{A}^\sigma$ (resp., $\mathscr{A}_k^\sigma$) and a characterization of the fibers of ϕ is also needed. This additional invariant can be described in terms of Weyl group elements of a maximal torus containing the maximal k-split torus. For more details we refer to [30].

Fortunately in many cases, including the standard pairs for $k = \mathbb{R}$ (see [28]), the map ϕ_k is actually one-to-one, so in these cases the fibers are trivial. This follows from the next result.

Theorem 4.5 ([30]). *Let $A \in \mathscr{A}_0^\sigma$ be a σ-stable maximal k-split torus of G with A^- a maximal (σ,k)-split torus of G and assume that $W(A^-)$ has representatives in H_k. Then there exists a one-to-one correspondence between the H_k-conjugacy classes of σ-stable maximal k-split tori and the $W(A)$-conjugacy classes of (σ,k)-singular involutions of $W(A)$.*

It remains to determine how the different H-conjugacy classes of the σ-stable maximal k-split tori break up in H_k-conjugacy classes. This is field dependent. For $k=\mathbb{R}$ a characterization and a classification is given in [33]. A different characterization of these conjugacy classes for the Lie algebra case was given by Matsuki [47]. This characterization is more difficult and it is very difficult to classify the conjugacy classes using this characterization. For other fields the characterization and classification are still open, except for $G=\mathrm{SL}(n,k)$ in which case a classification is given by Beun and Helminck [8, 9]. Essential in the characterization of $\mathscr{A}_k^\sigma/H_k$ are the H_k-conjugation classes of maximal (σ,k)-split tori. They are all H-conjugate, but not necessarily H_k-conjugate. In the case where they are all H_k-conjugate one can often show that the map ϕ_k is one-to-one. We illustrate the above results with a discussion of the classification for $\mathrm{SL}(2,k)$.

4.4 H_k-Conjugacy classes in $\mathscr{A}_k^\sigma$ for $G=\mathrm{SL}(2,k)$

The full classification of the double cosets $P_k\backslash G_k/H_k$ for $G=\mathrm{SL}(2,k)$ is given in [8]. Since the dimension of a maximal torus is one, any σ-stable maximal k-split torus is either contained in H or is (σ,k)-split. Before we characterize the (σ,k)-split tori we need some more notation. For $m\in k$ we use $\overline{m}$ to denote the entire square class of m in $k^*/(k^*)^2$. By abuse of notation, we use $m\in k^*/(k^*)^2$ to denote that m is the representative of the square class $\overline{m}$ of k. Recall from Section 2.1 that we may assume that

$$\sigma=\mathrm{Int}(A) \quad \text{with} \quad A=\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}.$$

Theorem 4.6 ([8]). *Let $G=\mathrm{SL}(2,k)$, T the diagonal matrices in G, and $U=\{q\in k^*/(k^*)^2 \mid x_1^2-m^{-1}x_2^2=q^{-1}$ has a solution in $k\}$. Then we have the following.*

1. *The number of H_k-conjugacy classes of (σ,k)-split maximal tori is $|U/\{1,-m\}|$.*
2. *For $y\in U$, let $r,s\in k$ such that $r^2-m^{-1}s^2=y^{-1}$ and let*

$$g=\begin{pmatrix} r & sym^{-1} \\ s & ry \end{pmatrix}.$$

Then $\{T_y=g^{-1}Tg \mid y\in U\}$ is a set of representatives of the H_k-conjugacy classes of maximal (σ,k)-split tori in G.

As for the maximal k-split tori contained in H we have the following.

Proposition 4.4. *Let G, σ be as above. Then H_k is k-anisotropic if and only if $m\notin\overline{1}$. If $m\in\overline{1}$, then H is a maximal k-split torus of G.*

It remains to determine which Weyl group elements of σ-stable maximal k-split tori have representatives in H_k. For this we have the following results.

Lemma 4.2. *Let T_i be a (σ,k)-split maximal torus. Then,*

1. $|W_{H_k}(T_i)| = 2$ *when* $m \in \overline{1}$ *and* $-1 \in (k^*)^2$.
2. $|W_{H_k}(T_i)| = 2$ *when* $m \in \overline{-1}$ *and* $-1 \notin (k^*)^2$.
3. $|W_{H_k}(T_i)| = 1$ *otherwise.*

Finally in the case where H is a maximal k-split torus we clearly have $W_{H_k}(H) = \{\mathrm{id}\}$, thus $|W_{G_k}(H)/W_{H_k}(H)| = 2$. The above results give us a detailed description of the double cosets $P_k\backslash G_k/H_k$ for $G = \mathrm{SL}(2,k)$. We illustrate the results with the following example.

Example 4.1. Let $G_k = \mathrm{SL}(2,k)$ with $k = \mathbb{Q}_p$, $p \neq 2$. Then $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 = \{1, s_p, p, ps_p\}$ where s_p is the smallest nonsquare in $\mathbb{F}_p$. Let $m \in k^*/(k^*)^2$ such that

$$\sigma = \mathrm{Int}(A) \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}.$$

The number of H_k-conjugacy classes of (σ,k)-split maximal tori is equal to

1. 1 for $m = p, ps_p$.
2. 1 for $p \cong 1 \mod 4$ and $m = s_p$.
3. 2 for $p \cong 3 \mod 4$ and $m = s_p$.
4. 2 for $p \cong 3 \mod 4$ and $m = 1$.
5. 4 for $p \cong 1 \mod 4$ and $m = 1$.

Moreover for $m = 1$ we also have a maximal k-split torus in H. So this leads to either two or six orbits:

1. $|P_k\backslash G_k/H_k| = 2$ for $m = s_p$, $m = p$, and $m = ps_p$.
2. $|P_k\backslash G_k/H_k| = 6$ for $m = 1$.

This example illustrates that maximal (σ,k)-split tori in G do not need to be conjugate under H_k and there is no unique open orbit.

5 Bruhat order and twisted involution posets

The previous sections focused on characterizing and classifying orbits of parabolic k-subgroups acting on symmetric k-varieties. There are many other properties of these orbit decompositions that play an important role in the study of these symmetric k-varieties and their applications. For example, there is a natural geometry associated with these double cosets related to the Zariski closures of the double cosets in the case of algebraically closed fields or the topological closures of the double cosets in the case of fields with a topology. This geometry plays a fundamental role in representation theory. For example, in the case where $k = \mathbb{C}$ a reductive group G with an involution σ can be viewed as the complexification of a reductive real Lie group G_0 such that σ is the complexification of a Cartan involution of G_0. Then G/H is the complexification of the symmetric space defined by G_0 and the Cartan involution. The H-action on G/B appears here in connection with the infinite-dimensional representation theory of the Lie group G_0. In particular, the geometry of H-orbits on G/B plays a fundamental role in the classification of Harish-Chandra modules for G_0 (see [61]).

There is a partial order on the double cosets defined by the Zariski (or topological) closures. If $\mathscr{O}_{v_1}$ and $\mathscr{O}_{v_2}$ are orbits, then $\mathscr{O}_{v_1} \leq \mathscr{O}_{v_2}$ if and only if $\mathscr{O}_{v_1}$ is contained in the closure of $\mathscr{O}_{v_1}$. This order is called the *Bruhat order on* V_A (or V_k) and it generalizes the usual Bruhat order on a connected reductive algebraic group defined by the Bruhat decomposition. In the case of the Bruhat decomposition of the group, Chevalley showed that this geometric Bruhat order corresponds with the combinatorially defined Bruhat order on the Weyl group (see [11] for the first published proof of this). The combinatorial Bruhat order on the Weyl group has been studied by many mathematicians and much is known about the corresponding poset. For symmetric varieties over algebraically closed fields, Richardson and Springer [53, 54] showed that for k algebraically closed there is a similar combinatorial description of the Bruhat order on V. However, in this case the combinatorics is considerably more complicated. In this case the combinatorics of the orbit closures corresponds to a combinatorial order on the set of twisted involutions in a Weyl group. Several of Richardson and Springer's results can be generalized to describe the Bruhat order on the sets V_A (or V_k) of orbits of minimal parabolic k-subgroups acting on symmetric k-varieties, but a full combinatorial description of the Bruhat order is still open.

In the following we give a survey of results about the geometry and combinatorics of these orbit decompositions that play an important role in the study of these symmetric k-varieties and their applications. We start with an overview of results about the combinatorics of the related set of twisted involutions in the Weyl group, which play an important role in all of this. The description of the combinatorics in [56, 53] for the case of groups defined over an algebraically closed field depends on the existence of a Borel subgroup invariant under the involution σ. For groups defined over non algebraically closed fields a σ-stable minimal parabolic k-subgroup does not need to exist. One can still obtain a similar combinatorial characterization, but one has to pass to another involution. We start this section by developing the framework and discussing some of the main results. After that we look at orbit closures and discuss the combinatorial description of the Bruhat order in the case of algebraically closed fields. The description given focuses on admissible sequences and is a slight simplification of the one in [53, 54]. We also indicate which results can be generalized to the nonalgebraically closed case. We conclude this section with a brief discussion of the combinatorics of the twisted involution poset related to the Bruhat order.

5.1 Combinatorics of twisted involutions

A first description of the twisted involutions, in the case where $\sigma(\Phi^+) = \Phi^+$, was given in [56]. For $k = \overline{k}$ there exists a σ-stable Borel subgroup (see [57]), so then this condition is satisfied. However, if $k \neq \overline{k}$ then G does not necessarily have σ-stable minimal parabolic k-subgroups. We discuss this in further detail in Section 5.3. One can generalize the description in [56] and give a similar description of the twisted involutions, when $\sigma(\Phi^+) \neq \Phi^+$. This follows essentially from the results in [39], although some of these results are not explicitly stated there. In the following we will review this characterization and prove a few additional results. First we need a few facts about real, complex, and imaginary roots.

5.1.1 Some notation

Let A be a σ-stable maximal quasi-k-split torus of G, $\Phi = \Phi(A)$ the root system of A with respect to G, Φ^+ a set of positive roots of Φ, Δ the corresponding basis, $W = W(A)$ the Weyl group of A, and $\Sigma = \{s_\alpha \mid \alpha \in \Delta\}$. The Weyl group W is generated by Σ. Let $E = X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. If $\phi \in \mathrm{Aut}(\Phi)$, then we denote the eigenspace of ϕ for the eigenvalue ξ, by $E(\phi, \xi)$. For a subset Π of Δ denote the subset of Φ consisting of integral combinations of Π by Φ_Π. Then Φ_Π is a subsystem of Φ with Weyl group W_Π. Let w^0_Π denote the longest element of W_Π with respect to Π.

The involution σ of G induces an automorphism of W, also denoted by σ, given by

$$\sigma(w) = \sigma \circ w \circ \sigma, \qquad w \in W.$$

If s_α is the reflection defined by α, then $\sigma(s_\alpha) = s_{\sigma(\alpha)}$, $\alpha \in \Phi$.

5.1.2 Real, complex, and imaginary roots

The roots of Φ can be divided into three subsets, related to the action of σ, as follows.

1. $\sigma(\alpha) \neq \pm\alpha$. Then α is called *complex* (relative to σ).
2. $\sigma(\alpha) = -\alpha$. Then α is called *real* (relative to σ).
3. $\sigma(\alpha) = \alpha$. Then α is called *imaginary* (relative to σ).

These definitions carry over to the Weyl group W and the set V_A. We first define real, complex, and imaginary elements for the Weyl group, which then will imply similar definitions for V_A.

Definition 5.1. Given $w \in \mathscr{I}_\sigma$, an element $\alpha \in \Phi$ is called *complex* (resp., *real*, *imaginary*) relative to w if $w\sigma(\alpha) \neq \pm\alpha$ (resp., $w\sigma(\alpha) = -\alpha$, $w\sigma(\alpha) = \alpha$). We use the following notation,

$$\begin{aligned} C'(w,\sigma) &= \{\alpha \in \Phi^+ \mid -\alpha \neq w\sigma(\alpha) < 0\} \\ C''(w,\sigma) &= \{\alpha \in \Phi^+ \mid \alpha \neq w\sigma(\alpha) > 0\} \\ R(w,\sigma) &= \{\alpha \in \Phi^+ \mid -\alpha = w\sigma(\alpha)\} \\ I(w,\sigma) &= \{\alpha \in \Phi^+ \mid \alpha = w\sigma(\alpha)\}. \end{aligned}$$

If $w = w_g$ is a twisted involution related to an element $g \in \mathscr{V}_A$ (i.e., w_g denotes the image in W of $g^{-1}\sigma(g) \in N_G(A)$), then this definition of real, complex, and imaginary roots is related to the involution $\psi_g = \operatorname{Int}(g^{-1}\sigma(g)) \circ \sigma$ (see Equation (4)), since $\psi_g(\alpha) = w_g\sigma(\alpha)$ for all $\alpha \in \Phi$. Thus, these are ψ_g-real, imaginary, complex roots.

5.1.3 Another characterization of $\mathscr{V}^P$

The combinatorics of the real, imaginary, and complex roots can be used to describe much of the geometry of these double cosets. As an example we give more detailed descriptions of the set $\mathscr{V}^P$. Similar results hold for $\mathscr{V}_A$ and $\mathscr{V}_k$. We use the notation from Section 3.2. In particular let B be a Borel subgroup of G, $T \subset B$ a σ-stable maximal torus, $\Phi = \Phi(T)$ the root system of T with respect to G, and $\Delta \subset \Phi$ the basis of simple roots corresponding to B. For a subset Π of Δ, let $P = P_\Pi$ be the parabolic subgroup such that $P_\alpha \subseteq P$ for all $\alpha \in \Pi$ and $P_\alpha \not\subseteq P$ for all $\alpha \in \Delta - \Pi$. We denote $\mathscr{V}^{P_\Pi}$ by $\mathscr{V}^\Pi$.

Proposition 5.1 ([12]). *Let $g \in \mathscr{V}$ and let $\Delta_c \subset \Delta$ be the set of all ψ_g-compact simple roots.*

1. *$G^\theta gB$ is open in $G^\theta gP_\Pi$ (i.e., $g \in \mathscr{V}^\Pi$) if and only if Π is contained in $\Delta_c \cup \psi_g(\Phi^-)$.*
2. *$G^\theta gB$ is closed in $G^\theta gP_\Pi$ if and only if Π is contained in $\psi_g(\Phi^+)$.*

One can also use these real, complex, and imaginary roots to characterize the affine orbits. For more details we refer to [12, Section 2]. The above results also lead to the following result in the case where $P = G$.

Proposition 5.2 ([39, Proposition 9.2 and Lemma 1.7]). *The double coset $G^\theta gB$ is open (resp., closed) in G if and only if each simple root is either ψ_g-compact or in $\psi_g(\Phi^-)$ (resp., each simple root is in $\psi_g(\Phi^+)$, i.e., B is ψ_g-stable).*

Remark 5.1. In the Examples 3.1 and 3.2 it is easy to determine the imaginary and compact roots. Recall that in Example 3.1 for $g = (g_1,g_2) \in \mathscr{V}$, ψ_g acts on G by $\psi_g(x_1,x_2) = (nx_2n^{-1}, n^{-1}x_1n)$, where $n = g_1^{-1}g_2 \in \mathbf{N}$, and on roots by $\psi_g(\alpha,0) = (0,w^{-1}(\alpha))$, $\psi_g(0,\alpha) = (w(\alpha),0)$. So in this case there are no ψ_g-imaginary roots. If $\Pi = (\Pi_1 \times 0) \cup (0 \times \Pi_2)$ is a subset of the set of simple roots, and $g \in \mathscr{V}$, then $g \in \mathscr{V}^\Pi$ if and only if $w(\Pi_1)$ and $w^{-1}(\Pi_2)$ are contained in Φ^-. This amounts to: w is the element of maximal length in its $(\mathbf{W}_{\Pi_1}, \mathbf{W}_{\Pi_2})$-double coset.

In Example 3.2 the pair (B,T) is standard, and hence all roots are imaginary. The compact roots in this case are the pairs (i,j) with $1 \le i,j \le n-1$. In particular, for a subset Π of Δ, we have: $g_{i,j} \in \mathscr{V}^{\Pi}$ if and only if α_{i-1} and α_j are not in Π. A more involved example is the following.

Example 5.1. Let $G = \mathrm{GL}_n$ with involution σ defined by $\sigma(g) = (g^{-1})^t$. Then G^{σ} is the orthogonal group O_n. Let B be the Borel subgroup of G consisting of upper triangular matrices, and let T be the maximal torus of diagonal matrices. Then T and B are σ-split. We assume G^{σ} acts from the left and define $\tau(g) = g^{-1}\sigma(g)$. For $g \in \mathscr{V}$, we have $w_g^2 = 1$, and the map $g \mapsto w_g$ induces a bijection from $G^{\sigma}\backslash G/B = G^{\sigma}\backslash \mathscr{V}/T$ onto the set of elements of W of order ≤ 2, see [53, 10.2]. We identify W with the symmetric group S_n, and Φ with the set of pairs (i,j) of distinct integers between 1 and n. Then Δ consists of the pairs $\alpha_i = (i, i+1)$, $1 \leq i \leq n-1$. We have $\psi_g(i,j) = (w_g(j), w_g(i))$, hence the ψ_g-imaginary roots are the pairs $(i, w_g(i))$. We claim that there are no ψ_g-compact roots. To see this, let Γ be the copy of GL_2 in G associated with the pair $(i, w_g(i))$. Then ψ_g stabilizes Γ, and acts there by inverse transpose followed with conjugation by a symmetric monomial matrix. A matrix computation shows that $\psi_g(E_{i,w_g(i)}) = -E_{i,w_g(i)}$ where $E_{i,j}$ denotes the elementary $n \times n$ matrix. As a consequence, the imaginary simple roots are the pairs $(i, i+1)$ such that $w_g(i) = i+1$; because $w_g^2 = 1$, these simple roots are pairwise orthogonal.

Let Π be a subset of Δ and let $g \in \mathscr{V}$. By Proposition 5.1, $g \in \mathscr{V}^{\Pi}$ if and only if $w_g(i) < w_g(i+1)$ for any $(i, i+1) \in \Pi$. If $g \in \mathscr{V}^{\Pi}$, then one easily shows that Φ_{Π} is ψ_g-stable if and only if w_g fixes Π pointwise. For more details, see [53, 10.2] and [12, 2.4].

5.1.4 Reduction to involutions that leave Φ^+ invariant

To get a similar description of the twisted involutions as in [56] which will also work in all cases and not only the algebraically closed case, we pass to another involution, which leaves Φ^+ invariant. Let $w_0 \in W$ such that

$$\sigma(\Phi^+) = w_0(\Phi^+), \tag{8}$$

and let $\sigma' = \sigma w_0$. Instead of working with σ we can work again with σ' and w'. As in [39] we get the following results for w_0, σ', and the sets of twisted involutions $\mathscr{I}_{\sigma}$ and $\mathscr{I}_{\sigma'}$:

Proposition 5.3. *Let Φ, Φ^+, σ, w_0, and σ' be as above. Then we have the following properties.*

1. $w_0 \in \mathscr{I}_{\sigma}$.
2. $\sigma'(\Phi^+) = \Phi^+$.
3. σ' *is an involution of* Φ.
4. $\mathscr{I}_{\sigma'} = \mathscr{I}_{\sigma} \cdot w_0$.

For the real, complex, and imaginary roots we again have the following.

Lemma 5.1. *If $w \in \mathscr{I}_\sigma$ and $w' = ww_0$, then $w'\sigma' = w\sigma$. In particular we have*

$$I(w,\sigma) = I(w',\sigma'), \qquad R(w,\sigma) = R(w',\sigma')$$
$$C'(w,\sigma) = C'(w',\sigma'), \qquad C''(w,\sigma) = C''(w',\sigma')$$

We also have again the following characterization of twisted involutions.

Proposition 5.4 ([39, 7.9]). *If $w \in \mathscr{I}_\sigma$ and $w' = ww_0 \in \mathscr{I}_{\sigma'}$, then there exist $s_1,\dots,s_h \in \Sigma$ and a σ'-stable subset Π of Δ satisfying the following conditions.*

(i) $w' = s_1,\dots,s_h w^0_\Pi \sigma'(s_h),\dots,\sigma'(s_1)$ and $l(w') = 2h + l(w^0_\Pi)$.
(ii) $w^0_\Pi \sigma'\alpha = -\alpha$, $\alpha \in \Phi_\Pi$ (i.e., $\Phi^+_\Pi \subset R(w^0_\Pi,\sigma')$).

Moreover if $w' = t_1,\dots,t_m w^0_\Lambda \sigma'(t_m),\dots,\sigma'(t_1)$, where $t_1,\dots,t_m \in \Sigma$ and Λ a σ'-stable subset of Δ satisfying conditions (i) and (ii), then $m = h$, $s_1,\dots,s_h\Pi = t_1,\dots,t_h\Lambda$ and

$$s_1,\dots,s_h\sigma'(s_h),\dots,\sigma'(s_1) = t_1,\dots,t_h\sigma'(t_h),\dots,\sigma'(t_1).$$

5.2 Lifting σ' to an involution of G

In this section we discuss how σ' can be lifted to an involution of G conjugate to σ. Unfortunately this involution is generally not a k-involution. All this involves the following class of parabolic subgroups of G.

Definition 5.2. A parabolic subgroup P in G is called a *quasi-parabolic k-subgroup* if there exist $x \in \mathscr{V}_A$ such that xPx^{-1} is a parabolic k-subgroup.

Using a similar argument as in [39, 2.4] one can show that every quasi minimal parabolic k-subgroup of G contains a σ-stable maximal quasi-k-split torus of G.

Let P be a quasi minimal parabolic k-subgroup of G, $A \subset P$ a σ-stable maximal quasi-k-split torus, $W = W(A)$, $\Phi = \Phi(A)$, and $\mathscr{I} = \mathscr{I}_\sigma$ the set of twisted involutions in W. Let $w \in \mathscr{I}$ and $\xi = w\sigma$. Then ξ is an involution of Φ. In the following we show when ξ can be lifted to an involution of G and when that involution is conjugate to σ.

Lemma 5.2. *Let $w \in W$ and $n \in N_G(A)$ a representative. Then* $\mathrm{Int}(n)\sigma$ *is an involution of G if and only if $\sigma(n) = n^{-1}z$, with $z \in Z(G)$.*

Lemma 5.3. *Let $\mathscr{V}_A$, V_A, $\varphi : V_A \to \mathscr{I}$ be as above. Let $w \in \mathscr{I}$, $n \in N_G(A)$ a representative of w and $\xi =$ $\mathrm{Int}(n)\sigma$. Assume that $\sigma(n) = n^{-1}z$, with $z \in Z(G)$. Then ξ is conjugate to σ if and only if $n \in \tau(\mathscr{V}_A)Z(G)$.*

Proof. Assume first that $x \in G$ such that $\xi = \mathrm{Int}(x)\sigma\,\mathrm{Int}(x^{-1})$. Then $\xi = \mathrm{Int}(n)\sigma = \mathrm{Int}(x\sigma(x)^{-1})\sigma$, so $n \in \tau(\mathscr{V}_A)Z(G)$. Conversely if $n \in \tau(\mathscr{V}_A)Z(G)$, then let $x \in \mathscr{V}_A$ and $z \in Z(G)$ such that $n = x\sigma(x)^{-1}z$. Then $\xi = \mathrm{Int}(n)\sigma = \mathrm{Int}(x)\sigma\,\mathrm{Int}(x^{-1})$ is conjugate to σ.

Combining the above results we get the following.

Proposition 5.5. *Let $\mathscr{V}_A$, V_A, $\varphi : V_A \to \mathscr{I}$ be as above and let $w \in \mathscr{I}$. Then the following are equivalent.*

1. *There exists a representative $n \in N_G(A)$ for w, such that $\xi = \mathrm{Int}(n)\sigma$ is an involution of G conjugate to σ.*
2. *$w \in \varphi(V_A) \subset \mathscr{I}$.*

5.2.1 Lifting σ'

Let P and A be as above. As in (8), let $w_0 \in W = W(A)$ be such that $\sigma(\Phi^+) = w_0(\Phi^+)$ and let $\sigma' = \sigma w_0 = w_0^{-1}\sigma$. We can solve now the question of when the involution σ' can be lifted to a conjugate of σ. By Proposition 5.5 it suffices to show that $w_0 \in \varphi(V_A)$. This is equivalent to the following:

Proposition 5.6. *Let $\mathscr{V}_A$, V_A, $\varphi : V_A \to \mathscr{I}$ and w_0 be as above. Then we have the following:*

(1) *$w_0 \in \varphi(V_A)$ if and only if G contains a σ-stable quasi minimal parabolic k-subgroup.*
(2) *$w_0 \in \varphi_k(V_k)$ if and only if G contains a σ-stable minimal parabolic k-subgroup.*

Proof. Assume first $w_0 \in \varphi(V_A)$. Let $v_0 \in V_A$ be such that $\varphi(v_0) = w_0$ and let $x_0 = x(v_0) \in \mathscr{V}_{\mathscr{A}}$ be a representative of v_0. Then $\tau(x_0)$ is a representative of w_0 in $N_G(A)$. Since $\sigma(\Phi^+) = w_0(\Phi^+)$, we have $\sigma(P) = \tau(x_0)P\tau(x_0^{-1})$. But then $P_1 = \sigma(x_0^{-1})P\sigma(x_0)$ is a σ-stable quasi-parabolic k-subgroup of G.

Conversely, assume $P_0 \subset G$ is a σ-stable quasi-parabolic k-subgroup. By [30, Theorem 3.11] there exists $x \in \mathscr{V}_A$ such that $P_0 = xPx^{-1}$. Since $\sigma(P_0) = P_0$ it follows that

$$\sigma(P) = \sigma(x)^{-1}xPx^{-1}\sigma(x) = \tau(\sigma(x)^{-1})P\tau(\sigma(x))$$

Now $\tau(\sigma(x)^{-1}) \in N_G(A)$. Let $w \in W$ be the corresponding Weyl group element. Then $\sigma(\Phi^+) = w(\Phi^+)$, so $w = w_0 \in \varphi(V_A)$. This shows (5.6). The proof of (5.6) follows with a similar argument replacing V_A and $\mathscr{V}_A$ by V_k and $\mathscr{V}_k$.

If $\xi \in \mathrm{Aut}(G)$ with $\xi(A) = A$, then by abuse of notation we write $\xi|\Phi$ for the action of ξ on Φ. Summarizing the above results we now get the following result.

Corollary 5.1. *Let w_0, σ' be as above. There exists a representative $n \in N_G(A)$ of w_0, such that $\xi = \mathrm{Int}(n)\sigma$ is an involution of G conjugate to σ satisfying $\xi|\Phi = \sigma'$. The involution $\sigma' \in \mathrm{Aut}(\Phi)$ can be lifted to a k-involution if and only if G has a σ-stable minimal parabolic k-subgroup*

Remark 5.2. It was shown in [30] that there always exists a σ-stable quasi minimal parabolic k-subgroup, but σ-stable minimal parabolic k-subgroups of G do not necessarily exist. We discuss these results in more detail in Section 5.3.

5.3 σ-Stable parabolic subgroups

The description of the combinatorics above depends on the existence of a σ-stable conjugate of a minimal parabolic k-subgroup. For k an algebraically closed field, Steinberg proved in [57] that there exists a σ-stable Borel subgroup B and if $T \subset B$ is a σ-stable maximal torus, then $(T \cap H)^0$ is a maximal torus of H. For minimal parabolic k-subgroups this result is not always true as can be seen from the following example.

Example 5.2. (1) $k = \mathbb{R}$, $G = \mathrm{SL}(2,\mathbb{R})$, $\sigma(x) = {}^t x^{-1}$, $H = \mathrm{SO}(2)$. Then H is anisotropic over $\mathbb{R}$, and G has no proper σ-stable parabolic $\mathbb{R}$-subgroup.
(2) $k = \mathbb{R}$, $G = \mathrm{SL}(2,\mathbb{R})$, $\sigma(x) = axa^{-1}$, $a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then H is the subgroup of diagonal matrices. The Borel subgroup of upper triangular matrices is σ-stable.

The minimal σ-stable parabolic k-subgroups can be characterized as follows.

Proposition 5.7 ([39]). *Let P be a σ-stable parabolic k-subgroup, L a σ-stable Levi k-subgroup of P, and A a σ-stable maximal k-split torus of L. Then P is a minimal σ-stable parabolic k-subgroup of G if and only if*

1. *A^+ is a maximal k-split torus of H.*
2. *$L = Z_G(A^+)$.*

P is a σ-stable minimal parabolic k-subgroup of G if and only if it satisfies (1) *and* (2) *and $L = Z_G(A) = Z_G(A^+)$.*

5.3.1 σ-stable quasi-parabolic k-subgroups

As we have seen in the above example, a minimal σ-stable parabolic k-subgroup of G is not necessarily a minimal parabolic k-subgroup of G. For this we need to expand the G_k-conjugacy to G-conjugacy. The existence of a σ-stable conjugate in G^P of a minimal parabolic k-subgroup was proven by Helminck [30]. Before we state the result we need a bit more notation. Let P be a minimal quasi-parabolic k-subgroup of G, $A \subset P$ a σ-stable maximal quasi-k-split torus of G, $X = X^*(A)$ the group of characters of A, $\Phi = \Phi(A)$ the set of roots of A with respect to G and Φ^+ the set of positive roots with respect to a basis Δ of Φ. Let $X_0(-\sigma) = \{\chi \in X \mid \sigma(\chi) = -\chi\}$, $\Phi_0(-\sigma) = \Phi \cap X_0(-\sigma)$ and π the natural projection from X to $X/X_0(-\sigma)$. A linear order $\succ$ on X is called a σ^--order if it has the following property:

$$\text{if } \chi \in X,\ \chi \succ 0, \text{ and } \chi \notin X_0(\sigma), \text{ then } \sigma(\chi) \succ 0. \tag{9}$$

A basis Δ of Φ with respect to a σ^--order on X will be called a σ^--basis of Φ. A σ^--order on X induces orders of $\Phi_0(-\sigma)$ and $\bar{\Phi}_\sigma := \pi(\Phi - \Phi_0(-\sigma))$. Conversely, linear orders on $\Phi_0(-\sigma)$ and $\bar{\Phi}_\sigma$ induce a unique σ^--order on X. We have the following result.

Theorem 5.1 ([30]). *Let P, A, and so on be as above. Then*

1. *There exists a $g \in \mathscr{V}_A$, such that $A_1 = g^{-1}Ag \in \mathscr{A}^\sigma$ is σ-stable and $\Phi_0(\sigma, A_1) := \{\alpha \in \Phi(A_1) \mid \sigma(\alpha) = -\alpha\} = \emptyset$.*
2. *Let Δ_1 be a σ^--basis of $\Phi(A_1)$. Then $\sigma(\Delta_1) = \Delta_1$.*
3. *$P_1 := g^{-1}Pg \supset A_1$ is σ-stable.*

Note that if G does not contain a σ-stable minimal parabolic k-subgroup, then the torus A_1 above is quasi-k-split and $A_1 \notin \mathscr{A}_0^\sigma$ and similarly $P_1H \in V_A$ but not in V_0.

5.4 Relation between σ and σ'

Now that we have shown σ' can be lifted to G and that it is conjugate to σ' we can show that corresponding orbit decompositions for V_A and $\varphi(V_A)$ are again similar. This all can be seen as follows.

Let P, A, $\mathscr{V}_A$, V_A, $\mathscr{I}$ and $\varphi : V_A \to \mathscr{I}$ be as above. Write $\Phi = \Phi(A)$ and $W = W(A)$. Take $w_0 \in W$ such that $\sigma(\Phi^+) = w_0(\Phi^+)$, $n_0 = x_0\sigma(x_0)^{-1} \in N_G(A) \cap \tau(G)$ a representative of w_0^{-1}, $\sigma' = \mathrm{Int}(n_0)\sigma$, and $H' = x_0Hx_0^{-1}$. Then H' is a closed reductive subgroup of G satisfying

$$G^0_{\sigma'} \subset H' \subset G_{\sigma'}.$$

Denote the actions of σ and σ' on Φ also by σ and σ'. Then $\sigma' = \sigma w_0 = w_0^{-1}\sigma$. As for σ let $\tau' : G \to G$ be the map defined by $\tau'(x) = x\sigma'(x)^{-1}$, $\mathscr{V}'_A = \{x \in G \mid \tau'(x) \in N_G(A)\}$, V'_A the set of $(Z_G(A) \times H')$-orbits in $\mathscr{V}'_A$, $\mathscr{I}_{\sigma'}$ the set of twisted involutions of W with respect to σ' and $\varphi' : V'_A \to \mathscr{I}_{\sigma'}$ as in Section 4.2.1.

Let $\iota_{w_0} : \mathscr{I} \to \mathscr{I}_{\sigma'}$ be the right translation by w_0 and let $\delta : V_A \to V'_A$ be the map induced by the map $g \to gx_0^{-1}$ from $\mathscr{V}_A$ to $\mathscr{V}'_A$. Then $\varphi' \circ \delta = \iota_{w_0} \circ \varphi$. So we obtained the following relation between the sets $\mathscr{V}_A$, $\mathscr{V}'_A$, $\mathscr{I}$ and $\mathscr{I}_{\sigma'}$.

Lemma 5.4. *Let $\mathscr{V}_A$, $\mathscr{V}'_A$, $\mathscr{I}$, $\mathscr{I}_{\sigma'}$, x_0, n_0, and w_0 be as above. Then we have the following.*

1. *$\mathscr{V}'_A = \mathscr{V}_A \cdot x_0^{-1}$ and $\tau'(\mathscr{V}'_A) = \tau(\mathscr{V}_A) \cdot n_0^{-1}$.*
2. *$\mathscr{I}_{\sigma'} = \mathscr{I}_\sigma \cdot w_0$.*

5.5 W-Action on $\mathscr{I}_{\sigma'}$

As in Section 4.2.2 there is also an action of the Weyl group W on $\mathscr{I}_{\sigma'}$. Namely if $w \in W$ and $a' \in \mathscr{I}_{\sigma'}$, then define an action $w *' a' = wa'\sigma'(w)^{-1}$. Since $\sigma' = \sigma w_0$ and $a' = aw_0$ for some $a \in \mathscr{I}_{\sigma}$, we get

$$\begin{aligned} w *' a' &= waw_0\sigma' w^{-1}\sigma' = waw_0w_0^{-1}\sigma w^{-1}\sigma w_0 \\ &= wa\sigma w^{-1}\sigma w_0 = (w * a)w_0. \end{aligned}$$

This means that the isomorphism ι_{w_0} is equivariant with respect to the actions of W on $\mathscr{I}$ and $\mathscr{I}_{\sigma'}$. This leads to the following result.

Proposition 5.8. *Let $\mathscr{V}_A$, $\mathscr{V}_A'$, $\mathscr{I}$, $\mathscr{I}_{\sigma'}$, n_0, and w_0 be as above. Then we have the following.*

1. *The map $\iota_{w_0} : \mathscr{I} \to \mathscr{I}_{\sigma'}$ induces an isomorphism between $\mathscr{I}/W$ and $\mathscr{I}_{\sigma'}/W$.*
2. *$\varphi'(V_A')/W \simeq \varphi(V_A)/W$.*

Remark 5.3. In the case where G contains a σ-stable parabolic k-subgroup we can get the same results for V_k and $\varphi_k(V_k)$.

5.6 Orbit closures

In this subsection we discuss a number of results about the orbit closures and the Bruhat type order induced by the closure relations. First some more notation.

5.6.1 Some notation

Let P be a fixed minimal parabolic k-subgroup of G and A a σ-stable maximal k-split torus of P. Let $\Phi(G,A)$ denote the set of roots of A in G and $\Phi^+ = \Phi(P,A)$ with basis Δ. Let s_α denote the reflection defined by $\alpha \in \Phi$ and write $\Sigma = \{s_\alpha \mid \alpha \in \Delta\}$ for the set of simple reflections in W. Then (W,Σ) is a Coxeter group. For $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the root subspace of the Lie algebra $L(G)$ of G corresponding to α. We have the decomposition $L(G) = L(Z_G(A)) \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$. Given $\alpha \in \Delta$, let P_α denote the standard parabolic k-subgroup of G containing P such that $\Phi(P_\alpha,A) = (\mathbb{Z}\alpha \cap \Phi) \cup \Phi^+$. It is easy to see that

$$\dim(P_\alpha) = \dim(P) + \dim(\bigoplus_{\gamma\in\mathbb{Z}\alpha\cap\Phi^+} \mathfrak{g}_\gamma). \tag{10}$$

Let $Q = \{g\sigma(g)^{-1} \mid g \in G\}$ be as in Section 2.1.1. The set Q is stable under the twisted action of G defined by $g * x = gx\sigma(g)^{-1}$, $g \in G, x \in Q$. From Proposition 2.2, there exists $v_0 \in V_k$ such that for $n_0 = x(v_0)\sigma(x(v_0))^{-1}$ the orbit $P * n_0$ is open in Q. Then $P * n_0$ is the unique open orbit of P in Q, called the big cell. This result is of importance in the description of the orbit closures. In the following we describe the closure $\mathrm{cl}(PgH)$ of PgH in G, where $g \in G_k$. In fact we assume $g \in \mathscr{V}^A$ and describe the closure of the orbit $P * n$ in Q, where $n = g\sigma(g)^{-1} \in N_G(A)$. Since $\mathrm{cl}(\mathscr{O}_v) = \tau^{-1}(\mathrm{cl}(P * n))$ this also gives a description of the closures of the orbits $\mathscr{O}_v = PxH$ in G. When k is a local field, G_k is endowed with the topology, called the t-topology, induced from that of k. We also discuss the t-closure of $P_k g H_k$ in G_k.

5.6.2 Closures and twisted involutions

Let w_0 be as in Equation (8), $v \in V_A$, $x = x(v)$, $\mathscr{O}_v = PxH$, $n = x\sigma(x)^{-1}$, w be the image of n in W, and $w' = ww_0$. We write w' as in Proposition 5.4,

$$w' = s_1, \dots, s_h w^0_\Pi \sigma'(s_h), \dots, \sigma'(s_1),$$

with $l(w') = 2h + l(w^0_\Pi)$. Choose $n_1, \dots, n_h \in N_G(T)$ with images $s_1, \dots, s_h$ in W, respectively. Set $s_i = s_{\alpha_i}$ and let $P_i = P_{\alpha_i}$, $1 \leq i \leq h$ be as in Section 5.6.1. If we write $u = n_h^{-1}, \dots, n_1^{-1} x$ and $m = u\sigma(u)^{-1}$, then we have the following description of the orbit closures.

Proposition 5.9 ([39, Proposition 9.5]). *Let P, v, x, n, w, w', u, m, Π be as above. Then we have the following.*

1. $\mathrm{cl}(P * n) = P_1 * \cdots * P_h * P_\Pi * m$.
2. $\mathrm{cl}(\mathscr{O}_v) = \mathrm{cl}(PxH) = \tau^{-1}(\mathrm{cl}(P * n)) = P_1 \dots P_h P_\Pi \mathscr{O}_u$.
3. $\dim(P * n) = \Sigma_{i=1}^h (\dim(P_i) - \dim(P)) + \dim(P_\Pi * m)$.

For the t-closure of a P_k orbit in Q_k when k is a local field we get the following result which bears a striking resemblance to Proposition 5.9.

Proposition 5.10 ([39]).
Let k be a local field, $v \in \tau^{-1} N_k$, $n = v\sigma(v)^{-1}$, w the image of n in W, and $w' = ww_0 = s_1, \dots, s_h w^0_\Pi \sigma'(s_h), \dots, \sigma'(s_1)$ as in Section 5.6.2. Then

$$\text{t-cl}(P(k) * n) = P_1(k) * \cdots * P_h(k) * \text{t-cl}(P(k) * m).$$

To characterize the open and closed orbits in this case we define the following. A minimal parabolic k-subgroup P of G is called *quasi-σ-stable* (resp., quasi-σ-split) if P is contained in a minimal σ-stable parabolic k-subgroup of G (resp., minimal σ-split parabolic k-subgroup of G).

Proposition 5.11 ([39]). *Let k be a local field, H an open k-subgroup of G^σ, and P a minimal parabolic k-subgroup of G. The following conditions are equivalent.*

1. *$P_k H_k$ is closed in G_k.*
2. $\dim(PH) = \min\{\dim(PgH) \mid g \in G_k\}$.
3. *P is quasi-σ-stable.*

Proposition 5.12 ([39]). *Let k, H, and P be as in Proposition 5.11. Then P_kH_k is open in G_k if and only if P is quasi σ-split.*

These results give a fairly detailed description of the orbit closures. It suggests that the closure of an orbit $\mathcal{O}_v$ is to a large extent determined by the corresponding twisted involution $\varphi(v)$. To actually compute the orbits contained in the closure of an orbit one typically uses the Bruhat order defined by the closure relations. We discuss this order in the next section.

5.7 Bruhat order on V_A and V_k

Let $v \in V_A$, $x = x(v)$, and let $\mathcal{O}_v = PxH$ denote the corresponding double coset. This is a smooth subvariety and the closure $\mathrm{cl}(\mathcal{O}_v)$ is a union of double cosets. The *Bruhat order on* V_A is the order $\leq$ on V_A defined by the closure relations on the double cosets $\mathcal{O}_v = Px(v)H$. Thus $v_1 \leq v$ if and only if $\mathcal{O}_{v_1} \subset \mathrm{cl}(\mathcal{O}_v)$.

Similarly if k is a local field the Bruhat order on V_k is the order $\leq_k$ on V_k defined by the closure relations on the double cosets $\mathcal{O}_v^k = P_kx(v)H_k$. Thus $v_1 \leq_k v$ if and only if $\mathcal{O}_{v_1}^k \subset \text{t-cl}(\mathcal{O}_v^k)$.

We can represent the closure relations on the set of double cosets $Px(v)H$ (resp., $P_kx(v)H_k$) by a poset, using the above Bruhat order. This is called the poset of V_A (resp., V_k).

The Bruhat order on V_A' is defined similar as for V_A and will be denoted by $\leq'$. Using the map $\delta : V_A \to V_A'$ as in Section 5.5, we get the following relation between the Bruhat order on V_A and V_A'. Let $v_1, v_2 \in V_A$, then

$$v_1 \leq v_2 \quad \text{if and only if} \quad \delta(v_1) \leq' \delta(v_2). \tag{11}$$

So the posets of the orbits for V_A and V_A' are the same. In particular, if $v \in V_A$ and $v' = \delta(v) \in V_A'$, then the orbit $\mathcal{O}_v$ in G is closed (resp., open) if and only if the orbit $\mathcal{O}_{v'}$ in G is closed (resp., open).

Remark 5.4. In the case of the Bruhat decomposition of the group G (i.e., B-orbits on G/B with B a Borel subgroup), Chevalley gave a combinatorial description of this geometrically defined order. Here the Bruhat order on the $B \times B$-orbits on G corresponds to the combinatorially defined Bruhat order on the Weyl group. In the case where k is algebraically closed and $P = B$ a Borel subgroup acting on the symmetric variety G/H, Richardson and Springer gave a similar combinatorial description of the Bruhat order on V_A (see [53, 54]). They used the map $\varphi : V_A \to \mathcal{I}$, as in Section 4.2.1, to map elements of V_A to twisted involutions in the Weyl group. An additional complication in this case, compared to the Bruhat decomposition of the group, is that this map is not always one-to-one. Note that the Bruhat order on V_A also induces an order on $\mathcal{I}$.

For k not algebraically closed and P a minimal parabolic k-subgroup it is still an open question to define a combinatorial description of the Bruhat order on V_A or

V_k in the case of local fields. For V_A a lot of the structure used by Richardson and Springer in [53] is the same. For example, one can also use the map $\varphi : V_A \to \mathscr{I}$, as in Section 4.2.1, and each orbit closure contains a unique open orbit. However for V_k the latter is certainly no longer true as can be seen from example 4.1, where there are several maximal (σ, k)-split tori, each corresponding with an open orbit.

In the next section we review the combinatorial description of the Bruhat order for the case where k is algebraically closed and $P = B$ a Borel subgroup. Our formulation of the combinatorial Bruhat orders on V_A and $\mathscr{I}_{\sigma'}$ differs slightly from the one in [53].

Remark 5.5. There are several open questions about the geometry of these orbit closures. For example, for connected H, characterize the singularities of the H-orbit closures in G/B and determine in which cases these orbit closures are normal. In the case of the Bruhat decomposition of the group the B-orbit closures in G/B are the Schubert varieties. Moreover, they are normal with rational singularities [50]. Similar results hold for the diagonal action of G on $G/B \times G/B$, since the diagonal is the fixed-point subgroup of the involution of $G \times G$ exchanging both factors (see Example 3.1). An example that the H-orbit closures in G/B need not be normal can be found in [6, page 281].

Other questions include the following. If X is an H-orbit closure in G/B and $\mathscr{L}$ a homogeneous line bundle on G/B having nonzero global sections, describe the H-module $H^0(X, \mathscr{L})$ and the image of the restriction map $\mathrm{res}_X : H^0(G/B, \mathscr{L}) \to H^0(X, \mathscr{L})$. In particular, in which cases is res_X surjective? In the case of the Bruhat decomposition of the group the spaces $H^0(X, \mathscr{L})$ are the Demazure modules; their character is given by the Demazure character formula, and the maps res_X are surjective. Moreover, the higher cohomology groups $H^i(X, \mathscr{L})$ vanish for $i \geq 1$. Similar results hold for the diagonal G-action on $G/B \times G/B$; see [51]. An example that the maps res_X need not be surjective can be found in [12]. In this paper it is also shown that for certain G^σ-orbit closures $X \subseteq G/B$ called *induced flag varieties* the map res_X is surjective and X is projectively normal in the embedding given by any ample line bundle on G/B.

5.8 Combinatorial Bruhat order on V_A

Throughout this section we assume k is algebraically closed and $P = B$ a Borel subgroup. Then $A = T \subset B$ is a σ-stable maximal torus and $\mathscr{V}_A = \mathscr{V}$, $V_A = V$. The combinatorial description of the Bruhat order on V and $\mathscr{I}$ in [53] is given with respect to a so called standard pair (B, T), i.e., B is a σ-stable Borel subgroup and $T \subset B$ is a σ-stable maximal torus. Using the results about twisted involutions in Section 5.1 and the above description of the orbit closures, one can generalize these results to an arbitrary pair (B, T). We show how the results in [53] carry over to this more general setting. Note that, if the pair (B, T) is not a standard pair, then the combinatorics of the orbit closures are actually related to $\mathscr{I}_{\sigma'}$ and not to $\mathscr{I}$.

So instead of $\varphi : V \to \mathscr{I}$ we need to consider $\iota_{w_0} \circ \varphi : V \to \mathscr{I}_{\sigma'}$, where $\iota_{w_0} : \mathscr{I} \to \mathscr{I}_{\sigma'}$ is as in Section 5.5.

5.8.1 Rank one analysis

To define the combinatorial Bruhat order on V, the first thing we need to do is to analyze the description of the orbit closure in Proposition 5.9 in more detail. The set $P_\Pi \mathscr{O}_u$ as in Proposition 5.9 can also be described by a sequence $P_{s_1}, \dots, P_{s_k}$ acting on a closed orbit, where $(s_1, \dots, s_k)$ is a sequence in Σ. To prove this, it suffices to analyze the rank one situation. A detailed description of this is given in [53]. Roughly the results are as follows. Let $v \in V$, $x = x(v)$ and $\mathscr{O}_v = BxH$ the corresponding double coset. If $s \in \Sigma$, then $P_s \mathscr{O}_v = P_s xH$ is a union of one, two, or three $(B \times H)$-orbits. The rank one analysis leads to the following result, which enables us to define an action of the Coxeter group (W, Σ) on V.

Lemma 5.5. *Let $v \in V$ and $s \in \Sigma$. Then we have the following.*

1. *$P_s \operatorname{cl}(\mathscr{O}_v)$ is closed.*
2. *$P_s \mathscr{O}_v$ contains a unique dense $(B \times H)$-orbit.*

The first statement follows from [58, Lemma 2] and the second statement can be found in [53, Section 4].

5.9 Admissible sequences in V

Related to a sequence $\mathbf{s} = (s_1, \dots, s_k)$ in Σ we can now define a sequence $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_k)$ in V as follows. Let $\mathscr{O}_{v_0}$ be a closed orbit and for $i \in [1, k]$ let $\mathscr{O}_{v_i}$ be the unique dense orbit in $P_{s_i} \mathscr{O}_{v_{i-1}}$. We call $\mathbf{s}$ an *admissible sequence for $v \in V$* if there exists a closed orbit $\mathscr{O}_{v_0}$, such that the sequence $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_k)$ in V satisfies

$$\dim P_{s_{i+1}} \dots P_{s_1} \mathscr{O}_{v_0} > \dim P_{s_i} \dots P_{s_1} \mathscr{O}_{v_0} \quad \text{for } i = 1, \dots, k-1, \tag{12}$$

and $v = v_k$. In this case we also call the pair $(\mathscr{O}_{v_0}, \mathbf{s})$ an *admissible pair for v*.

The above sequences start at a closed orbit and build up from there. The question arises of whether one could start at the opposite end with the open orbit and build down from there. Unfortunately this is not possible. The reason for this is that $P_s \mathscr{O}_v$ can consist of three $(B \times H)$-orbits (one open and two closed). Later on we show that for the twisted involutions in $\mathscr{I}_{\sigma'}$ we can actually go up and down.

The existence of an admissible pair for each $v \in V$ is stated in the following result.

Theorem 5.2. *Let $v \in V$. Then there exists a closed orbit $\mathscr{O}_{v_0}$ and a sequence $\mathbf{s} = (s_1, \dots, s_k)$ in Σ such that* $\operatorname{cl}(\mathscr{O}_v) = P_{s_k} \cdots P_{s_1} \mathscr{O}_{v_0}$ *and* $\dim \mathscr{O}_v = k + \dim \mathscr{O}_{v_0}$.

This result is a refinement of [53, Theorem 4.6] to arbitrary pairs (B,T) and follows easily from Proposition 5.9 and the rank one analysis.

Definition 5.3. If $\mathbf{s} = (s_1,\dots,s_k)$ is a sequence in Σ, then we write $\ell(\mathbf{s}) = k$ for the length of the sequence. The length $L(v)$ of an element $v \in V$ now can be defined as follows. Let $(\mathscr{O}_{v_0}, \mathbf{s})$ be an admissible pair for v. Define $L(v) = k = \dim \mathscr{O}_v - \dim \mathscr{O}_{v_0}$. Since all closed orbits have the same dimension the above definition of length does not depend on the admissible pair for v.

Remark 5.6. The above discussion shows how a sequence $\mathbf{s} = (s_1,\dots,s_k)$ in Σ can be used to define a sequence $\mathbf{v}(\mathbf{s}) = (v_0, v_1,\dots,v_k)$ in V. Using the map $\varphi : V \to \mathscr{I}$ the sequence $\mathbf{s}$ in Σ also defines sequences $\mathbf{a}(\mathbf{s}) = (a_0,a_1,\dots,a_k)$ in $\mathscr{I}$ and $\mathbf{a}'(\mathbf{s}) = (a'_0,a'_1,\dots,a'_k)$ in $\mathscr{I}_{\sigma'}$, where $a_i = \varphi(v_i)$ and $a'_i = (\iota_{w_0} \circ \varphi)(v_i)$. In Section 5.5 we give a combinatorial description of these sequences in $\mathscr{I}$ and $\mathscr{I}_{\sigma'}$.

The set of admissible sequences in V has natural order. This order is defined as follows.

Definition 5.4. Let $x, y \in V$. Then we write $x \preceq y$ if there exist admissible pairs $(\mathscr{O}_{v_0}, \mathbf{s} = (s_1,\dots,s_k))$ for x and $(\mathscr{O}_{v_0}, \mathbf{t} = (t_1,\dots,t_r))$ for y with $k \le r$ and $s_i = t_i$ for $i = 1,\dots k$. It is easy to see that $\preceq$ defines a partial order on V. The order $\preceq$ on V is the same as the standard order on V as defined in [53, Section 5]. In the following we define an order on the set $\mathscr{I}_{\sigma'}$ of twisted involutions similar to this order on V.

5.10 Bruhat order on $\mathscr{I}$ and $\mathscr{I}_{\sigma'}$

In order to give combinatorial definitions of the admissible sequences in $\mathscr{I}$ and $\mathscr{I}_{\sigma'}$ as in Section 5.6, we first have to define actions of (W,Σ) on $\mathscr{I}$ and $\mathscr{I}_{\sigma'}$. We begin with the former following [53, Section 3.1].

If $s \in \Sigma$, define a map $\eta(s) : \mathscr{I} \to \mathscr{I}$ as follows. Let $a \in \mathscr{I}$. If $s * a = a$ then set $\eta(s)(a) = sa$ and if $s * a \neq a$ then set $\eta(s)(a) = s * a$. The map $\eta(s)$ is a bijection of $\mathscr{I}$ of period two which does not have any fixed points. Write $s \circ a$ for $\eta(s)(a)$. This operation extends to an action of (W,Σ) on $\mathscr{I}$ as follows. Let $w \in W$ and $s_1 \cdots s_k$ a reduced expression of w with respect to Δ. Then for $a \in \mathscr{I}$ define $w \circ a = s_k \circ \cdots \circ s_1 \circ a$. One easily shows that this definition does not depend on the reduced expression for $w \in W$.

The action of (W,Σ) on $\mathscr{I}_{\sigma'}$ can be defined similarly. One can also induce this action from the above action of (W,Σ) on $\mathscr{I}$ using the bijection $\iota_{w_0} : \mathscr{I} \to \mathscr{I}_{\sigma'}$ as in Section 5.5. Let $\eta'(s) = \iota_{w_0}\eta(s)\iota_{w_0}^{-1}$ be the bijection of $\mathscr{I}_{\sigma'}$ induced by $\eta(s) : \mathscr{I} \to \mathscr{I}$. Recall that by Lemma 5.4 $a \in \mathscr{I}$ if and only if $aw_0 \in \mathscr{I}_{\sigma'}$. So, if $s * aw_0 = aw_0$ then $\eta'(s)(aw_0) = saw_0$ and if $s * aw_0 \neq aw_0$ then $\eta'(s)(aw_0) = s * (aw_0)$. We write $s \circ' aw_0$ for $\eta'(s)(aw_0)$.

5.10.1 Admissible sequences in $\mathscr{I}_{\sigma'}$

As in V, a sequence $\mathbf{s} = (s_1, \dots, s_k)$ in Σ induces sequences in $\mathscr{I}_{\sigma'}$ and $\mathscr{I}$. These sequences, which we call Σ-sequences in $\mathscr{I}_{\sigma'}$ or $\mathscr{I}$, are defined by induction as follows. The sequence in $\mathscr{I}_{\sigma'}$ is $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$, where $a_0 = 1$ and $a_i = s_i \circ' a_{i-1}$ for $i \in [1, k]$; the sequence in $\mathscr{I}$ is defined similarly. We mainly use sequences in $\mathscr{I}_{\sigma'}$.

These Σ-sequences start at the identity in $\mathscr{I}_{\sigma'}$ or $\mathscr{I}$ and build up from there. One could also start with the longest element in the Weyl group with respect to Δ and build the sequence in $\mathscr{I}_{\sigma'}$ down from there. In that case we get the following. Let $w_\Delta^0 \in W$ be the longest element with respect to the basis Δ. If $\mathbf{s} = (s_1, \dots, s_k)$ is a sequence in Σ, then define a sequence $\mathbf{b}(\mathbf{s}) = (b_0, b_1, \dots, b_k)$ in $\mathscr{I}_{\sigma'}$ by induction as follows. Let $b_0 = w_\Delta^0$ and for $i \in [1, k]$ let $b_i = s_i \circ' b_{i-1}$. Such a sequence is called a w_Δ^0-sequence in $\mathscr{I}_{\sigma'}$.

To define an order on $\mathscr{I}_{\sigma'}$ compatible with this action we need first to define admissible sequences. These are defined as follows. For $w \in W$ let $l(w)$ denote the length of w with respect to the Bruhat order on W.

Definition 5.5. Let $\mathbf{s} = (s_1, \dots, s_k)$ be a sequence in Σ and let $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$ (resp., $\mathbf{b}(\mathbf{s}) = (b_0, b_1, \dots, b_k)$) be the Σ-sequence (resp., w_Δ^0-sequence) in $\mathscr{I}_{\sigma'}$ induced by $\mathbf{s}$. Then $\mathbf{s}$ is called an *admissible sequence* (or an admissible Σ-sequence) if $0 = l(a_0) < l(a_1) < \cdots < l(a_k)$. The sequence $\mathbf{s}$ is called an *admissible w_Δ^0-sequence* if $l(w_\Delta^0) = l(b_0) > l(b_1) > \cdots > l(b_k)$. If $a \in \mathscr{I}_{\sigma'}$, then the sequence $\mathbf{s}$ in Σ is called an *admissible Σ-sequence for a* (resp., an *admissible w_Δ^0-sequence for a*) if $\mathbf{s}$ is an admissible Σ-sequence (resp., admissible w_Δ^0-sequence) and $a_k = a$ (resp., $b_k = a$).

If $a \in \mathscr{I}_{\sigma'}$ and $\mathbf{s} = (s_1, \dots, s_k)$ an admissible Σ-sequence for a, then we define the length of a as $\ell(a) = \ell(\mathbf{s}) = k$.

Using a similar argument to that in [53, Lemma 3.4], it follows that every element of $\mathscr{I}_{\sigma'}$ has an admissible Σ-sequence and an admissible w_Δ^0-sequence.

Lemma 5.6. *For every $a \in \mathscr{I}_{\sigma'}$, there exist sequences $\mathbf{s}$ and $\mathbf{t}$ in Σ, such that $\mathbf{s}$ is an admissible Σ-sequence for a and $\mathbf{t}$ is an admissible w_Δ^0-sequence for a.*

From this result we get the following relation between the length of an admissible Σ-sequence for a and an admissible w_Δ^0-sequence for a; see [53, Lemma 8.18].

Lemma 5.7. *Let $a \in \mathscr{I}_{\sigma'}$, $\mathbf{s}$ be an admissible Σ-sequence for a and $\mathbf{t}$ an admissible w_Δ^0-sequence for a. Then $\ell(\mathbf{t}) = \ell(w_\Delta^0) - \ell(a) = \ell(w_\Delta^0) - \ell(\mathbf{s})$.*

As in V the admissible sequences in $\mathscr{I}_{\sigma'}$ induce partial orders on $\mathscr{I}_{\sigma'}$. In fact the Σ-sequences and w_Δ^0-sequences in $\mathscr{I}_{\sigma'}$ lead to opposite orders in $\mathscr{I}_{\sigma'}$. These orders are defined as follows.

5.10.2 Bruhat order on $\mathscr{I}_{\sigma'}$

Let $a, b \in \mathscr{I}_{\sigma'}$. Define the order $\preceq_1$ on $\mathscr{I}_{\sigma'}$ with respect to the admissible Σ-sequences as follows. Write $a \preceq_1 b$ if there exist admissible Σ-sequences $\mathbf{s} = (s_1, \dots, s_k)$ for a and $\mathbf{t} = (t_1, \dots, t_r)$ for b with $k \le r$ and $s_i = t_i$ for $i = 1, \dots, k$.

Similarly, using admissible w_Δ^0-sequences we get an order $\preceq_2$ on $\mathscr{I}_{\sigma'}$ as follows. If $a, b \in \mathscr{I}_{\sigma'}$, then write $a \preceq_2 b$ if there exist admissible w_Δ^0-sequences $\mathbf{s} = (s_1, \dots, s_k)$ for a and $\mathbf{t} = (t_1, \dots, t_r)$ for b with $k \le r$ and $s_i = t_i$ for $i = 1, \dots, k$.

The orders $\preceq_1$ and $\preceq_2$ define partial orders on $\mathscr{I}_{\sigma'}$. The order $\preceq_1$ is also be called the *Bruhat order on* $\mathscr{I}_{\sigma'}$. The corresponding poset is called the *Richardson–Springer involution poset*. Using a combinatorial argument one can show that this order on $\mathscr{I}_{\sigma'}$ is compatible with the partial order on $\mathscr{I}_{\sigma'}$ induced by the Bruhat order on the Weyl group W. The orders $\preceq_1$ and $\preceq_2$ define opposite orders on $\mathscr{I}_{\sigma'}$. We summarize this in the following result; see [53, Section 8].

Proposition 5.13. *Let $a, b \in \mathscr{I}_{\sigma'}$. Then we have the following.*

1. *An admissible sequence for a can be extended to an admissible sequence for w_Δ^0.*
2. *w_Δ^0 is the longest element of $\mathscr{I}_{\sigma'}$ with respect to $\preceq_1$.*
3. *$a \preceq_1 b$ if and only if $b \preceq_2 a$.*

From the above results it easily follows now that the admissible sequences in V lead to admissible sequences in $\mathscr{I}_{\sigma'}$.

Lemma 5.8. *Let $v \in V$. If $(\mathscr{O}_{v_0}, \mathbf{s} = (s_1, \dots, s_k))$ is an admissible pair for v and $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_r)$ is the corresponding sequence in V, then $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$, with $a_i = \iota_{w_0} \varphi(v_i)$ $(i = 1, \dots, k)$, is an admissible sequence for $a = \iota_{w_0} \varphi(v)$.*

Combining the above results we now obtain the following relation between the Bruhat order on V and the Bruhat order on $\mathscr{I}_{\sigma'}$.

Theorem 5.3. *Let $v \in V$, $a = \iota_{w_0} \varphi(v) \in \mathscr{I}_{\sigma'}$, and $\mathbf{s} = (s_1, \dots, s_k)$ a sequence in Σ. Then $\mathbf{s}$ is an admissible sequence for a if and only if there exists a closed orbit $\mathscr{O}_{v_0}$ such that $(\mathscr{O}_{v_0}, \mathbf{s})$ is an admissible pair for v.*

Although the sets V and $\mathscr{I}_{\sigma'}$ have compatible orders, we can not always identify V as a subset of $\mathscr{I}$. From the above characterization of the elements of V in terms of admissible pairs, we get the following.

Corollary 5.2. *The map $\varphi : V \to \mathscr{I}$ is injective if and only if there is a unique closed orbit.*

An example of a case for which the map φ is injective is the case that $G = \mathrm{SL}(2n+1, k)$ and $\sigma(g) = {}^t g^{-1}$ (see Example 5.1).

5.11 Combinatorics of the Richardson–Springer involution poset

As we have seen above, the Richardson–Springer involution poset plays an important role in the description of the orbit closures. In the remainder of this section we review some results about this poset. In [26] it was shown that one can restrict to the case that $\sigma' = \mathrm{id}$, and $\mathscr{I}_{\mathrm{id}}$ is the set of involutions in W. In particular if G is simple and $\sigma' \neq \mathrm{id}$, then the poset $\mathscr{I}_{\sigma'}$ is the opposite of the poset of $\mathscr{I}_{\mathrm{id}}$:

Theorem 5.4. *If G is simple and $\sigma' \neq \mathrm{id}$, then $\mathbf{s}$ is an admissible Σ-sequence (resp., w_Δ^0-sequence) for $a \in \mathscr{I}_{\sigma'}$ if and only if $\mathbf{s}$ is an admissible w_Δ^0-sequence (resp., Σ-sequence) for an element $b \in \mathscr{I}_{\mathrm{id}}$ and conversely.*

We illustrate this result with the following example.

Example 5.3. The poset in Figure 5.3 gives the poset of $\mathscr{I}_{\mathrm{id}}$ and $\mathscr{I}_{\sigma'}$ for the root system of type A_3. The Weyl group of this root system is the symmetric group S_4. In the figure below we write $\overline{s}_i$ if $s_i w\sigma'(s_i) \neq w$ and s_i otherwise (both in the case that $\sigma' \neq \mathrm{id}$ and $\sigma' = \mathrm{id}$).

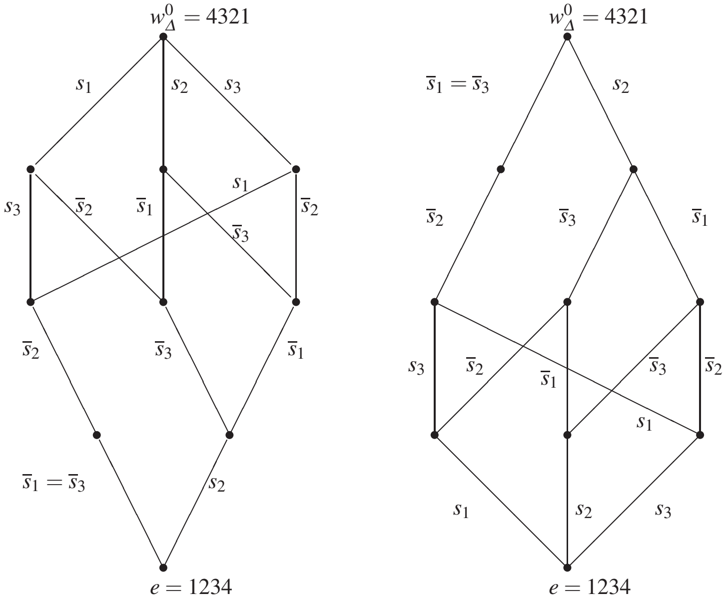

Figure 1. Poset of $\mathscr{I}_{\sigma'}$ and $\mathscr{I}_{\mathrm{id}}$ for $W(A_3) = S_4$.

In addition to the above result there are several results about the combinatorics of this poset as well as algorithms to compute them. This includes a combinatorial description of the length $L(v)$ of an element $v \in V$. For more details we refer to [25, 17].

6 Other orbit decompositions

In addition to the orbits of parabolic k-subgroups acting on symmetric k-varieties as discussed in the previous sections, there are many other orbit decompositions of these symmetric k-varieties that play an important role in the study of these symmetric k-varieties and their applications. Below we briefly review a couple of these orbit decompositions. Examples are orbits of symmetric subgroups acting on symmetric varieties and in the case of local fields orbits of parahoric subgroups acting on symmetric k-varieties.

6.1 Orbits of symmetric subgroups

In the case where k is algebraically closed the orbits of symmetric subgroups acting on symmetric varieties are of importance in representation theory. For H acting on the symmetric variety G/H these orbits were studied by Vust [62] and Richardson [52] and for an arbitrary symmetric subgroup K acting on a symmetric variety these orbits were studied by Helminck and Schwarz in [37]. In this section we briefly review some of the results.

In the following let H and K be the fixed point groups of involutions σ and θ. The groups H and K are reductive and the double coset space $H\backslash G/K$ is, in general, no longer finite. Thus we are led to study the invariant theoretic quotient of G by $H\times K$. For this we need some preliminaries on quotients. If X is an affine G-variety, G-reductive, then $\mathscr{O}(X)^G$, the algebra of invariant functions on X, is finitely generated. Let $X/\!/G$ denote the affine variety corresponding to $\mathscr{O}(X)^G$ and let π (or π_X) denote the morphism $X\to X/\!/G$ dual to the inclusion $\mathscr{O}(X)^G\subset\mathscr{O}(X)$. For $x\in X$, let T_xX denote the tangent space at x, Gx the G-orbit through x and G_x the isotropy group at x. We say that the G-action on X is *stable* if there is a nonempty open subset of X consisting of closed orbits. If all the G-orbits in X are closed (e.g., G is finite), then the quotient is called *geometric*, in which case the notation X/G is also used.

6.1.1 $H=K$ is a symmetric subgroup

This case was studied extensively in [62] and [52]. We assume that G is connected, and that we are given an involution σ of G with $H=G^\sigma$. Write $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ where $\mathfrak{q}$ is the (-1)-eigenspace of σ acting on the Lie algebra $\mathfrak{g}$ of G. There is an isomorphism $\tau\colon G/H\simeq Q\subset G$, where $\tau(gH)=g\sigma(g)^{-1}$ and $Q:=\operatorname{Im}\tau$. The left-action of H on G/H, under the isomorphism τ, becomes conjugation. Let $A\subset Q$ be a maximally σ-split torus, and let W denote the Weyl group of A relative to the action of H, i.e., the normalizer of A in H divided by the centralizer of A in H. Then Vust and Richardson show the following variant of the Chevalley Restriction Theorem for symmetric varieties.

Theorem 6.1 ([52, 62]). *Let G, σ, H, Q, and so on be as above. Then*

1. *The inclusion $A \to Q$ induces an isomorphism $A/W \simeq Q//H$.*
2. *Each fiber of $Q \to Q//H$ consists of finitely many orbits.*

The results of Vust and Richardson also generalize similar results by Kostant and Rallis for real Riemannian symmetric spaces [44].

6.1.2 *H* and *K* are symmetric subgroups

This case was studied extensively in [37]. One has analogous results to those of Vust and Richardson above. We assume that G is connected and that we have two commuting involutions σ and θ. Set $K = G^\theta$ and $H = G^\sigma$. Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{q}$ is the (-1)-eigenspace of σ and $\mathfrak{p}$ is the (-1)-eigenspace of θ. We have $G/H \simeq Q$ as before, but the action of K on Q is not by conjugation. The action is $(x,q) \mapsto x * q := xq\sigma(x)^{-1}$, $x \in K$, $q \in Q$. Let A denote a torus of G which is maximal among tori which are simultaneously σ and θ split. Let $W_K^*(A)$ denote the Weyl group for the $*$-action of K, i.e., $W_K^*(A) = N_K^*(A)/Z_K^*(A)$ where $N_K^*(A) = \{x \in K \mid x * A \subset A\}$ and $Z_K^*(A) = \{x \in K \mid x * a = a \text{ for all } a \in A\}$. We have the following variant of the Chevalley restriction theorem.

Theorem 6.2 ([37]). *Let H, K, and so on be as above.*

1. *The inclusion $A \to Q$ induces an isomorphism $A/W_K^*(A) \simeq Q//K$.*
2. *Each fiber of $Q \to Q//K$ consists of finitely many orbits.*

In [37, 38] we prove many other properties of these double cosets and also give a classification of the groups $W_K^*(A)$ above. In recent work we have expanded some of these results to commuting involutions of real reductive groups. For other fields a characterization of these double cosets is still open, even in the case where $H = K$. The latter case is especially of interest in the study of p-adic symmetric k-varieties and their representations.

6.2 *Orbits of parahoric subgroups*

The analogue of the Riemannian symmetric spaces for p-adic groups are the Euclidean buildings. Bruhat and Tits showed in [13] that most properties of the Riemannian symmetric spaces carry over to these Euclidean buildings. For example they showed that a compact group of isometries of a Euclidean building has a fixed point. They used this to show that in a simply connected simple p-adic group the maximal compact subgroups are parahoric subgroups. There are many other similarities between the Riemannian symmetric spaces and the Euclidean buildings. For example for both a geodesic joining two points is unique. There is also a

p-adic curvature on the building. The Euclidean building also replaces the role of the Riemannian symmetric spaces in the cohomology of discrete subgroups.

In analogy to the double coset space $H_k\backslash G_k/P_k$, it is worthwhile to study the double coset space $H_k\backslash G_k/I_k$, where I is a minimal parahoric subgroup of G, in case of a non-Archimedean local field k. One of the advantages of $H_k\backslash G_k/I_k$ compared to $H_k\backslash G_k/P_k$ is that the Euclidean building carries information about k, whereas the spherical building only carries information about G.

This double coset decomposition is of importance for the study of p-adic symmetric k-varieties and their representations. The structure of $H_k\backslash G_k/I_k$ is still an open problem. We hope to address this problem and many related ones in the near future.

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Root Systems for Levi Factors and Borel–de Siebenthal Theory

Bertram Kostant[1]

Summary. Let $\mathfrak{m}$ be a Levi factor of a proper parabolic subalgebra $\mathfrak{q}$ of a complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{t} = \operatorname{cent}\mathfrak{m}$. A nonzero element $\nu \in \mathfrak{t}^*$ is called a $\mathfrak{t}$-root if the corresponding adjoint weight space $\mathfrak{g}_\nu$ is not zero. If ν is a $\mathfrak{t}$-root, some time ago we proved that $\mathfrak{g}_\nu$ is $\operatorname{ad}\mathfrak{m}$ irreducible. Based on this result we develop in the present paper a theory of $\mathfrak{t}$-roots which replicates much of the structure of classical root theory (case where $\mathfrak{t}$ is a Cartan subalgebra). The results are applied to obtain new results about the structure of the nilradical $\mathfrak{n}$ of $\mathfrak{q}$. Also applications in the case where $\dim\mathfrak{t} = 1$ are used in Borel–de Siebenthal theory to determine irreducibility theorems for certain equal rank subalgebras of $\mathfrak{g}$. In fact the irreducibility results readily yield a proof of the main assertions of the Borel–de Siebenthal theory.

Key words: Levi factors, Borel-de Siebenthal theory, root structures, linear algebraic groups

Mathematics Subject Classification (2000): 20Cxx, 20G05, 17B45, 12xx, 22xx

Dedicated to Gerry Schwarz on the occasion of his 60th birthday

0 Introduction

0.1

Let $\mathfrak{m}$ be a Levi factor of a proper parabolic subalgebra $\mathfrak{q}$ of a complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{t} = \operatorname{cent}\mathfrak{m}$ and let $\mathfrak{s} = [\mathfrak{m},\mathfrak{m}]$ so that one has a direct sum

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$\mathfrak{m} = \mathfrak{t} + \mathfrak{s}$. Let $\mathfrak{r}$ be the Killing form orthocomplement of $\mathfrak{m}$ in $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{m} + \mathfrak{r}$ and $[\mathfrak{m}, \mathfrak{r}] \subset \mathfrak{r}$. A nonzero element $\nu \in \mathfrak{t}^*$ is called a $\mathfrak{t}$-root if $\mathfrak{g}_\nu \neq 0$ where $\mathfrak{g}_\nu = \{z \in \mathfrak{g} \mid \operatorname{ad} x(z) = \nu(x) z, \forall x \in \mathfrak{t}\}$. One readily has $\mathfrak{g}_\nu \subset \mathfrak{r}$ and a direct sum

$$\mathfrak{r} = \sum_{\nu \in R} \mathfrak{g}_\nu \tag{0.1}$$

where $R \subset \mathfrak{t}^*$ is the set of all $\mathfrak{t}$-roots. It is immediate that $\mathfrak{g}_\nu$ is an $\operatorname{ad}\mathfrak{m}$-submodule of $\mathfrak{r}$ for any $\nu \in R$. Some time ago we proved

Theorem 0.1. $\mathfrak{g}_\nu$ *is an irreducible* $\operatorname{ad}\mathfrak{m}$*-module for any* $\nu \in R$ *and any irreducible* $\mathfrak{m}$*-submodule of* $\mathfrak{r}$ *is of this form. In particular* $\mathfrak{r}$ *is a multiplicity-free* $\operatorname{ad}\mathfrak{m}$*-module and equation 0.1 is the unique decomposition of* $\mathfrak{r}$ *as a sum of irreducible* $\operatorname{ad}\mathfrak{m}$*-modules.*

Our Theorem 0.1 appeared, with the apppropriate citations, as Theorem 8.13.3 in [Wol] and Theorem 2.1 in [Jos]. In the present paper we will use Theorem 0.1 (reproved for convenience) to develop a theory of $\mathfrak{t}$-roots which in many ways replicates results in the usual root theory, i.e., the case where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. For example, it is established that if $\mu, \nu \in R$ and $\mu + \nu \in R$, then one has the equality

$$[\mathfrak{g}_\mu, \mathfrak{g}_\nu] = \mathfrak{g}_{\mu+\nu}.$$

Also with respect to a natural inner product on $\mathfrak{t}^*$ if $\mu, \nu \in R$ and $(\mu, \nu) < 0$, then $\mu + \nu \in R$, and if $(\mu, \nu) > 0$, then $\mu - \nu \in R$. (See Theorem 2.2)

The nilradical $\mathfrak{n}$ of $\mathfrak{q}$ is contained in $\mathfrak{r}$, and one introduces a set R^+ of positive $\mathfrak{t}$-roots so that

$$\mathfrak{n} = \sum_{\nu \in R^+} \mathfrak{g}_\nu.$$

As in the Cartan subalgebra case one can similarly define the set $R_{\text{simp}} \subset R^+$ of simple positive $\mathfrak{t}$-roots and prove that if, by definition, $\ell(\mathfrak{t}) = \dim \mathfrak{t}$, then $\operatorname{card} R_{\text{simp}} = \ell(\mathfrak{t})$. (See Theorem 2.7.) In fact if $R_{\text{simp}} = \{\beta_1, \ldots, \beta_{\ell(\mathfrak{t})}\}$, then the β_i are a basis of $\mathfrak{t}^*$ and $(\beta_i, \beta_j) \leq 0$ if $i \neq j$. In addition one proves that $\mathfrak{n}$ is generated by $\mathfrak{g}_{\beta_i}$ for $i = 1, \ldots, \ell(\mathfrak{t})$. In fact we prove that for the nilradical $\mathfrak{n}$ of a parabolic subalgebra of $\mathfrak{g}$ one has

Theorem 0.2. *Except for indexing, the upper central series of* $\mathfrak{n}$ *is the same as the lower central series of* $\mathfrak{n}$*.*

In Section 3 of the paper we deal with the case where $\ell(\mathfrak{t}) = 1$ so that $\mathfrak{q}$ is a maximal parabolic subalgebra. The results are applied here to Borel–de Siebenthal theory and irreducibility theorems are obtained for the adjoint action of equal (to that of $\mathfrak{g}$) rank subalgebras $\mathfrak{g}^{a_j}$ of $\mathfrak{g}$ on the Killing form orthocomplement of $\mathfrak{g}^{a_j}$ in $\mathfrak{g}$. In Remark 3.9 we also show that these results provide a proof of the main statements of the Borel–de Siebenthal theory.

1 Levi Factor Foot System

1.1

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let ℓ be the rank of $\mathfrak{g}$. Let $\mathfrak{h}^*$ be the dual space to $\mathfrak{h}$ and let $\Delta \subset \mathfrak{h}^*$ be the set of roots for the pair $(\mathfrak{h},\mathfrak{g})$. For each $\varphi \in \Delta$, let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector. Let $\Delta_+ \subset \Delta$ be a choice of positive roots and let $\Pi \subset \Delta$ be the set of simple positive roots. Let $\Delta_- = -\Delta_+$. For any $\varphi \in \Delta$ and $\alpha \in \Pi$, let $n_\alpha(\varphi) \in \mathbb{Z}$ be the integer (nonnegative if $\varphi \in \Delta_+$ and nonpositive if $\varphi \in \Delta_-$) so that

$$\varphi = \sum_{\alpha\in\Pi} n_\alpha(\varphi)\,\alpha. \tag{1.1}$$

If $\mathfrak{u} \subset \mathfrak{g}$ is any subspace which is stable under $\operatorname{ad}\mathfrak{h}$, let

$$\begin{aligned} \Delta(\mathfrak{u}) &= \{\varphi \in \Delta \mid e_\varphi \in \mathfrak{u}\} \\ \Delta_+(\mathfrak{u}) &= \Delta(\mathfrak{u}) \cap \Delta_+ \\ \Delta_-(\mathfrak{u}) &= \Delta(\mathfrak{u}) \cap \Delta_-. \end{aligned} \tag{1.2}$$

Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$, containing $\mathfrak{h}$ such that $\Delta(\mathfrak{b}) = \Delta_+$. Let $\mathfrak{n}_\mathfrak{b} = [\mathfrak{b},\mathfrak{b}]$ be the nilradical of $\mathfrak{b}$. A standard parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is any Lie subalgebra of $\mathfrak{g}$ which contains $\mathfrak{b}$.

Let B be the Killing form on $\mathfrak{g}$. Assume that $\mathfrak{q}$ is some fixed standard parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{q}$ admits a unique Levi decomposition

$$\mathfrak{q} = \mathfrak{m} + \mathfrak{n} \tag{1.3}$$

where $\mathfrak{n} = \mathfrak{n}_\mathfrak{q}$ $(\subset \mathfrak{n}_\mathfrak{b})$ is the nilradical of $\mathfrak{q}$ and $\mathfrak{m} = \mathfrak{m}_\mathfrak{q}$ is the unique Levi factor of $\mathfrak{q}$ which contains $\mathfrak{h}$. We will assume throughout that $\mathfrak{q} \neq \mathfrak{g}$ so that $\mathfrak{n} \neq 0$. Let $\mathfrak{s} = [\mathfrak{m},\mathfrak{m}]$ so that $\mathfrak{s}$ is the unique maximal semisimple ideal in $\mathfrak{m}$. Let $\mathfrak{t}$ be the center of $\mathfrak{m}$ so that B is nonsingular on both $\mathfrak{t}$ and $\mathfrak{m}$ and

$$\mathfrak{m} = \mathfrak{t} \oplus \mathfrak{s} \tag{1.4}$$

is a B-orthogonal decomposition of $\mathfrak{m}$ into a direct sum of ideals. Let $\mathfrak{h}(\mathfrak{s}) = \mathfrak{h} \cap \mathfrak{s}$ so that $\mathfrak{h}(\mathfrak{s})$ is a Cartan subalgebra of $\mathfrak{s}$. B is nonsingular on $\mathfrak{h}(\mathfrak{s})$ and

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{h}(\mathfrak{s}) \tag{1.5}$$

is a B-orthogonal decomposition.

The nonsingular bilinear form, $B|\mathfrak{h}$, on $\mathfrak{h}$ induces a nonsingular bilinear form on $\mathfrak{h}^*$ which we denote by $B|\mathfrak{h}^*$. We may embed the dual spaces $\mathfrak{t}^*$ and $\mathfrak{h}(\mathfrak{s})^*$ to $\mathfrak{t}$ and $\mathfrak{h}(\mathfrak{s})$, respectively, in $\mathfrak{h}^*$ so that $\mathfrak{t}^*$ is the orthocomplement of $\mathfrak{h}(\mathfrak{s})$ and $\mathfrak{h}(\mathfrak{s})^*$ is the orthocomplement to $\mathfrak{t}$. Then $B|\mathfrak{h}^*$ is nonsingular on both $\mathfrak{t}^*$ and $\mathfrak{h}(\mathfrak{s})^*$ and

$$\mathfrak{h}^* = \mathfrak{t}^* \oplus \mathfrak{h}(\mathfrak{s})^* \tag{1.6}$$

is a $B|\mathfrak{h}^*$ orthogonal direct sum.

Let $\mathfrak{h}^*_{\mathbb{R}}$ be the real form of $\mathfrak{h}^*$ spanned over $\mathbb{R}$ by Δ. As one knows, $B|\mathfrak{h}^*$ is positive definite on $\mathfrak{h}^*_{\mathbb{R}}$. On the other hand, similarly, $\Delta(\mathfrak{s})$ clearly spans over $\mathbb{R}$, a real form, $\mathfrak{h}(\mathfrak{s})^*_{\mathbb{R}}$, of $\mathfrak{h}(\mathfrak{s})^*$. Of course $\mathfrak{h}(\mathfrak{s})^*_{\mathbb{R}}$ is a real subspace of $\mathfrak{h}^*_{\mathbb{R}}$ and, clearly, if $\mathfrak{t}^*_{\mathbb{R}}$ is the $B|\mathfrak{h}^*$ orthocomplement of $\mathfrak{h}(\mathfrak{s})^*_{\mathbb{R}}$ in $\mathfrak{h}^*_{\mathbb{R}}$, then $\mathfrak{t}^*_{\mathbb{R}}$ is a real form of $\mathfrak{t}^*$ and

$$\mathfrak{h}^*_{\mathbb{R}} = \mathfrak{t}^*_{\mathbb{R}} \oplus \mathfrak{h}(\mathfrak{s})^*_{\mathbb{R}} \tag{1.7}$$

is a real Hilbert space orthogonal direct sum. For any $\gamma \in \mathfrak{h}^*_{\mathbb{R}}$ we let $\gamma_{\mathfrak{t}} \in \mathfrak{t}^*_{\mathbb{R}}$ and $\gamma_{\mathfrak{s}} \in \mathfrak{h}(\mathfrak{s})^*_{\mathbb{R}}$, respectively, be the components of γ with respect to the decomposition (1.7) so that

$$\gamma = \gamma_{\mathfrak{t}} + \gamma_{\mathfrak{s}} \tag{1.8}$$

and

$$(\gamma_{\mathfrak{t}}, \gamma_{\mathfrak{s}}) = 0 \tag{1.9}$$

where (μ,λ) denotes the $B|\mathfrak{h}^*$-pairing of any $\mu,\lambda \in \mathfrak{h}^*$.

Let $\overline{\mathfrak{n}}$ be the span of all $e_{-\varphi}$, for $\varphi \in \Delta(\mathfrak{n})$ so that one has a triangular decomposition

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{n} + \overline{\mathfrak{n}}. \tag{1.10}$$

Now put $\mathfrak{r} = \mathfrak{n} + \overline{\mathfrak{n}}$ so that $\mathfrak{r}$ is $\operatorname{ad}\mathfrak{m}$-stable and one has a B-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{r}. \tag{1.11}$$

Remark 1.1. From the general properties of Levi factors of parabolic subalgebras one knows that $\mathfrak{m}$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$ so that

$$\Delta(\mathfrak{r}) = \{\varphi \in \Delta \mid \varphi_{\mathfrak{t}} \neq 0\}. \tag{1.12}$$

1.2

Let V be a $\mathfrak{t}$-module and let $\mu \in \mathfrak{t}^*$. Put

$$V_\mu = \{v \in V \mid x \cdot v = \langle \mu, x\rangle\, v,\ \forall x \in \mathfrak{t}\}. \tag{1.13}$$

The subspace V_μ is called the μ-weight space (for $\mathfrak{t}$) of V. If $V_\mu \neq 0$, then μ is called a $\mathfrak{t}$-weight of V and any $v \in V_\mu$ is called a μ-weight vector.

If V is a finite-dimensional $\mathfrak{g}$-module, then necessarily $\gamma \in \mathfrak{h}^*_{\mathbb{R}}$ where γ is any $\mathfrak{h}$-weight of V. One then notes $\mu \in \mathfrak{t}^*$ is a $\mathfrak{t}$-weight of V if and only if

$$\mu = \gamma_{\mathfrak{t}} \text{ where } \gamma \text{ is an } \mathfrak{h}\text{-weight of } V. \tag{1.14}$$

An important special case is when $V = \mathfrak{g}$ and the module structure is defined by the adjoint action. Let R' be the set of all $\mathfrak{t}$-weights of $\mathfrak{g}$. If V is any $\mathfrak{g}$-module and ξ is a $\mathfrak{t}$-weight of V, it is obvious that, for any $\mu \in R'$,

$$\mathfrak{g}_\mu \cdot V_\xi \subset V_{\mu+\xi}. \tag{1.15}$$

Clearly $0 \in R'$ and

$$\mathfrak{g}_0 = \mathfrak{m} \tag{1.16}$$

so that V_ξ is an $\mathfrak{m}$-module.

Remark 1.2. If V is finite dimensional then, since $\mathfrak{s} \subset \mathfrak{m}$ and $\mathfrak{s}$ is semisimple, note that V_ξ is a completely reducible $\mathfrak{s}$-module.

If V is equal to the adjoint $\mathfrak{g}$-module $\mathfrak{g}$ in (1.15) one has

$$[\mathfrak{g}_\nu, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\nu+\mu} \tag{1.17}$$

for any $\nu, \mu \in R'$. In particular, if $\mu \in R'$, then

$$[\mathfrak{m}, \mathfrak{g}_\mu] \subset \mathfrak{g}_\mu \tag{1.18}$$

and

$$\mathfrak{g}_\mu \text{ is a completely reducible } \mathfrak{s}\text{-module for the maximal semisimple ideal } \mathfrak{s} \text{ of } \mathfrak{m}. \tag{1.19}$$

Let $R = R' \setminus \{0\}$ so that, recalling Remark 1.1,

$$R = \{\nu \in \mathfrak{t}^* \mid \nu = \varphi_\mathfrak{t}, \text{ for some } \varphi \in \Delta(\mathfrak{r})\}, \tag{1.20}$$

and one readily has (see (1.16)) the direct sum

$$\mathfrak{g} = \mathfrak{m} + \sum_{\nu \in R} \mathfrak{g}_\nu. \tag{1.21}$$

We refer to the elements in R as $\mathfrak{t}$-roots (in $\mathfrak{g}$) and $\mathfrak{g}_\nu$ as the $\mathfrak{t}$-root space corresponding to $\nu \in R$. Partially summarizing one readily has

Proposition 1.3. *One has the two disjoint unions*

$$\begin{aligned} \Delta &= \sqcup_{\mu \in R'} \Delta(\mathfrak{g}_\mu) \\ \Delta(\mathfrak{r}) &= \sqcup_{\nu \in R} \Delta(\mathfrak{g}_\nu). \end{aligned} \tag{1.22}$$

Furthermore if $\nu \in R$, *then*

$$\begin{aligned} &\Delta(\mathfrak{g}_\nu) = \{\varphi \in \Delta \mid \varphi_\mathfrak{t} = \nu\}, \textit{ and} \\ &\{e_\varphi \mid \varphi \in \Delta(\mathfrak{g}_\nu), \textit{ is a basis of } \mathfrak{g}_\nu\}. \end{aligned} \tag{1.23}$$

Let (x, y) denote the pairing of $x, y \in \mathfrak{g}$ defined by B.

Remark 1.4. Note that $\nu \in R$ if and only if $-\nu \in R$ and

$$\Delta(\mathfrak{g}_{-\nu}) = -\Delta(\mathfrak{g}_\nu). \tag{1.24}$$

Furthermore if $\mu, \nu \in R$ and $\nu \neq -\mu$, then

$$(\mathfrak{g}_\mu, \mathfrak{g}_\nu) = 0 \tag{1.25}$$

and

$$\mathfrak{g}_\nu \text{ and } \mathfrak{g}_{-\nu} \text{ are nonsingularly paired by } B. \tag{1.26}$$

Let $\tau : \mathfrak{h} \to \mathfrak{h}^*$ be the linear isomorphism defined by $B|\mathfrak{h}$. Thus for $x \in \mathfrak{h}$ and $\mu \in \mathfrak{h}^*$,

$$\begin{aligned} \langle \mu, x \rangle &= (\mu, \tau(x)) \\ &= (\tau^{-1}(\mu), x). \end{aligned} \tag{1.27}$$

Thus if $\mathfrak{t}_{\mathbb{R}} = \tau^{-1}(\mathfrak{t}^*_{\mathbb{R}})$, then one readily has

Proposition 1.5.

$$\begin{aligned} &(1)\ \mathfrak{t}_{\mathbb{R}} \textit{ is a real form of } \mathfrak{t} \\ &(2)\ B \textit{ is real and positive definite on } \mathfrak{h}_{\mathbb{R}} \\ &(3)\ \mathfrak{t}_{\mathbb{R}} = \{x \in \mathfrak{t} \mid \langle \nu, x \rangle \textit{ is real } \forall \nu \in R\} \end{aligned} \tag{1.28}$$

Let $\nu \in R$. Clearly $\operatorname{Ker}\nu$ has codimension 1 in $\mathfrak{t}$. It follows from Proposition 1.5 that there exists a unique element $h_\nu \in \mathfrak{t}_{\mathbb{R}}$ which is B-orthogonal to $\operatorname{Ker}\nu$ and such that

$$\langle \nu, h_\nu \rangle = 2. \tag{1.29}$$

Note that it follows from 1.27 that

$$\tau(h_\tau) = 2\nu/(\nu, \nu). \tag{1.30}$$

Let $\nu \in R$ and put $\mathfrak{m}(\nu) = [\mathfrak{g}_\nu, \mathfrak{g}_{-\nu}]$ so that $\mathfrak{m}(\nu)$ is an ideal of $\mathfrak{m}$. Let $\mathfrak{t}(\nu) = \mathfrak{m}(\nu) \cap \mathfrak{t}$ and $\mathfrak{s}(\nu) = \mathfrak{m}(\nu) \cap \mathfrak{s}$. Decomposing the adjoint action representation of $\mathfrak{s}$ on $\mathfrak{m}(\nu)$ into its primary components, it is clear that $\mathfrak{t}(\nu)$ is the primary component corresponding to the trivial representation and $\mathfrak{s}(\nu)$ is the sum of the remaining components. Consequently one has the direct sum

$$\mathfrak{m}(\nu) = \mathfrak{t}(\nu) \oplus \mathfrak{s}(\nu). \tag{1.31}$$

Remark 1.6. If $\mathfrak{s}_i$ is a simple component of $\mathfrak{s}$, one has $\Delta(\mathfrak{s}_i) = -\Delta(\mathfrak{s}_i)$ so that $B|\mathfrak{s}_i$ is nonzero and hence nonsingular by simplicity. On the other hand if $\mathfrak{s}_i$ and $\mathfrak{s}_j$ are distinct simple components then, of course, $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$. Consequently $\mathfrak{s}_i$ and $\mathfrak{s}_j$ are B-orthogonal by the invariance of B and the equality $[\mathfrak{s}_i, \mathfrak{s}_i] = \mathfrak{s}_i$.

Proposition 1.7. *Let $\nu \in R$. Then*

$$\mathfrak{t}(\nu) = \mathbb{C}h_\nu. \tag{1.32}$$

In addition $B|\mathfrak{m}(\nu)$ is nonsingular and the kernel of the adjoint action of $\mathfrak{m}$ on $\mathfrak{g}_\nu$ is the orthocomplement of $\mathfrak{m}(\nu)$ in $\mathfrak{m}$. In particular $\mathfrak{m}(\nu)$ operates faithfully on $\mathfrak{g}_\nu$.

Proof. Let $\mathfrak{m}(\nu)^\perp$ be the B-orthogonal subspace to $\mathfrak{m}(\nu)$ in $\mathfrak{m}$. Let $x \in \mathfrak{m}$, $y \in \mathfrak{g}_\nu$, and $z \in \mathfrak{g}_{-\nu}$. But $(x,[y,z]) = ([x,y],z)$. Since $\mathfrak{g}_\nu$ and $\mathfrak{g}_{-\nu}$ are orthogonally paired by B this proves that $\mathfrak{m}(\nu)^\perp$ is the kernel of the adjoint action of $\mathfrak{m}$ on $\mathfrak{g}_\nu$. But then $\operatorname{Ker}\nu = \mathfrak{m}(\nu)^\perp \cap \mathfrak{t}$. Recalling the definition and properties of h_ν (see (1.29) and Proposition 1.5) it follows immediately that $B|\operatorname{Ker}\nu$ is nonsingular. It then follows that $\mathfrak{t}(\nu)$ must be the one-dimensional B-orthocomplement of $\operatorname{Ker}\nu$ in $\mathfrak{t}$. But then one has (1.32) by definition of h_ν. But now $\mathfrak{s}(\nu)$ is clearly an ideal in $\mathfrak{s}$. Hence $\mathfrak{s}(\nu)$ is a sum of simple components of $\mathfrak{s}$. Thus $B|\mathfrak{m}(\nu)$ is nonsingular by Remark 1.6. □

1.3

In this section we will mainly be concerned with decomposing $\mathfrak{r}$ into irreducible $\mathfrak{m}$-modules. Effectively this comes down to understanding the action of $\mathfrak{s}$ on $\mathfrak{g}_\nu$ for any $\nu \in R$.

Obviously $\Delta_+(\mathfrak{s})$ defines a choice of positive roots in $\Delta(\mathfrak{s})$ so that

$$\mathfrak{b}(\mathfrak{s}) = \mathfrak{h}(\mathfrak{s}) + \sum_{\varphi \in \Delta_+(\mathfrak{s})} \mathbb{C}e_\varphi$$

is a Borel subalgebra of $\mathfrak{s}$. Highest weights and highest weight vectors for $\mathfrak{s}$-modules will be defined with respect to $\mathfrak{b}(\mathfrak{s})$.

Let $C(\mathfrak{s}) \subset \mathfrak{h}(\mathfrak{s})^*_\mathbb{R}$ be the dominant Weyl chamber.

Proposition 1.8. *Let $\xi, \eta \in C(\mathfrak{s})$. Then*

$$(\xi,\eta) \geq 0. \tag{1.33}$$

Proof. Let I be an index set for the simple components of $\mathfrak{s}$ where, if $i \in I$, then $\mathfrak{s}_i$ is the corresponding component. One readily has

$$C(\mathfrak{s}) = \sum_{i \in I} C(\mathfrak{s}_i)$$

where $\mathfrak{h}(\mathfrak{s}_i) = \mathfrak{h} \cap \mathfrak{s}_i$ and $C(\mathfrak{s}_i) \subset \tau(\mathfrak{h}(\mathfrak{s}_i))$ is the dominant Weyl chamber for $\mathfrak{s}_i$. But then, if $i, j \in I$ are distinct, $C(\mathfrak{s}_i)$ and $C(\mathfrak{s}_j)$ are $B|\mathfrak{h}^*$-orthogonal by (1.27) and Remark 1.6. Thus it suffices to prove that, if $i \in I$, and $\xi, \eta \in C(\mathfrak{s}_i)$ are nonzero, then

$$(\xi,\eta) > 0. \tag{1.34}$$

But $B|\mathfrak{s}_i$ is a positive multiple of the $\mathfrak{s}_i$-Killing form and for the $\mathfrak{s}_i$-Killing form (1.34) is known (see e.g., Lemma 2.4 in [Kos]). □

We established the following theorem some time ago. It appears in the literature in works of J. Wolf and A. Joseph with proper citations in both cases. See Theorem 8.13.3 in [Wol] and in a closer reproduction of my argument, Theorem 2.1 in [Jos].

Theorem 1.9. *Let $\nu \in R$. Then the $\mathfrak{t}$-root subspace $\mathfrak{g}_\nu$ is an irreducible $\mathfrak{m}$ and irreducible $\mathfrak{s}$-module under the adjoint action. In fact it is a faithful irreducible $[\mathfrak{g}_\nu, \mathfrak{g}_{-\nu}] \subset \mathfrak{m}$-module and in the notation of (1.31) an irreducible $\mathfrak{s}(\nu)$-module. In addition*

$$\mathfrak{r} = \sum_{\nu \in R} \mathfrak{g}_\nu \tag{1.35}$$

is a multiplicity-one representation of $\mathfrak{m}$ and the summands (i.e., the $\mathfrak{t}$-root spaces) on the right hand side of (1.35) are the irreducible components.

Proof. Each of the summands on the right side of (1.35) affords a different character of $\mathfrak{t}$ and hence these summands are inequivalent as $\mathfrak{m}$-modules. Recalling Proposition 1.7 it suffices only to prove that $\mathfrak{g}_\nu$ is an irreducible $\mathfrak{s}$-module. The elements $\{e_\varphi \mid \varphi \in \Delta(\mathfrak{g}_\nu)\}$ are a weight basis of $\mathfrak{g}_\nu$ for the Cartan subalgebra $\mathfrak{h}(\mathfrak{s})$. Moreover, since root spaces for $\mathfrak{h}$ have multiplicity-one, the $\mathfrak{h}(\mathfrak{s})$-weights in $\mathfrak{g}_\nu$ have multiplicity-one since $\varphi_{\mathfrak{t}} = \varphi'_{\mathfrak{t}}$ for $\varphi, \varphi' \in \Delta(\mathfrak{g}_\nu)$.

Assume $\mathfrak{g}_\nu$ is not $\mathfrak{s}$-irreducible. Then there exists distinct $\varphi, \varphi' \in \Delta(\mathfrak{g}_\nu)$ such that e_φ and $e_{\varphi'}$ are $\mathfrak{s}$-highest weight vectors. In particular $\varphi_{\mathfrak{s}}$ and $\varphi'_{\mathfrak{s}}$ are in $C(\mathfrak{s})$. But then

$$(\varphi_{\mathfrak{s}}, \varphi'_{\mathfrak{s}}) \geq 0 \tag{1.36}$$

by Proposion 1.8. But

$$\begin{aligned} \varphi &= \nu + \varphi_{\mathfrak{s}} \\ \varphi' &= \nu + \varphi'_{\mathfrak{s}}. \end{aligned} \tag{1.37}$$

Hence

$$(\varphi, \varphi') > 0. \tag{1.38}$$

Thus $\beta = \varphi - \varphi'$ is a root. Furthermore $\beta_{\mathfrak{t}} = 0$ so that $\beta \in \Delta(\mathfrak{m}) = \Delta(\mathfrak{s})$. Without loss of generality we may choose the ordering so that $\beta \in \Delta_+(\mathfrak{s})$. But then $[e_\beta, e_{\varphi'}]$ is a nonzero multiple of e_φ. This contradicts the fact that $e_{\varphi'}$ is an $\mathfrak{s}$-highest weight vector. □

2 Properties of the $\mathfrak{t}$-root System

2.1

We will utilize Theorem 1.9 to establish some properties of R. To begin with

Lemma 2.1. *Assume $\nu, \mu \in R$ and $\nu + \mu \neq 0$. Assume also that $[\mathfrak{g}_\nu, \mathfrak{g}_\mu] \neq 0$. Then $\nu + \mu \in R$ (obvious) and one has the equality*

$$[\mathfrak{g}_\nu, \mathfrak{g}_\mu] = \mathfrak{g}_{\nu+\mu}. \tag{2.1}$$

Proof. The left side of (2.1) is a nonzero $\mathfrak{m}$-submodule of the right side. One therefore has the equality (2.1) by irreducibility. □

Let $p, q \in \mathbb{Z}$ where $p \leq q$. Let $I_{p,q}$ denote the set of integers m such that $p \leq m \leq q$. A finite nonempty subset $I \subset \mathbb{Z}$ will be called an interval if it is of the form $I_{p,q}$.

Theorem 2.2. *Let $\nu \in R$ and assume V is a finite-dimensional $\mathfrak{g}$-module with respect to a representation π. Let γ be a $\mathfrak{t}$-weight of V and let*

$$I = \{j \in \mathbb{Z} \mid \gamma + j\nu \text{ be a } \mathfrak{t}\text{-weight of } V\},$$

noting that I is of course finite and not empty since $0 \in I$. Then there exist $p \leq 0 \leq q$, $p, q \in \mathbb{Z}$ such that

$$I = I_{p,q}. \tag{2.2}$$

Moreover if I has only one element (i.e., $p = q = 0$), then $(\gamma, \nu) = 0$. Furthermore if I has more than one element (i.e., $p < q$), then

$$\begin{aligned} &(\gamma + q\nu, \nu) > 0 \text{ and} \\ &(\gamma + p\nu, \nu) < 0. \end{aligned} \tag{2.3}$$

Finally let $m \in I_{p,q}$. If $m < q$. Then

$$\pi(\mathfrak{g}_\nu)(V_{\gamma+m\nu}) \neq 0, \tag{2.4}$$

and if $p < m$, then

$$\pi(\mathfrak{g}_{-\nu})(V_{\gamma+m\nu}) \neq 0. \tag{2.5}$$

Proof. Let $X = \sum_{j\in I} V_{\gamma+j\nu}$ so that X is stable under $\pi(\mathfrak{g}_\nu)$ and $\pi(\mathfrak{g}_{-\nu})$ as well as $\pi(\mathfrak{m})$. One notes that, by (1.30), if $j \in I$, then

$$V_{\gamma+j\nu} \text{ is the eigenspace of } \pi(h_\nu)|X \text{ corresponding to the eigenvalue } \langle\gamma, h_\nu\rangle + 2j. \tag{2.6}$$

Now since $h_\nu \in [\mathfrak{g}_\nu, \mathfrak{g}_{-\nu}]$ one must have

$$\operatorname{tr} \pi(h_\nu)|Y = 0 \tag{2.7}$$

where $Y \subset X$ is any subspace which is stable under $\pi(\mathfrak{g}_\nu)$ and $\pi(\mathfrak{g}_{-\nu})$. If I has one element, then obviously $I = I_{0,0}$ and one has $(\gamma, \nu) = 0$ by (2.7). Thus it suffices to consider the case where I has more than one element. Now if Y_1, Y_2 are two nonzero subspaces of X that are both stable under $\pi(\mathfrak{g}_\nu)$ and $\pi(\mathfrak{g}_{-\nu})$, it follows from (2.7) that one cannot have that the

$$\text{maximal eigenvalue of } \pi(h_\nu) \text{ in } Y_1 < \text{ the minimal eigenvalue of } \pi(h_\nu) \text{ in } Y_2. \tag{2.8}$$

Now assume that $p, q \in I$ and $m \in \mathbb{Z}$ is such that $m \notin I$ and $p < m < q$. If we define Y_1 (resp. Y_2) to be the sum of all $V_{\gamma+j\nu}$, where $j \in I$ and $j < m$ (resp. $j > m$), the conditions of (2.8) are satisfied which, as noted above, is a contradiction. Thus $I = I_{p,q}$ for some $p,q \in \mathbb{Z}$ where $q > p$. But then (2.3) follows from (2.7) where $Y = X$.

Now let $m \in I_{p,q}$ where $m < q$. Assume that $\pi(\mathfrak{g}_\tau)(V_{\gamma+m\nu}) = 0$. That is,

$$\pi(e_\varphi)(V_{\gamma+m\nu}) = 0,\ \forall\varphi \in \Delta(\mathfrak{g}_\nu). \tag{2.9}$$

Thus for any $v \in V_{\gamma+(m+1)\nu}$ and $\varphi \in \Delta(\mathfrak{g}_\nu)$, one has

$$\pi(e_\varphi)\pi(e_{-\varphi})v = 0. \tag{2.10}$$

But from the representation theory of three-dimensional simple Lie algebras, (2.10) implies that $\pi(e_{-\varphi})v = 0$. That is

$$\pi(\mathfrak{g}_{-\nu})(V_{\gamma+(m+1)\nu}) = 0. \tag{2.11}$$

But then if Y_1 (resp. Y_2) is the sum of all $V_{\gamma+j\nu}$ for $j \in I_{p,q}$ where $j \leq m$ (resp. $j \geq m+1$), one defines Y_1 and Y_2 satisfying the contradictory (2.8). This proves (2.4). Clearly a similar argument proves (2.5). □

Applying Theorem 2.2 and Lemma 2.1 to the case where $V = \mathfrak{g}$ and π is the adjoint representation, one immediately has the following result asserting that some familar properties of ordinary roots still hold for $\mathfrak{t}$-roots.

Theorem 2.3. *Let $\nu,\mu \in R$. If $\mu+\nu \in R$ (resp. $\mu-\nu \in R$), then*

$$\begin{aligned}[][\mathfrak{g}_\mu, \mathfrak{g}_\nu] &= \mathfrak{g}_{\nu+\mu}\ (\textit{resp.}\\ [\mathfrak{g}_\mu, \mathfrak{g}_{-\nu}] &= \mathfrak{g}_{\mu-\nu}).\end{aligned} \tag{2.12}$$

Furthermore, one indeed has

$$\begin{aligned}\mu+\nu \in R\ (\textit{resp. } \mu-\nu \in R)\ \textit{if}\ (\mu,\nu) < 0\ (\textit{resp. } (\mu,\nu) > 0)\\ \textit{and}\ \mu+\nu \neq 0\,(\textit{resp. } \mu-\nu \neq 0).\end{aligned} \tag{2.13}$$

Moreover

$$\textit{if}\ (\mu,\nu) = 0,\ \textit{then}\ \mu+\nu \in R\ \textit{if and only if}\ \mu-\nu \in R. \tag{2.14}$$

2.2

Recalling (1.3) let $\delta_\mathfrak{n}$ be in the dual space $\mathfrak{m}^*$ to $\mathfrak{m}$ defined so that if $x \in \mathfrak{m}$, then

$$\langle \delta_\mathfrak{n}, x\rangle = \operatorname{tr}\operatorname{ad} x|\mathfrak{n}. \tag{2.15}$$

Since $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ (see (1.10)) are clearly stable under $\operatorname{ad}\mathfrak{m}$ one has a partition $R = R_\mathfrak{n} \cup R_{\overline{\mathfrak{n}}}$ so that

$$\begin{aligned}\mathfrak{n} &= \sum_{\nu \in R_{\mathfrak{n}}} \mathfrak{g}_\nu \\ \overline{\mathfrak{n}} &= \sum_{\nu \in R_{\overline{\mathfrak{n}}}} \mathfrak{g}_\nu .\end{aligned} \tag{2.16}$$

Clearly Remark 1.4 implies

$$R_{\overline{\mathfrak{n}}} = -R_{\mathfrak{n}}. \tag{2.17}$$

Lemma 2.4. *One has $\delta_{\mathfrak{n}} \in \mathfrak{t}^*_{\mathbb{R}}$ (see (1.7)). Furthermore*

$$\begin{aligned}(\delta_{\mathfrak{n}}, \nu) &> 0, \text{ if } \nu \in R_{\mathfrak{n}} \text{ and} \\ (\delta_{\mathfrak{n}}, \nu) &< 0, \text{ if } \nu \in R_{\overline{\mathfrak{n}}}.\end{aligned} \tag{2.18}$$

Proof. Since $\mathfrak{s}$ is semisimple, $\delta_{\mathfrak{n}}$ must vanish on $\mathfrak{s}$ and hence

$$\delta_{\mathfrak{n}} \in \mathfrak{t}^* \tag{2.19}$$

where, besides regarding $\mathfrak{t}^* \subset \mathfrak{h}^*$ as in (1.6), we also regard $\mathfrak{t}^* \subset \mathfrak{m}^*$, using (1.4). Let $\varphi \in \Delta$. Normalize the choice of root vectors so that $(e_\varphi, e_{-\varphi}) = 1$. Let $\mathfrak{a}_\varphi$ be the root TDS corresponding to φ and let $x_\varphi = [e_\varphi, e_{-\varphi}]$. Then as one knows $x_\varphi \in \mathfrak{h}^*_{\mathbb{R}}$ and (see (1.27))

$$\tau(x_\varphi) = \varphi. \tag{2.20}$$

For some index set P, let

$$\mathfrak{g} = \sum_{p \in P} \mathfrak{u}_p \tag{2.21}$$

be the decomposition of $\mathfrak{g}$ into a sum of irreducible $\operatorname{ad}(\mathfrak{a}_\varphi + \mathfrak{h})$-submodules. One clearly has the direct sum (with possibly 0-dimensional summands)

$$\mathfrak{n} = \sum_{p \in P} \mathfrak{n}_p \tag{2.22}$$

where $\mathfrak{n}_p = \mathfrak{n} \cap \mathfrak{u}_p$. Now assume that $\varphi \in \Delta(\mathfrak{n})$. Then since each $\mathfrak{n}_p$ is stable under $\operatorname{ad} e_\varphi$ for any $p \in P$, it is immediate from the representation theory of a TDS that

$$\operatorname{tr} \operatorname{ad} x_\varphi | \mathfrak{n}_p \geq 0.$$

But there exists $p_o \in P$ such that $\mathfrak{n}_{p_o} = \mathbb{C} e_\varphi$ so that $\operatorname{tr} \operatorname{ad} x_\varphi | \mathfrak{n}_{p_o} > 0$. Thus

$$\delta_{\mathfrak{n}}(x_\varphi) > 0. \tag{2.23}$$

But then

$$(\delta_{\mathfrak{n}}, \varphi) > 0 \tag{2.24}$$

by (2.20). Now let $\nu \in R_{\mathfrak{n}}$ and let $\varphi \in \Delta(\mathfrak{g}_\nu)$. But clearly $\nu = \varphi_{\mathfrak{t}}$ (see (1.23)) so that (2.24) and (2.19) imply the first line of (2.18). The second line is implied by (2.17). Since $\mathfrak{h}^*_{\mathbb{R}}$ is clearly spanned by $R_{\mathfrak{n}}$ (see (1.28)), it follows from (2.18) and (2.10) that $\delta_{\mathfrak{n}} \in \mathfrak{t}^*_{\mathbb{R}}$. $\square$

Introduce the lexicographical ordering in $\mathfrak{t}^*_{\mathbb{R}}$ with respect to an orthogonal ordered basis of $\mathfrak{t}^*_{\mathbb{R}}$ having $\delta_{\mathfrak{n}}$ as its first element. It follows from Lemma 2.4 that if R_+ is the set of positive $\mathfrak{t}$-roots with respect to this ordering, one has

$$R_+ = R_{\mathfrak{n}}. \tag{2.25}$$

Remark 2.5. One recalls that since $\mathfrak{t}^*_{\mathbb{R}}$ is a lexicographically ordered real euclidean space if $\xi_i \in \mathfrak{t}^*_{\mathbb{R}}$, $i = 1,\ldots,k$, are positive elements such that, for $i \neq j$,

$$(\xi_i, \xi_j) \leq 0,$$

then the ξ_i are linearly independent. (See e.g., [Hum], §10, Theorem′, (3), p. 48.)

2.3

Let $\ell(\mathfrak{t}) = \dim \mathfrak{t}$ and $\ell(\mathfrak{s}) = \dim \mathfrak{h}(\mathfrak{s})$ so that $\ell(\mathfrak{s})$ is the rank of $\mathfrak{s}$ and

$$\ell = \ell(\mathfrak{t}) + \ell(\mathfrak{s}). \tag{2.26}$$

Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be the set of simple positive roots in Δ_+. If $\varphi \in \Delta_+(\mathfrak{s})$ and $\varphi = \varphi_1 + \varphi_2$ where φ_1, $\varphi_2 \in \Delta_+$, obviously $(\varphi_1)_{\mathfrak{t}} = -(\varphi_2)_{\mathfrak{t}}$. But then $(\varphi_i)_{\mathfrak{t}}$, $i = 1,2$, vanish by Lemma 2.4 so that $\varphi_1, \varphi_2 \in \Delta_+(\mathfrak{s})$. Hence if $\varphi \in \Delta_+(\mathfrak{s})$ is simple with respect to $\Delta_+(\mathfrak{s})$, it is simple with respect to Δ_+. We may therefore order Π so that $\alpha_{\ell(\mathfrak{t})+i} \in \Delta_+(\mathfrak{s})$, for $i = 1\ldots,\ell(\mathfrak{s})$, and hence if $\Pi_{\mathfrak{s}} = \{\alpha_{\ell(\mathfrak{t})+1}, \ldots, \alpha_\ell\}$, then

$$\Pi_{\mathfrak{s}} \text{ is a basis of } \mathfrak{h}(\mathfrak{s})^*\text{; see (1.6).} \tag{2.27}$$

A $\mathfrak{t}$-root $\nu \in R_+$ is called simple if ν cannot be written $\nu = \nu_1 + \nu_2$ where $\nu_1, \nu_2 \in R_+$. Let R_{simp} be the set of simple $\mathfrak{t}$-roots in R_+.

Lemma 2.6. *Assume $\xi_1, \xi_2 \in R_{\mathrm{simp}}$ are distinct. Then*

$$(\xi_1, \xi_2) \leq 0 \tag{2.28}$$

so that, by Remark 2.5, the elements in R_{simp} are linearly independent. In particular

$$\operatorname{card} R_{\mathrm{simp}} \leq \ell(\mathfrak{t}). \tag{2.29}$$

Proof. Assume $(\xi_1, \xi_2) > 0$. Then, by Theorem 2.3, $\xi_1 - \xi_2$ and $\xi_2 - \xi_1$ are in R. Without loss assume $\nu \in R_+$ where $\nu = \xi_1 - \xi_2$. Then $\xi_1 = \nu + \xi_2$. This contradicts the simplicity of ξ_1. Hence one has (2.28). □

As noted in (1.6), $\mathfrak{h}(\mathfrak{s})^*$ is the orthocomplement of $\mathfrak{t}$ in $\mathfrak{h}$. Thus, by (2.27), if

$$\beta_j = (\alpha_j)_{\mathfrak{t}} \tag{2.30}$$

for $j = 1, \dots, \ell(\mathfrak{t})$, then clearly

$$\begin{aligned}&\beta_j \in R_+ \text{ and}\\&\beta_j,\ j = 1, \dots, \ell(\mathfrak{t}), \text{ are a basis of } \mathfrak{t}^*_{\mathbb{R}}.\end{aligned} \tag{2.31}$$

Recalling (1.1), for any $\nu \in R$ and $j = 1, \dots, \ell(\mathfrak{t})$, let $n_j(\nu) = n_{\alpha_j}(\varphi)$ where $\varphi \in \Delta(\mathfrak{g}_\nu)$. This is independent of the choice of φ by (2.27).

Theorem 2.7. *One has*

$$R_{\mathrm{simp}} = \{\beta_1, \dots, \beta_{\ell(\mathfrak{t})}\} \tag{2.32}$$

so that R_{simp} is a basis of $\mathfrak{t}^*_{\mathbb{R}}$ and for $i \neq j$,

$$(\beta_i, \beta_j) \leq 0. \tag{2.33}$$

Furthermore for $\nu \in R_+$ and $j = 1, \dots, \ell(\mathfrak{t})$, one has

$$\nu = \sum_{j=1}^{\ell(\mathfrak{t})} n_j(\nu)\, \beta_j. \tag{2.34}$$

Proof. Let $j \in \{1, \dots, \ell(\mathfrak{t})\}$. Assume that $\beta_j \notin R_{\mathrm{simp}}$. Then there exists $\nu_1, \nu_2 \in R_+$ such that $\beta_j = \nu_1 + \nu_2$. But then by (2.12) in Theorem 2.3 one has

$$[\mathfrak{g}_{\nu_1}, \mathfrak{g}_{\nu_2}] = \mathfrak{g}_{\beta_j}. \tag{2.35}$$

But $e_{\alpha_j} \in \mathfrak{g}_{\beta_j}$ by (1.23). Thus there exist, for $i = 1, 2$, $\varphi_i \in \Delta(\mathfrak{g}_{\nu_i})$ such that $\varphi_1 + \varphi_2 = \alpha_j$. This contradicts the simplicity of α_j since $\varphi_1, \varphi_2 \in \Delta_+$. But then (2.32) follows from (2.29). Also (2.33) and the fact that R_{simp} is a basis of $\mathfrak{t}^*_{\mathbb{R}}$ follow from Lemma 2.6.

Now let $\nu \in R_+$ and let $\varphi \in \Delta(\mathfrak{g}_\nu)$. Recalling the expansion (1.1) one immediately has

$$\nu = \sum_{\alpha \in \Pi} n_\alpha(\varphi)\, \alpha_{\mathfrak{t}}, \tag{2.36}$$

but this yields (2.34). □

Remark 2.8. Note that if $j = 1, \dots, \ell(\mathfrak{t})$, one has that $e_{\alpha_j} \in \mathfrak{g}_{\beta_j}$ and e_{α_j} is a lowest weight vector for the irreducible $\mathfrak{m}$-module $\mathfrak{g}_{\beta_j}$. Indeed this is clear since for $i = 1, \dots, \ell(\mathfrak{s})$,

$$[e_{-\alpha_{\ell(\mathfrak{t})+1}}, e_{\alpha_j}] = 0. \tag{2.37}$$

2.4

Henceforth we will assume that $\mathfrak{g}$ is simple. Let $\psi \in \Delta_+$ be the highest root so that, by the simplicity of $\mathfrak{g}$,

$$\operatorname{cent}\mathfrak{n}_{\mathfrak{b}} = \mathbb{C}\, e_{\psi} \tag{2.38}$$

where we recall (§1.1 $\mathfrak{n}_{\mathfrak{b}} = [\mathfrak{b},\mathfrak{b}]$.

Remark 2.9. For any $\alpha \in \Pi$ one knows that

$$n_{\alpha}(\psi) > 0 \tag{2.39}$$

and, for any $\varphi \in \Delta_+$,

$$n_{\alpha}(\psi) \geq n_{\alpha}(\varphi). \tag{2.40}$$

Indeed (2.39) and (2.40) are consequences of the immediate fact that

$$e_{\psi} \in U(\mathfrak{n}_{\mathfrak{b}})\, e_{\varphi} \tag{2.41}$$

where any Lie algebra $\mathfrak{a}$, $U(\mathfrak{a})$ is the enveloping algebra of $\mathfrak{a}$.

For any $\nu \in R$ let $o(\nu) = \sum_{j=1}^{\ell(\mathfrak{t})} n_{\beta_j}(\nu)$, and for any $k \in \mathbb{Z}_+$ let

$$\mathfrak{n}(k) = \sum_{\nu\in R,\, o(\nu)=k} \mathfrak{g}_{\nu}. \tag{2.42}$$

For $j,k \in \mathbb{Z}$, clearly

$$[\mathfrak{n}(j),\mathfrak{n}(k)] \;\subset\; \mathfrak{n}(j+k). \tag{2.43}$$

Now $\psi_{\mathfrak{t}} \neq 0$ by Remark 2.9 (one has $\ell(\mathfrak{t}) < \ell$ by our assumption in §1.1). Let $\nu(\text{cent}) \in R$ be defined by putting $\nu(\text{cent}) = \psi_{\mathfrak{t}}$. Put $k(\text{cent}) = o(\nu(\text{cent}))$ so that

$$k(\text{cent}) = \sum_{i=1}^{\ell(\mathfrak{t})} n_{\alpha_i}(\psi). \tag{2.44}$$

Remark 2.9 clearly also implies

Proposition 2.10. $\mathfrak{n}(k) = 0$ *if* $k > k(\text{cent})$ *and*

$$\mathfrak{n}(k(\text{cent})) = \mathfrak{g}_{\nu(\text{cent})}. \tag{2.45}$$

Furthermore one has the direct sum

$$\mathfrak{n} = \sum_{k=0}^{k(\text{cent})} \mathfrak{n}(k). \tag{2.46}$$

The upper central series of $\mathfrak{n}$ (defined for any nilpotent Lie algebra) is a sequence of distinct ideals $\mathfrak{n}_1 \;\subset\; n_2 \;\subset\; \cdots \;\subset\; \mathfrak{n}_d = \mathfrak{n}$ where $\mathfrak{n}_1 = \operatorname{cent}\mathfrak{n}$ and for $i \geq 2$,

$$n_i/\mathfrak{n}_{i-1} = \operatorname{cent}\mathfrak{n}/\mathfrak{n}_{i-1}. \tag{2.47}$$

See (14) on p. 29 in [Jac]. We refer to d as the length of the upper central series.

Theorem 2.11. *If $i < k(\mathrm{cent})$ and $\nu \in \mathfrak{n}(i)$, there exists $j \in \{1, \ldots, \ell(\mathfrak{t})\}$ such that $\nu + \beta_j \in \mathfrak{n}(i+1)$ so that*

$$[\mathfrak{g}_\nu, \mathfrak{g}_{\beta_j}] = \mathfrak{g}_{\nu+\beta_j}. \tag{2.48}$$

In particular $[\mathfrak{g}_\nu, \mathfrak{g}_{\beta_j}] \neq 0$. Furthermore

$$\mathfrak{g}_{\nu(\mathrm{cent})} = \mathrm{cent}\,\mathfrak{n} \tag{2.49}$$

and the upper central series $\mathfrak{n}_i$, of the nilradical $\mathfrak{n}$ of the general proper parabolic subalgebra $\mathfrak{q}$, is given as follows:

$$\mathfrak{n}_i = \sum_{j=1}^{i} \mathfrak{n}(k(\mathrm{cent}) - j + 1) \tag{2.50}$$

noting that $k(\mathrm{cent})$ is the length of the upper central series.

Proof. Let $\varphi \in \Delta(\mathfrak{g}_\nu)$ be the highest weight of the $\mathfrak{m}$-irreducible module $\mathfrak{g}_\nu$. Thus $[e_{\alpha_{i+\ell(\mathfrak{t})}}, e_\varphi] = 0$ for $i = 1, \ldots, \ell(\mathfrak{s})$. But $\varphi \neq \psi$. Hence by the uniqueness of the $\mathfrak{g}$-highest weight there must exist $j = 1, \ldots, \ell(\mathfrak{t})$ such that $[e_{\alpha_j}, e_\varphi] \neq 0$. But this implies (2.48). Now obviously $\mathfrak{g}_{\nu(\mathrm{cent})} \subset \mathrm{cent}\,\mathfrak{n}$ by (2.43) and (2.46). But clearly $\mathrm{cent}\,\mathfrak{n}$ is stable under $\mathrm{ad}\,\mathfrak{m}$. Therefore to prove (2.49) it suffices to show that $\mathfrak{g}_\nu \not\subset \mathrm{cent}\,\mathfrak{n}$ for $\nu \neq \nu(\mathrm{cent})$. But this is established by (2.48). Let $i > 1$. Assume inductively that one has (2.50) where $i-1$ replaces i. Then, returning to (2.50) as stated, if $\mathfrak{n}_{(i)}$ is given by the right side of (2.50), one has $\mathfrak{n}_{(i)} \subset \mathfrak{n}_i$ by (2.43) and (2.46). But the upper central series (e.g., by induction) is stabilized by $\mathrm{ad}\,\mathfrak{m}$ so that in particular $\mathfrak{n}_i$ is stabilized by $\mathrm{ad}\,\mathfrak{m}$. But again (2.48) implies that if $\mathfrak{g}_\nu \subset \mathfrak{n}(j)$ where $j < k(\mathrm{cent}) - i + 1$, then $\mathfrak{g}_\nu \not\subset \mathfrak{n}_i$. This implies that $\mathfrak{n}_i = \mathfrak{n}_{(i)}$. $\square$

Using the notation in [Jac], see page 23, the lower central series $\mathfrak{n}^i$ of $\mathfrak{n}$ is a sequence of ideals defined inductively so that $\mathfrak{n}^1 = \mathfrak{n}$ and for $i > 1, \mathfrak{n}^i = [\mathfrak{n}, \mathfrak{n}^{i-1}]$. See also p. 11 in [Hum]. The indexing in [Hum] differs by 1 from the indexing in [Jac]. We will call the maximum k such that $\mathfrak{n}^k \neq 0$ the length of the lower central series.

Theorem 2.12. *Let i be any integer where $2 \leq i \leq k(\mathrm{cent})$ and let $\nu \in R$ where $\mathfrak{g}_\nu \in \mathfrak{n}(i)$. Then there exists $j \in \{1, \ldots, \ell(\mathfrak{t})\}$ and $\mu \in R$ where $\mathfrak{g}_\mu \in \mathfrak{n}(i-1)$ such that*

$$[\mathfrak{g}_{\beta_j}, \mathfrak{g}_\mu] = \mathfrak{g}_\nu \tag{2.51}$$

so that

$$[\mathfrak{n}(1), \mathfrak{n}(i-1)] = \mathfrak{n}(i). \tag{2.52}$$

In particular the lower central series $\mathfrak{n}^i$ of the nilradical $\mathfrak{n}$ of the arbitrary proper parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is given by

$$\mathfrak{n}^i = \sum_{j=i}^{k(\mathrm{cent})} n(j) \tag{2.53}$$

so that (see (2.44)) $k(\text{cent})$ *is the length of the lower central series of* $\mathfrak{n}$ *(as well as the upper central series, see Theorem 2.11. The lower and upper central series of* $\mathfrak{n}$ *are therefore, except for indexing, the same*

$$\mathfrak{n}^i = \mathfrak{n}_{k(\text{cent})-i+1} \tag{2.54}$$

for $i = 1,\dots,k(\text{cent})$.

Proof. It suffices only to prove (2.51). But, by Remark 2.5 and Theorem 2.7, $\beta_j \in R_{\text{simp}}$ exists so that

$$(\nu,\beta_j) > 0. \tag{2.55}$$

Thus $\mu \in R$ where $\mu = \nu - \beta_j$ by Theorem 2.3. But also necessarily $n_j(\nu) > 0$ by (2.33) so that $\mathfrak{g}_\mu \in \mathfrak{n}(i-1)$. But $\mu + \beta_j = \nu$. Thus by (2.12) of Theorem 2.3 one has (2.51). $\square$

3 Borel–de Siebenthal Theory, Special Elements, and the Lie Subalgebras they Define

3.1

We continue to assume (starting in §2.4) that $\mathfrak{g}$ is simple. In this section we will apply the results of §1 and §2 to the case where $\ell(\mathfrak{t}) = 1$. It will be convenient to change some notation and earlier indexing. In particular we now fix an ordering in Π so that $\Pi = \{\alpha_1,\dots,\alpha_\ell\}$. Also recalling (1.1) we will write $n_i(\varphi)$ for $n_{\alpha_i}(\varphi)$ so that for the highest root ψ one has

$$\psi = \sum_{i=1}^{\ell} n_i(\psi)\,\alpha_i. \tag{3.1}$$

Now let x_j, $j = 1,\dots,\ell$, be the basis of $\mathfrak{h}_{\mathbb{R}}$ so that

$$\langle \alpha_l, x_j\rangle = \delta_{ij}. \tag{3.2}$$

Now for $j = 1,\dots,\ell$, let $\mathfrak{q}[j]$ be the maximal standard parabolic subalgebra defined by x_j. The standard Levi factor $\mathfrak{m}[j]$ of $\mathfrak{q}[j]$ is the centralizer $\mathfrak{g}^{x_j}$ of x_j in $\mathfrak{g}$. The decomposition (1.3) now becomes

$$\mathfrak{m}[j] = \mathfrak{t}[j] + \mathfrak{s}[j] \tag{3.3}$$

where the one-dimensional center $\mathfrak{t}[j]$ of $\mathfrak{m}[j]$ is given by

$$\mathfrak{t}[j] = \mathbb{C}x_j. \tag{3.4}$$

If $\Pi_{\mathfrak{s}[j]}$ is the set of simple positive roots of the (rank $\ell-1$)-semisimple Lie algebra $\mathfrak{s}[j]$, defined as in (2.27), one now has

$$\Pi_{\mathfrak{s}[j]} = \Pi \setminus \{\alpha_j\}. \tag{3.5}$$

The nilradical $\mathfrak{n}[j]$ of $\mathfrak{q}[j]$ is given by

$$\mathfrak{n}[j] = \text{span of } \{e_\varphi \mid n_j(\varphi) > 0\}. \tag{3.6}$$

Write $R[j]$ for R, $R[j]_+$ for R_+ and $R[j]_{\text{simp}}$ for R_{simp}. Let $\beta[j] = (\alpha_j)_{\mathfrak{t}[j]}$ so that

$$\langle \beta[j], x_j\rangle = 1. \tag{3.7}$$

Proposition 3.1. *One has*

$$R[j]_{\text{simp}} = \{\beta[j]\} \tag{3.8}$$

and

$$R[j]_+ = \{\beta[j], 2\beta[j], \ldots, n_j(\psi)\beta[j]\} \tag{3.9}$$

and

$$R[j] = \{\pm\beta[j], \pm 2\beta[j], \ldots, \pm n_j(\psi)\beta[j]\}. \tag{3.10}$$

In particular

$$\begin{aligned} \operatorname{card} R[j]_+ &= n_j(\psi) \\ \operatorname{card} R[j] &= 2n_j(\psi). \end{aligned} \tag{3.11}$$

Proof. The proof is immediate from (2.17), (2.25), Theorem 2.7, Theorem 2.12 and (2.44) which implies here that

$$k(\text{cent}) = n_j(\psi). \tag{3.12}$$

□

Let $I[j] = \{\pm 1, \ldots, \pm n_j(\psi)\}$ so that if $\nu \in R[j]$, then $\nu = k\beta[j]$ for $k \in I[j]$. We recall (see §1.2) that for $k \in I[j]$,

$$\begin{aligned} \mathfrak{g}_{k\beta[j]} &= \text{ span of } \{e_\varphi \mid n_j(\varphi) = k\} \\ &= \{z \in \mathfrak{g} \mid [x_j, z] = kz\}. \end{aligned} \tag{3.13}$$

One has the direct sums

$$\begin{aligned} \mathfrak{n}[j] &= \sum_{k=1}^{n_j(\psi)} \mathfrak{g}_{k\beta[j]} \\ \mathfrak{g} &= \mathfrak{m}[j] + \sum_{k\in I[j]} \mathfrak{g}_{k\beta[j]}. \end{aligned} \tag{3.14}$$

One now has

Theorem 3.2. *Let $j \in \{1,\dots,\ell\}$. Then $\mathfrak{g}_{k\beta[j]}$ is $\operatorname{ad}\mathfrak{m}[j]$-irreducible for any $k \in I[j]$. In particular the nilradical $\mathfrak{n}[j]$ of $\mathfrak{q}[j]$ is, as an $\operatorname{ad}\mathfrak{m}[j]$-module, multiplicity-free with $n_j(\psi)$-irreducible components. Also $\mathfrak{g}/\mathfrak{m}[j]$ is, as an $\operatorname{ad}\mathfrak{m}[j]$-module, multiplicity-free with $2n_j(\psi)$-irreducible components. Finally, if $p,q \in I[j]$ and $p+q \in I[j]$, then*

$$[\mathfrak{g}_{p\beta[j]}, \mathfrak{g}_{q\beta[j]}] = \mathfrak{g}_{(p+q)\beta[j]}. \tag{3.15}$$

Proof. The first statement is just the present application of Theorem 1.9. The equality (3.15) is given by (2.12) of Theorem 2.3. □

3.2

Let $C \subset \mathfrak{h}_{\mathbb{R}}$ be the fundamental Weyl chamber corresponding to $\mathfrak{b}$ so that

$$C = \{x \in \mathfrak{h}_{\mathbb{R}} \mid \langle \alpha_i, x\rangle \geq 0, \text{ for } i = 1,\dots,\ell\}, \tag{3.16}$$

and let $A \subset C$ be the fundamental alcove so that A is the simplex defined by

$$A = \{x \in C \mid \langle \psi, x\rangle \leq 1\}. \tag{3.17}$$

Let G be a simply-connected complex group for which $\mathfrak{g} = \operatorname{Lie} G$. Let K be a maximal compact subgroup of G. We may choose K so that if $\mathfrak{k} = \operatorname{Lie} K$, then $i h_{\mathbb{R}} = \operatorname{Lie} H_{\text{comp}}$ where H_{comp} is a maximal torus of K.

A classical result of Cartan and Weyl is the statement

Proposition 3.3. *For any element $g \in K$ there exists a unique element $x \in A$ such that*

$$g \text{ is } K\text{-conjugate to } \exp 2\pi i\, x. \tag{3.18}$$

Clearly the $\ell+1$ vertices v_j, $j = 0,\dots,\ell$, of A are then given as $v_0 = 0$ and for $j > 0$,

$$v_j = x_j / n_j(\psi). \tag{3.19}$$

For $j = 1,\dots,\ell$, let $a_j \in K$ be defined by putting

$$a_j = \exp 2\pi\, i v_j. \tag{3.20}$$

Let $\omega[j]$ be the $n_j(\psi)$-root of unity given by putting $\omega[j] = e^{2\pi i/n_j(\psi)}$. For $g \in G$ let G^g (resp. $\mathfrak{g}^g$) be the centralizer of g in G (resp. $\mathfrak{g}$). As an immediate consequence of (3.14) the well-known adjoint action of a_j on $\mathfrak{g}$ is given by

Proposition 3.4. $\operatorname{Ad} a_j$ *has order $n_j(\psi)$ on $\mathfrak{g}$. In fact one has*

$$\begin{aligned} \operatorname{Ad} a_j &= 1 \textit{ on } \mathfrak{m}[j] \\ &= \omega[j]^k \textit{ on } \mathfrak{g}_{k\beta[j]} \textit{ for all } k \in I[j]. \end{aligned} \tag{3.21}$$

In particular one has the direct sum

$$\mathfrak{g}^{a_j} = \mathfrak{m}[j] + \mathfrak{g}_{n_j(\psi)\beta[j]} + \mathfrak{g}_{-n_j(\psi)\beta[j]}. \tag{3.22}$$

An element $a \in K$ is called special if the reductive subalgebra $\mathfrak{g}^a$ of $\mathfrak{g}$ is in fact semisimple (i.e., $\operatorname{cent}\mathfrak{g}^a = 0$). The following is also well known but proved here for completeness.

Proposition 3.5. *For $j = 1,\dots,\ell$, the element a_j is special.*

Proof. Since $\mathfrak{h} \subset \mathfrak{m}[j]$ one obviously has $\operatorname{cent}\mathfrak{g}^{a_j} \subset \operatorname{cent}\mathfrak{m}[j]$. But $\operatorname{cent}\mathfrak{m}[j] = \mathbb{C}x_j$. However $\operatorname{ad}x_j = n_j(\psi)$ on $\mathfrak{g}_{n_j(\psi)\beta[j]}$. Thus $\operatorname{cent}\mathfrak{g}^{a_j} = 0$. □

Remark 3.6. One can readily prove $a \in K$ is special if and only if either $a = 1$ or there exists $j \in \{1,\dots,\ell\}$, necessarily unique, such that a is K-conjugate to a_j. We also remark that special elements arise in connection with distinguished nilpotent conjugacy classes in $\mathfrak{g}$. Indeed if e is a distinguished nilpotent element then, where G^e_o is the identity component of G^e, the component group G^e/G^e_o is isomorphic to the finite group F where F is the centralizer in G of a TDS containing e. Since F is finite we may make choices so that $F \subset K$. But then the elements of F are special. Indeed if $a \in F$, then $e \in \mathfrak{g}^a$. But if $x \in \operatorname{cent}\mathfrak{g}^a$, then x is a semisimple element that commutes with e. Thus $x = 0$ since e is distinguished. Hence $\operatorname{cent}\mathfrak{g}^a = 0$.

3.3

Let $j = 1,\dots,\ell$. For completeness in this section we wish to give the proof of the Borel–de Siebenthal algorithm for determining the Dynkin diagram of $\mathfrak{g}^{a_j}$. We recall that the extended Dynkin diagram of $\mathfrak{g}$ is the usual Dynkin diagram (whose nodes are identified with Π) of $\mathfrak{g}$, together with an additional node α_0, where if $i = 1,\dots,\ell$, then α_0 is linked to α_i with

$$m_i = 2\,(\alpha_i,\psi)/(\alpha_i,\alpha_i) \tag{3.23}$$

lines and arrowhead at α_i, directed at α_i, if $m_i > 1$.

Theorem 3.7 (Borel–de Siebenthal algorithm). *The Dynkin diagram of $\mathfrak{g}^{a_j}$ is the subdiagram of the extended Dynkin diagram remaining after deleting α_j (and all lines linked to α_j) from the extended Dynkin diagram.*

Proof. Let $\mathfrak{m}[j]_+ = \mathfrak{m}[j] \cap \mathfrak{b}$ so that $\mathfrak{m}[j]_+$ is a Borel subalgebra, containing $\mathfrak{h}$, of $\mathfrak{m}[j]$. Now $\mathfrak{g}_{-n_j(\psi)\beta[j]}$ is clearly a commutative nilpotent subalgebra of $\mathfrak{g}^{a_j}$ which is stable (and in fact irreducible) under $\operatorname{ad}\mathfrak{m}[j]$. But then of course $\mathfrak{g}_{-n_j(\psi)\beta[j]}$ is stable under $\operatorname{ad}\mathfrak{m}[j]_+$. Thus

$$\mathfrak{b}[j] = \mathfrak{m}[j]_+ + \mathfrak{g}_{-n_j(\psi)\beta[j]} \tag{3.24}$$

is a solvable Lie subalgebra of $\mathfrak{g}^{a_j}$. But, by dimension, $\mathfrak{b}[j]$ is then a Borel subalgebra of $\mathfrak{g}^{a_j}$. Regarding $\Delta(\mathfrak{b}[j])$ as a system of positive roots for $\mathfrak{g}^{a_j}$, let $\Pi[j]$ be the corresponding set of simple positive roots. But now $-\psi \in \Delta(\mathfrak{g}_{-n_j(\psi)\beta[j]})$. (In fact clearly $e_{-\psi}$ is a lowest weight vector for the irreducible action of $\operatorname{ad}\mathfrak{m}[j]$ on $\mathfrak{g}_{-n_j(\psi)\beta[j]}$.) But one notes it is immediate that $-\psi \in \Pi[j]$. Since $\mathfrak{g}^{a_j}$ has rank ℓ one has, recalling (3.5),

$$\Pi[j] = \Pi_{\mathfrak{s}[j]} \cup \{-\psi\}. \tag{3.25}$$

But then the Dynkin diagram of $\mathfrak{g}^{a_j}$ is the Dynkin diagram of $\mathfrak{s}[j]$, together with the node defined by $-\psi$, where if $\alpha_i \in \Pi_{\mathfrak{s}[j]}$, then $-\psi$ (a long root) is linked to α_i by (see (3.23)) m_i lines and arrowhead at α_i, directed at α_i, if $m_i > 1$. But this is the Borel–de Siebenthal algorithm. $\square$

3.4

Let $j = 1,\ldots,\ell$, and let $I_o[j] = I[j] \setminus \{\pm n_j(\psi)\}$. Let

$$\mathfrak{r}[j] = \sum_{k\in I_o[j]} \mathfrak{g}_{k\beta[j]} \tag{3.26}$$

so that by (3.14) and (3.22) one has the B-orthogonal direct sum

$$\mathfrak{g} = \mathfrak{g}^{a_j} + \mathfrak{r}[j]. \tag{3.27}$$

For $k = 1,\ldots,n_j(\psi)-1$, let

$$\mathfrak{g}[a_j]^k = \{z \in \mathfrak{g} \mid \operatorname{Ad} a_j(z) = \omega[j]^k z\}. \tag{3.28}$$

Obviously

$$\mathfrak{g}[a_j]^k \text{ is stable under ad } \mathfrak{g}^{a_j} \text{ for } k = 1,\ldots,n_j(\psi)-1. \tag{3.29}$$

But clearly, by Proposition 3.4, for $k = 1,\ldots,n_j(\psi)-1$,

$$\mathfrak{g}[a_j]^k = \mathfrak{g}_{k\beta[j]} + \mathfrak{g}_{(k-n_j(\psi))\beta[j]}, \tag{3.30}$$

and hence by (3.26) one has the direct sum

$$\mathfrak{r}[j] = \sum_{k=1}^{n_j(\psi)-1} \mathfrak{g}[a_j]^k. \tag{3.31}$$

Consequently one notes that not only is $\mathfrak{r}[j]$ stable under $\operatorname{ad} g^{a_j}$ but (3.31) isolates $n_j(\psi)-1$ $\operatorname{ad} g^{a_j}$-submodules of $\mathfrak{r}[j]$.

Theorem 3.8. *Let $j = 1,\ldots,j$, so that a_j, defined by the vertex v_j of the fundamental alcove, is a special element of K. In particular, its centralizer $\mathfrak{g}^{a_j}$ is a maximal semisimple Lie subalgebra of $\mathfrak{g}$ if $n_j(\psi)$ is prime—by Borel–de Siebenthal theory. Let $\mathfrak{r}[j]$ be the B-orthocomplement of $\mathfrak{g}^{a_j}$ in $\mathfrak{g}$. Then $\mathfrak{r}[j]$ is a multiplicity-free $\mathrm{ad}\,\mathfrak{m}[j]$-module where $\mathfrak{m}[j] \subset \mathfrak{g}^{a_j}$ is the centralizer of v_j in $\mathfrak{g}$ and (3.26) is the decomposition of $r[j]$ into a sum of $2(n_j(\psi)-1)$-irreducible $\mathrm{ad}\,\mathfrak{m}[j]$-submodules. Next (3.30) defines the decomposition of the $\mathrm{Ad}\,a_j$-weight space $\mathfrak{g}[a_j]^k$ into a sum of two irreducible $\mathrm{ad}\,m[j]$-submodules. Furthermore, and mainly, the $\mathrm{Ad}\,a_j$-weight space $\mathfrak{g}[a_j]^k$ is an irreducible $\mathrm{ad}\,\mathfrak{g}^{a_j}$-submodule. In addition $\mathfrak{r}[j]$ is a multiplicity-free $\mathrm{Ad}\,\mathfrak{g}^{a_j}$-module and (3.31) is the decomposition of $r[j]$ into a sum of $n_j(\psi)-1$-irreducible $\mathrm{ad}\,\mathfrak{g}^{a_j}$-submodules. Finally, if $p,q,r \in \{1,\ldots,n_j(\psi)-1\}$ and $r \cong p+q \bmod n_j(\psi)$, then*

$$[\mathfrak{g}[a_j]^p, \mathfrak{g}[a_j]^q] = \mathfrak{g}[a_j]^r. \tag{3.32}$$

Proof. Up until the sentence beginnning with "Furthermore" the stated results have been established in Theorem 3.2 But now

$$\left[\mathfrak{g}_{(k-n_j(\psi))\beta[j]}, \mathfrak{g}_{n_j(\psi)\beta[j]}\right] = \mathfrak{g}_{k\beta[j]}$$

and

$$\left[\mathfrak{g}_{k\beta[j]}, \mathfrak{g}_{-n_j(\psi)\beta[j]}\right] = \mathfrak{g}_{(k-n_j(\psi))\beta[j]}$$

by (3.15). Hence recalling (3.22) and (3.30) it follows that $\mathfrak{g}[a_j]^k$ is $\mathfrak{g}^{a_j}$-irreducible for all $k = 1,\ldots,n_j(\psi)-1$.

Clearly the left side of (3.32) is contained in the right side of (3.32). But by $\mathrm{ad}\,g^{a_j}$ irreducibility one has (3.32) as soon as one observes that the left side is nonzero. But this is clear from (3.15) if $p+q=r$. If $p+q > n_j(\psi)$, then $r = p+q-n_j(\psi)$. But then (3.32) follows from (3.15) where q is replaced by $q - n_j(\psi)$. □

Remark 3.9. One of the main results of Borel–de Siebenthal theory is the statement that $\mathfrak{g}_1$ is a maximal proper (i.e., $g_1 \neq \mathfrak{g}$) semisimple subalgebra such that $\mathrm{rank}\,\mathfrak{g}_1 = \mathrm{rank}\,\mathfrak{g}$ if and only if

$$\mathfrak{g}_1 \cong \mathfrak{g}^{a_j} \tag{3.33}$$

where $n_j(\psi)$ is a prime number. This may be proved as follows: In one direction assume (3.33) where $n_j(\psi)$ is prime. Then if $\mathfrak{g}_o$ is a subalgebra where $\mathfrak{g}^{a_j} \subset \mathfrak{g}_o$ and $\mathfrak{g}_o \neq \mathfrak{g}^{a_j}$, there must exist, by (3.27), $0 \neq x \in \mathfrak{r}[j] \cap \mathfrak{g}_o$. But since $\mathfrak{g}_o$ is stable under $\mathrm{ad}\,\mathfrak{g}^{a_j}$ it follows from Theorem 3.8 that there exists $k \in \{1,\ldots,n_j(\psi)-1\}$ such that $\mathfrak{g}[a_j]^k \subset \mathfrak{g}_o$. But $\mathfrak{g}$ is generated by $\mathfrak{g}^{a_j}$ and $\mathfrak{g}[a_j]^k$, by (3.32), since $n_j(\psi)$ is prime. Thus $\mathfrak{g}^{a_j}$ is maximal as a proper Lie subalgebra of $\mathfrak{g}$. Conversely assume $\mathfrak{g}_1$ is a maximal proper semisimple subalgebra where $\mathrm{rank}\,\mathfrak{g}_1 = \mathrm{rank}\,\mathfrak{g}$. Let $G_1 \subset G$ be the subgroup corresponding to $\mathfrak{g}_1$. Let γ denote the adjoint representation of G_1 on $\mathfrak{g}/\mathfrak{g}_1$. By the equal rank condition 0 is not a weight of γ. Thus γ does not descend to the adjoint group of $\mathfrak{g}_1$. Thus there exists $1 \neq c \in \mathrm{cent}\,G_1$ such that $c \notin \mathrm{Ker}\,\gamma$. But c has finite order since $\mathfrak{g}_1$ is semisimple. We may therefore make choices so that $c \in K$. Of course $\mathfrak{g}_1 \subset \mathfrak{g}^c$. By maximality

$$\mathfrak{g}^c = \mathfrak{g}_1. \tag{3.34}$$

But then c is special and by Remark 3.6 choices can be made so that $c = a_j$ for some $j \in \{1,\ldots,\ell\}$. But if $n_j(\psi)$ is not prime there exists an integer $1 < k < n_j(\psi)$ such that k divides $n_j(\psi)$. But then, by (3.32), $\mathfrak{g}_1$ and $\mathfrak{g}[a_j]^k$ generate a proper semisimple subalgebra of $\mathfrak{g}$, contradicting the maximality of $\mathfrak{g}_1$.

4 Example

4.1

In this section we consider the example of the theory above for the case where, for a positive integer $n > 1$, $\mathfrak{g} =$ Lie $Sl(n,\mathbb{C})$. With the usual meaning of matrix units, e_{ij}, $x \in \mathfrak{g}$, when we can write

$$x = \sum_{i,j=1}^{n} a_{ij}(x)\, e_{ij}$$

where $\sum_{i=1}^{n} a_{ii}(x) = 0$. For k a positive integer, where $1 < k \leq n$, let

$$\delta = \{d_1,\ldots,d_k\} \tag{4.1}$$

where the d_p are positive integers such that

$$\sum_{p=1}^{k} d_p = n. \tag{4.2}$$

For $q \in \{1,\ldots,k\}$, put

$$f_q = \sum_{p=1}^{q} d_p$$

and hence

$$1 \leq f_1 < \cdots < f_k = n. \tag{4.3}$$

Now for any $i \in \{1,\ldots,n\}$, let $q(i) \in \{1,\ldots,k\}$ be the minimum value of q such that $i \leq f_q$. Thus if we put $f_0 = 0$, and we let I_q be the half-open interval of integers given by putting $I_q = (f_{q-1}, f_q]$, then one has the disjoint union

$$(0,n] = \sqcup_{q=1}^{k} I_q. \tag{4.4}$$

Clearly one has

$$\operatorname{card} I_q = d_q \tag{4.5}$$

and for any $i \in (0,n]$ one has

$$i \in I_{q(i)}. \tag{4.6}$$

Next put

$$\mathfrak{n}(\delta) = \{x \in \mathfrak{g} \mid a_{ij}(x) = 0, \text{ unless } j > f_{q(i)}\}.$$

In addition for $r, s \in \{1, \ldots, q\}$, where $r \neq s$ let

$$\mathfrak{g}_{r,s}(\delta) = \{x \in \mathfrak{g} \mid a_{ij}(x) = 0, \text{ unless } i \in I_r, \text{and } j \in I_s\}.$$

One readily notes that $\mathfrak{n}(\delta)$ is a nilpotent Lie algebra and one has the vector space direct sum

$$\mathfrak{n}(\delta) = \oplus_{r<s}\, \mathfrak{g}_{r,s}(\delta). \tag{4.7}$$

Let $\overline{\mathfrak{n}}(\delta)$ be the transpose of $\mathfrak{n}(\delta)$. One then has the direct sum

$$\overline{\mathfrak{n}}(\delta) = \oplus_{s<r}\, \mathfrak{g}_{r,s}(\delta). \tag{4.8}$$

Also for $q \in \{1, \ldots, k\}$, let

$$\mathfrak{s}_q(\delta) = \{x \in \mathfrak{g} \mid a_{ij}(x) = 0, \text{ unless } i, j \in I_q\}.$$

One readily has that $\mathfrak{s}_q(\delta) = 0$ if $d_q = 1$ and otherwise $\mathfrak{s}_q(\delta)$ is a simple Lie subalgebra of $\mathfrak{g}$ where in fact

$$\mathfrak{s}_q(\delta) \cong \text{ Lie}\, Sl(d_q, \mathbb{C}). \tag{4.9}$$

Let $\mathfrak{s}(\delta)$ be the semisimple Lie subalgebra given by putting

$$\mathfrak{s}(\delta) = \oplus_{q=1}^{k} \mathfrak{s}_q(\delta).$$

Now let $\mathfrak{h}$ be the space of all diagonal matrices in $\mathfrak{g}$ so that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let

$$\mathfrak{t}(\delta) = \{x \in \mathfrak{h} \mid a_{ii}(x) = a_{jj}(x) \text{ if } q(i) = q(j)\}. \tag{4.10}$$

Let

$$\mathfrak{m}(\delta) = \mathfrak{t}(\delta) + \mathfrak{s}(\delta)$$

and put $\mathfrak{q}(\delta) = \mathfrak{m}(\delta) + \mathfrak{n}(\delta)$. The following proposition is straightforward and is left as an exercise.

Proposition 4.1. $\mathfrak{q} = \mathfrak{q}(\delta)$ *is a parabolic subalgebra of* $\mathfrak{g}$ *and, up to conjugacy, every proper parabolic subalgebra is of this form. Moreover* $\mathfrak{q} = \mathfrak{m} + \mathfrak{n}$ *is a Levi decomposition of* $\mathfrak{q}$ *with* $\mathfrak{n}$ *as a nilradical and* $\mathfrak{m}$ *as a Levi factor where* $\mathfrak{m} = \mathfrak{m}(\delta)$ *and* $\mathfrak{n} = \mathfrak{n}(\delta)$. *Furthermore (4.10) is the decomposition (1.4) where* $\mathfrak{t} = \mathfrak{t}(\delta)$ *and* $\mathfrak{s} = \mathfrak{s}(\delta)$. *Next the set, R, of* $\mathfrak{t}$*-roots* ν *is parameterized by all pairs* $r, s \in \{1, \ldots, k\}$, *where* $r \neq s$, *and if the parameterization is denoted by* $\nu(r,s)$, *then for any* $x \in \mathfrak{t}$ *one has*

$$\nu(r,s)(x) = a_{ii}(x) - a_{jj}(x) \tag{4.11}$$

for $i \in I_r$ *and* $j \in I_s$. *In addition the* $\mathfrak{t}$*-root space corresponding to* $\nu(r,s)$ *is given by*

$$\mathfrak{g}_{\nu(r,s)} = \mathfrak{g}_{r,s}(\delta). \tag{4.12}$$

The irreducible adjoint action of $\mathfrak{m}$ *on* $\mathfrak{g}_{\nu(r,s)}$ *is given as follows: Put* $\mathfrak{s}_q = \mathfrak{s}_q(\delta)$*. Then* $\mathfrak{s}_p$ *operates trivially if* $p \notin \{r,s\}$*. Furthermore* $\mathfrak{g}_{\nu(r,s)}$ *is one-dimensional if and only if* $d_r = d_s = 1$*. If* $d_r = 1$ *and* $d_s > 1$ *(resp.* $d_r > 1$ *and* $d_s = 1$*), then* $\mathfrak{g}_{\nu(r,s)}$ *is* d_s *(resp.* d_r*)-dimensional and affords a fundamental irreducible* d_s *(resp.* d_r*)-dimensional of* $\mathfrak{s}_s$ *(resp.* $\mathfrak{s}_r$*). Moreover if* d_r *and* d_s *are both greater than* 1*, then* $\dim \mathfrak{g}_{\nu(r,s)} = d_r d_s$ *and* $\mathfrak{g}_{\nu(r,s)}$ *affords the direct product of a fundamental irreducible* d_r*-dimensional representation of* $\mathfrak{s}_r$ *and a fundamental irreducible* d_s*-dimensional representation of* $\mathfrak{s}_s$*.*

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Polarizations and Nullcone of Representations of Reductive Groups

Hanspeter Kraft[1] and Nolan R. Wallach[2]

Summary. We start with the following simple observation. Let V be a representation of a reductive group G, and let $f_1, f_2, \ldots, f_n$ be homogeneous invariant functions. Then the polarizations of $f_1, f_2, \ldots, f_n$ define the nullcone of $k \leq m$ copies of V if and only if every linear subspace L of the nullcone of V of dimension $\leq m$ is annhilated by a one-parameter subgroup (shortly a 1-PSG). This means that there is a group homomorphism $\lambda\colon \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t)x = 0$ for all $x \in L$. This is then applied to many examples. A surprising result is about the group SL_2 where almost all representations V have the property that all linear subspaces of the nullcone are annihilated. Again, this has interesting applications to the invariants on several copies. Another result concerns the n-qubits which appear in quantum computing. This is the representation of a product of n copies of SL_2 on the n-fold tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. Here we show just the opposite, namely that the polarizations never define the nullcone of several copies if $n \geq 3$. (An earlier version of this paper, distributed in 2002, was split into two parts; the first part with the title "On the nullcone of representations of reductive groups" is published in Pacific J. Math. **224** (2006), 119–140).

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1 Linear subspaces of the nullcone

In this chapter we study finite-dimensional complex representations of a reductive algebraic group G. It is a well-known and classical fact that the nullcone $\mathcal{N}_V$ of such

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a representation V plays a fundamental role in the geometry of the representation. Recall that $\mathcal{N}_V$ is defined to be the union of all G-orbits in V containing the origin 0 in their closure. Equivalently, $\mathcal{N}_V$ is the zero set of all non-constant homogeneous G-invariant functions on V.

In a previous paper [KrW06] we have seen that certain linear subspaces of the nullcone play a central role for understanding its irreducible components. In this chapter we discuss arbitrary linear subspaces of the nullcone $\mathcal{N}_V$ of a representation V of a reductive group G and show how they relate to questions about generator systems of and parameter systems for the invariants.

We first recall the definition of a polarization of a regular function $f \in \mathcal{O}(V)$. For $k \geq 1$ and arbitrary parameters $t_1, \dots, t_k$ we write

$$f(t_1 v_1 + t_2 v_2 + \cdots + t_k v_k) = \sum_{i_1, i_2, \dots, i_k} P_{i_1,\dots,i_k} f(v_1, \dots, v_k) \cdot t_1^{i_1} t_2^{i_2} \cdots t_k^{i_k}. \tag{1}$$

Then the regular functions $P_{i_1,\cdots,i_k} f$ defined on the sum $V^{\oplus k}$ of k copies of the original representation V are called *polarizations* of f. Here are a few well-known and easy facts.

a. If f is homogeneous of degree d then $P_{i_1,\dots,i_k} f$ is multihomogeneous of multidegree $(i_1, \dots, i_k)$ and thus $i_1 + \cdots + i_k = d$ unless $P_{i_1,\dots,i_k} f = 0$.
b. If f is G-invariant then so are the polarizations.
c. For a subset $A \subset \mathcal{O}(V)$ the algebra $\mathbb{C}[PA] \subset \mathcal{O}(V^{\oplus k})$ generated by the polarizations Pa, $a \in A$, contains all polarizations Pf for $f \in \mathbb{C}[A]$.

It is easily seen from examples that, in general, the polarizations of a system of generators do not generate the invariant ring of more than one copy (see [Sch07]). However, we might ask the following question.

Main Question 1.1. Given a set of invariant functions $f_1, \dots, f_m$ defining the nullcone of a representation V, when do the polarizations define the nullcone of a direct sum of several copies of V?

From now on let G denote a connected reductive group. An important tool in the context is the Hilbert–Mumford criterion which says that a vector $v \in V$ belongs to the nullcone $\mathcal{N}_V$ if and only if there is a one-parameter subgroup (abbreviated: 1-PSG) $\lambda^* \colon \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t)v = 0$ ([Kr85, Kap. II]). We say that a 1-PSG λ *annihilates a subset* $S \subset V$ if $\lim_{t\to 0} \lambda(t)v = 0$ for all $v \in S$.

Proposition 1.1. *Let V be a representation of G and let $f_1, f_2, \dots, f_r$ be homogeneous invariants defining the nullcone $\mathcal{N}_V$. For every integer $m \geq 1$ the following statements are equivalent.*

(i) Every linear subspace $L \subset \mathcal{N}_V$ of dimension $\leq m$ is annihilated by a 1-PSG of G.
(ii) The polarizations Pf_i define the nullcone of $V^{\oplus k}$ for all $k \leq m$.

Proof. By the very definition (1), the polarizations $P_{i_1,\dots,i_k} f_i$ vanish in a tuple $(v_1,\dots,v_k) \in V^{\oplus k}$ if and only if the linear span $\langle v_1,\dots,v_k\rangle$ consists of elements of the nullcone $\mathcal{N}_V$.

A first application is the following result about commutative reductive groups.

Proposition 1.2. *Let D be a commutative reductive group and let V be a representation of D. Assume that $\mathcal{O}(V)^D$ is generated by the homogeneous invariants $f_1,\dots,f_r$. Then the polarizations Pf_i define the nullcone of $V^{\oplus k}$ for any number k of copies of V.*

Proof. The represention V has a basis $(v_1,\dots,v_n)$ consisting of eigenvectors of D, i.e., there are characters $\chi_i \in X(D)$ $(i = 1,\dots,n)$ such that $hv_i = \chi_i(h)\cdot v_i$ for all $h \in D$. Denote by $x_1,\dots,x_n$ the dual basis so that $\mathcal{O}(V) = \mathbb{C}[x_1,\dots,x_n]$. It is well-known that the invariants are generated by the invariant monomials in the x_i. Hence, the nullcone is a union of linear subspaces: $\mathcal{N}_V = \bigcup_j L_j$, where L_j is spanned by a subset of the basis $(v_1,\dots,v_n)$. If $v \in L_j$ is a general element (i.e., all coordinates are non-zero) and if $\lim_{t\to 0} \lambda(t)v = 0$, then λ also annihilates the subspace L_j. Thus every linear subspace of $\mathcal{N}_V$ is annihilated by a 1-PSG.

Remark 1.1. The example of the representation of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $t(x,y) := (tx,t^{-1}y)$ shows that the polarizations of the invariants do not generate the ring of invariants of more than one copy of $\mathbb{C}^2$.

For the study of linear subspaces of the nullcone the following result turns out to be useful.

Proposition 1.3. *If there is a linear subspace L of $\mathcal{N}_V$ of a certain dimension d, then there is also a B-stable linear subspace of $\mathcal{N}_V$ of the same dimension where B is a Borel subgroup of G.*

Proof. The set of linear subspaces of the nullcone of a given dimension d is easily seen to form a closed subset Z of the Grassmanian $\mathrm{Gr}_d(V)$. Since Z is also stable under G it has to contain a closed G-orbit. Such an orbit always contains a point which is fixed by B, and this point corresponds to a B-stable linear subspace of V of dimension d.

2 Some examples

Let us give some instructive examples.

Example 2.1 (Orthogonal representations). Consider the standard representation of SO_n on $V = \mathbb{C}^n$. Then a subspace $L \subset V$ belongs to the nullcone if and only if L is totally isotropic with respect to the quadratic form q on V. Then V can be decomposed in the form $V = V_0 \oplus (L \oplus L')$ such that $q|_{V_0}$ is non-degenerate, L' is totally isotropic, and $L \oplus L'$ is the orthogonal complement of V_0. It follows that the 1-PSG λ of $\mathrm{GL}(V)$ given by

$$\lambda(t)v := \begin{cases} t \cdot v & \text{for } v \in L, \\ t^{-1} \cdot v & \text{for } v \in L', \\ v & \text{for } v \in V_0, \end{cases}$$

belongs to SO_n and annihilates L. Therefore, the polarizations of q define the null-cone of any number of copies of $\mathbb{C}^n$. Here the polarizations of q are given by the quadratic form q applied to each copy of V in $V^{\oplus m}$ and the associated bilinear form $\beta(v,w) := \frac{1}{2}(q(v+w) - q(v) - q(w))$ applied to each pair of copies in $V^{\oplus m}$.

Of course, this result is also an immediate consequence of the First Fundamental Theorem for O_n or SO_n (see [GoW98, Theorem 4.2.2] or [Pro07, 11.2.1]).

Example 2.2 (Conjugacy classes of matrices). Let GL_3 act on the 3×3-matrices $M_3(\mathbb{C})$ by conjugation and consider the following two matrices:

$$J := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad N := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

It is easy to see that $sJ + tN$ is nilpotent for all $s,t \in \mathbb{C}$. However, JN is a non-zero diagonal matrix and so there is no 1-PSG which annihilates the two-dimensional subspace $L := \langle J, N \rangle$ of the nullcone of M_3. It follows that the polarizations of the functions $X \mapsto \operatorname{tr} X^k$ ($1 \le k \le 3$) do not define the nullcone of two and more copies of M_3.

The polarizations for two copies are the following nine homogeneous invariant functions defined for $(A,B) \in M_3 \oplus M_3$:

$$\operatorname{tr} A,\ \operatorname{tr} B,\ \operatorname{tr} A^2,\ \operatorname{tr} AB,\ \operatorname{tr} B^2,\ \operatorname{tr} A^3,\ \operatorname{tr} A^2B,\ \operatorname{tr} AB^2,\ \operatorname{tr} B^3.$$

(Use the fact that $\operatorname{tr} ABA = \operatorname{tr} A^2B$ etc.) It is an interesting fact that these 9 functions define a subvariety Z of $M_3 \oplus M_3$ of codimension 9 and so the nullcone of $M_3 \oplus M_3$ is an irreducible component of Z. However, the invariant ring of $M_3 \oplus M_3$ has dimension 10 $(= 18 - 8)$ and so a system of parameters must contain 10 elements. It was shown by Teranishi [Te86] that one obtains a system of parameters by adding the function $\operatorname{tr} ABAB$, and a system of generators by adding, in addition, the function $\operatorname{tr} ABA^2B^2$.

Conjecture 2.1.[1] The polarizations of the functions $X \mapsto \operatorname{tr} X^j$ ($j = 1, \ldots, n$) for two copies of M_n define a subvariety Z of codimension $n^2 + 3n/2$ which is a set-theoretic complete intersection and has the nullcone as an irreducible component.

[1] Note added in proof: this conjecture was recently proved by J.-Y. Charbonnel and A. Moreau: "Nilpotent bicone and characteristic submodule in a reductive Lie algebra" http://arxiv.org/abs/0705.2685.

(Note that the number of polarizations of these n functions is $2+3+\cdots+(n+1) = n^2+3n/2$ *and that this number is also equal to the codimension of the nullcone (see [KrW06, Example 2.1]).*

Remark 2.1. It has been shown by Gerstenhaber [Ge58] that a linear subspace L of the nilpotent matrices $\mathcal{N}$ in M_n of maximal possible dimension $\binom{n}{2}$ (see Proposition 1.3) is conjugate to the nilpotent upper triangular matrices, hence annihilated by a 1-PSG. Jointly with Jan Draisma and Jochen Kuttler we have generalized this result to arbitrary semisimple Lie algebras, see [DKK06].

Example 2.3 (Symmetric matrices). [KrW06, Example 2.4] Consider the representation of $G := \mathrm{SO}_4$ on $S_0^2(\mathbb{C}^4)$, the space of trace zero symmetric 4×4-matrices. This is equivalent to the representation of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on $V_2 \otimes V_2$ where V_2 is the space of quadratic forms in 2 variables. The invariant ring is a polynomial ring generated by the functions $f_i := \operatorname{tr} X^i$, $2 \le i \le 4$. A direct calculation shows that every two-dimensional subspace of the nullcone is annihilated by a 1-PSG. This implies that the polarizations of the functions f_2, f_3, f_4 define the nullcone for two copies of $S_0^2(\mathbb{C}^4)$. Since the number of polarizations is $12 = 3+4+5$ which is the dimension of the invariant ring (i.e., of the quotient $(S_0^2(\mathbb{C}^4) \oplus S_0^2(\mathbb{C}^4))/\!/\,\mathrm{SO}_4$), we see that these 12 polarizations form a system of parameters. (This completes the analysis given in [WaW00].)

These examples show that there are two basic questions in this context.

Question 2.2. What are the linear subspaces of the nullcone of a representation V?

Question 2.3. Given a linear subspace $U \subset \mathcal{N}_V$ of the nullcone of a representation V, is there a 1-PSG which annihilates U?

We now give a general construction where we get a negative answer to Question 2.3 above. Denote by $\mathbb{C}^2 = \mathbb{C}e_0 \oplus \mathbb{C}e_1$ the standard representation of SL_2.

Proposition 2.1. *Let V be a representation of a reductive group H. Consider the representation $W := \mathbb{C}^2 \otimes V$ of $G := \mathrm{SL}_2 \times H$.*

a. *For every $v \in V$ the subspace $\mathbb{C}^2 \otimes v$ belongs to the nullcone $\mathcal{N}_W$.*
b. *If $v \in V \setminus \mathcal{N}_V$ then there is no 1-PSG λ of G such that* $\lim_{t\to 0} \lambda(t)w = 0$ *for all $w \in \mathbb{C}^2 \otimes v$.*

Proof. (a) Clearly, $e_0 \otimes v \in \mathcal{N}_W$ for any $v \in V$. Hence $\{g e_0 \otimes v \mid g \in \mathrm{SL}_2\} \subset N_W$, and the claim follows since $\mathbb{C}^2 \otimes v = \{g e_0 \otimes v \mid g \in \mathrm{SL}_2\} \cup \{0\}$.

(b) Assume that $\lim_{t\to 0} \lambda(t)w = 0$ for all $w \in \mathbb{C}^2 \otimes v$. Write $v = \sum v_j$ such that $\lambda(t)v_j = t^j \cdot v_j$ and choose $f \in \mathbb{C}^2$ such that $\lambda(t)f = t^s \cdot f$ where $s \le 0$. Since $v \notin \mathcal{N}_V$ there exists a $k \le 0$ such that $v_k \ne 0$. Then $\lambda(t)(f \otimes v_k) = t^{s+k} \cdot (f \otimes v_k)$ which leads to a contradiction since $s+k \le 0$.

Corollary 2.4. *If the representation V admits non-constant G-invariants, then the polarizations of the invariants of $W := \mathbb{C}^2 \otimes V$ do not define the nullcone of 2 or more copies of W.*

Corollary 2.5. *For $n \geq 3$ the polarizations of the invariants of the n-qubits $Q_n := \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ (n factors) under $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2$ do not define the nullcone of two or more copies of Q_n.*

3 General polarizations

For our applications we have to generalize the notion of polarization introduced in Section 1. Let V be a finite-dimensional vector space and $f \in \mathscr{O}(V^{\oplus k})$ a (multihomogeneous) regular function on k copies of V. Fixing $m \geq k$ and using parameters t_{ij}, $1 \leq i \leq k$, $1 \leq j \leq m$ where $m \geq k$ we write, for $(v_1, v_2, \ldots, v_m) \in V^{\oplus m}$,

$$f(\sum_j t_{1j}v_j, \sum_j t_{2j}v_j, \cdots, \sum_j t_{kj}v_j) = \sum_A t^A P_A f(v_1, v_2, \ldots, v_m). \tag{2}$$

where $A = (a_{ij})$ runs through the $k \times m$-matrices with non-negative integers a_{ij} and $t^A := \prod_{ij} t_{ij}^{a_{ij}}$. The regular (multihomogeneous) functions $P_A f \in \mathscr{O}(V^{\oplus m})$ obtained in this way are again called *polarizations* of f. As before, if V is a representation of G and f a G-invariant function, then so are the polarizations $P_A f$. The next lemma is an immediate consequence of the definition.

Lemma 3.1. *Let $f \in \mathscr{O}(V^{\oplus k})$, $v_1, \ldots, v_m \in V$ where $m \geq k$ and denote by $U := \langle v_1, v_2, \ldots, v_m \rangle \subset V$ the linear span of $v_1, \ldots, v_m$. Then the following two statements are equivalent.*

(i) f vanishes on $U^{\oplus m} \subset V^{\oplus m}$.
(ii) $P_A f(v_1, \ldots, v_m) = 0$ for all polarizations $P_A f$ of f.

Let us go back to the general situation of a representation of a connected reductive group G on a vector space V. Denote by $\mathscr{L}_V$ the set of linear subspaces of V which are annihilated by a 1-PSG of G and which are maximal under this condition, and by $\mathscr{M}_V$ the set of all maximal linear subspaces of the nullcone $\mathscr{N}_V$ of V.

We can regard $\mathscr{L}_V$ and $\mathscr{M}_V$ as closed G-stable subvarieties of the Grassmannian $\mathrm{Gr}(V) = \bigcup_{1 \leq d \leq \dim V} \mathrm{Gr}_d(V)$. We have seen in [KrW06] that $\mathscr{L}_V$ consists of a finite number of closed orbits. In particular, $\dim \mathscr{L}_V \leq \dim G/B$.

Proposition 3.1. *Let $k < m$ be positive integers and assume that the invariant functions $f_1, \ldots, f_n \in \mathscr{O}(V^{\oplus k})^G$ define the nullcone $\mathscr{N}_{V^{\oplus k}}$. If every linear subspace $U \subset \mathscr{N}_V$ with $k < \dim U \leq m$ is annihilated by a 1-PSG, then the polarizations $P_A f_i$ define the nullcone $\mathscr{N}_{V^{\oplus m}}$ of $V^{\oplus m}$.*

Proof. Assume that for a given $v = (v_1, \ldots, v_m)$ we have $P_A f_i(v_1, \ldots, v_m) = 0$ for all polarizations $P_A f_i$. Define $U := \langle v_1, \ldots, v_m \rangle$. By the lemma above $U^{\oplus k}$ belongs to the nullcone of $V^{\oplus k}$, hence $U \subset \mathscr{N}_V$. If $\dim U > k$, then by assumption U is annihilated by a 1-PSG and so $(v_1, \ldots, v_m) \in \mathscr{N}_{V^{\oplus m}}$.

If $\dim U \le k$, then, after possible rearrangement of $\{v_1,\dots,v_m\}$, we can assume that $U = \langle v_1,\dots,v_k\rangle$. Since $(v_1,\dots,v_k) \in \mathscr{N}_{V^{\oplus k}}$, by assumption, it follows again that U is annihilated by a 1-PSG.

Example 3.1. For the standard representation of SL_n on $V := \mathbb{C}^n$ there are no invariants for less than n copies, and $\mathscr{O}(V^{\oplus n})^{\mathrm{SL}_n} = \mathbb{C}[\det]$. Therefore, the determinants $\det(v_{i_1} v_{i_2} \cdots v_{i_n})$ define the nullcone on any number of copies of V. In fact, one knows that they even generate the ring of invariants, by the so-called "First Fundamental Theorem for SL_n" (see [Pro07, 11.1.2]).

Example 3.2. For the standard representation of Sp_{2n} on $V := \mathbb{C}^{2n}$ there are no invariants on one copy, and $\mathscr{O}(V \oplus V)^{\mathrm{Sp}_{2n}} = \mathbb{C}[f]$ where $f(u,v)$ is the skew form defining $\mathrm{Sp}_{2n} \subset \mathrm{GL}_{2n}$. As in the orthogonal case (see Example 2.1), one easily sees that every linear subspace of the nullcone is annihilated by a 1-PSG. Hence, the skew forms $f_{ij} = f(v_i,v_j)$ define the nullcone of any number of copies of V. Again, the "First Fundamental Theorem" shows that these invariants even generate the invariant ring (see [GoW98, Theorem 4.2.2] or [Pro07, 11.2.1]).

Example 3.3 (see Example 2.2). Applying the proposition to the case of the adjoint representation of GL_n on the matrices M_n we get the following result. *If the invariants $f_1,\dots,f_k$ define the nullcone of $\binom{n}{2} - 1$ copies of M_n, then the polarizations $P_A f_i$ define the nullcone of any number of copies of M_n.*

For $n = 3$ this implies (see Example 2.2) that the traces $\{\mathrm{tr} A_i, \mathrm{tr} A_i A_j, \mathrm{tr} A_i A_j A_k, \mathrm{tr} A_i A_j A_k A_\ell\}$ define the nullcone of any number of copies of M_3.

Let m_V denote the maximal dimension of a linear subspace of the nullcone $\mathscr{N}_V$.

Corollary 3.1. *If $f_1,\dots,f_n \in \mathscr{O}(V^{\oplus m_V})^G$ define the nullcone $\mathscr{N}_{V^{\oplus m_V}}$, then the polarizations $P_A f_i$ define the nullcone of any number of copies of V.*

4 Nullcone of several copies of binary forms

In this section we study the invariants and the nullcone of representations of the group SL_2. We denote by $V_n := \mathbb{C}[x,y]_n$ the binary forms of degree n considered as a representation of SL_2. Recall that in this setting the form $y^n \in V_n$ is a highest weight vector with respect to the standard Borel subgroup $B \subset \mathrm{SL}_2$ of upper triangular matrices.

The main result of this section is the following.

Theorem 4.1. *Consider the irreducible representation V_n of SL_2. Assume that $n > 1$ and that the homogeneous invariant functions $f_1, f_2, \dots, f_m \in \mathscr{O}(V_n)^{\mathrm{SL}_2}$ define the nullcone of V_n. Then the polarizations of the f_is for any number N of copies of V_n define the nullcone of $V_n^{\oplus N}$.*

The following result is a main step in the proof.

Lemma 4.1. *Let $h_1, h_2 \in V_n$ be two non-zero binary forms. Assume that every non-zero linear combination $\alpha h_1 + \beta h_2$ has a linear factor of multiplicity $> n/2$. Then h_1 and h_2 have a common linear factor of multiplicity $> n/2$.*

Proof. We can assume that h_1 and h_2 are linearly independent. Fix a number $k \in \mathbb{N}$ such $n/2 < k \leq n$ and define the following subsets of $V_n \oplus V_1$:

$$Y_k := \{(f, \ell) \in V_n \oplus V_1 \mid \ell^k \text{ divides } f\}.$$

This is a closed subset of $V_n \oplus V_1$, because $Y_k = \mathrm{SL}_2 \cdot (W \oplus \mathbb{C}y)$ where $W := \bigoplus_{i=k}^n \mathbb{C}x^{n-i}y^i$, and $W \oplus \mathbb{C}y$ is a B-stable linear subspace of $V_n \oplus V_1$. Moreover, Y_k is stable under the action of $\mathbb{C}^*$ by scalar multiplication on V_1. Therefore, the quotient $Y_k \setminus (W \times \{0\})/\mathbb{C}^*$ is a vector bundle $p\colon \mathscr{V}_k \to \mathscr{P}(V_1)$, namely the subbundle of the trivial bundle $V_n \times \mathscr{P}(V_1)$ whose fiber over $[\ell]$ is the subspace $\ell^k \cdot V_{n-k} \subset V_n$. It is clear that this vector bundle can be identified with the associated bundle $\mathrm{SL}_2 \times^B W \to \mathrm{SL}_2/B = \mathscr{P}^1$.

Now consider the following subset of $\mathbb{C}^2 \times \mathscr{P}(V_1)$,

$$\mathscr{L}_k := \{((\alpha, \beta), [\ell]) \in \mathbb{C}^2 \times \mathscr{P}(V_1) \mid \ell^k \text{ divides } \alpha h_1 + \beta h_2\}.$$

$\mathscr{L}_k$ is the inverse image of $\mathscr{V}_k$ under the morphism $\varphi\colon \mathbb{C}^2 \times \mathscr{P}(V_1) \to V_n \times \mathscr{P}(V_1)$ given by $((\alpha, \beta), [\ell]) \mapsto (\alpha h_1 + \beta h_2, [\ell])$, and so $\mathscr{L}_k$ is a closed subvariety of $\mathbb{C}^2 \times \mathscr{P}(V_1)$. Since φ is a closed immersion we can identify $\mathscr{L}_k$ with a closed subvariety of the $\mathscr{V}_k$.

If two linearly independent members f_1, f_2 of the family $\alpha h_1 + \beta h_2$ have the same linear factor ℓ of multiplicity $\geq k$, then all the members of the family have this factor and we are done. Otherwise, the morphism $p\colon \mathscr{L}_k \to \mathscr{P}(V_1)$ induced by the projection is surjective and the fibers are lines of the form $\mathbb{C}f \times \{[\ell]\}$. Hence $\mathscr{L}_k$ is a subbundle of $\mathscr{V}_k$. It follows from the construction of $\mathscr{L}_k$ as a subbundle of the trivial bundle of rank 2 that $\mathscr{L}_k$ is isomorphic to $\mathscr{O}(-1)$. The following Lemma 4.2 shows that this bundle cannot occur as a subbundle of $\mathrm{SL}_2 \times^B W \to \mathrm{SL}_2/B = \mathscr{P}^1$ provided that $n > 1$.

Remark 4.1. It was shown by MATTHIAS BÜRGIN in his thesis (see [Bü06]) that the following generalization of Lemma 4.1 holds. *Let $f, h \in \mathbb{C}[t]$ be two polynomials and k an integer ≥ 2. Assume that every linear combination $\lambda f + \mu h$ has a root of multiplicity $\geq k$. Then f and h have a common root of multiplicity $\geq k$.*

Lemma 4.2. *Denote by V_n^+ the B-stable subspace of V_n consisting of positive weights. Then we have*

$$\mathrm{SL}_2 \times^B V_n^+ \simeq \begin{cases} \mathscr{O}(-k)^k & \text{if } n = 2k-1, \\ \mathscr{O}(-k-1)^k & \text{if } n = 2k. \end{cases}$$

Proof. If M is a B-module we denote by $M(i)$ the module obtained from M by tensoring with the character $\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \mapsto t^i$. If $\mathscr{V}(M) := \mathrm{SL}_2 \times^B M$ then $\mathscr{V}(M(i)) = \mathscr{V}(M)(-i)$. With this notation we have the following isomorphisms as B-modules:

$$V_{2k-1}^+ \simeq V_{k-1}(k) \quad \text{and} \quad V_{2k}^+ \simeq V_{k-1}(k+1).$$

Since $\mathrm{SL}_2 \times^B V_m$ is the trivial bundle of rank $m+1$ the claim follows.

Now we can give the proof of our Main Theorem of this section.

Proof (Proof of Theorem 4.1). Let $h = (h_1, h_2, \dots, h_N) \in V_n^N$ an n-tuple of forms such that all polarizations of all f_i vanish on h. This implies that $f_i(\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_N h_N) = 0$ for all $(\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{C}^N$ and all i's. It follows that $\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_N h_N$ belongs to the nullcone of V_n for all $(\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{C}^N$, hence they all have a linear factor ℓ of multiplicity $> n/2$. Using Lemma 2 above, an easy induction shows that the h_i must have a common linear factor ℓ of multiplicity $> n/2$. Thus h belongs to the nullcone of V_n^N.

From the proof above we immediately get the following generalization of our Theorem 4.1.

Theorem 4.2. *Consider the representation $V = V_{n_1} \oplus V_{n_2} \oplus \cdots \oplus V_{n_k}$ of SL_2, where $1 < n_1 < n_2 < \cdots < n_k$. Assume that the multihomogeneous invariant functions $f_1, f_2, \dots, f_m \in \mathscr{O}(V)^{\mathrm{SL}_2}$ define the nullcone of V. Then the polarizations of the f_i's to the representation $\tilde{V} = V_{n_1}^{N_1} \oplus V_{n_2}^{N_2} \oplus \cdots \oplus V_{n_k}^{N_k}$ for any k-tuple $(N_1, N_2, \dots, N_k)$ define the nullcone of $\tilde{V}$.*

Remark 4.2. One can also include the case $n_1 = 1$ by either assuming that $N_1 = 1$ or by adding the invariants $[i,j]$ of $V_1^{N_1}$ to the set of polarizations. (Recall that $[i,j](\ell_1, \dots, \ell_N) := [\ell_i, \ell_j] := \alpha_i \beta_j - \alpha_j \beta_i$ where $\ell_i = \alpha_i x + \beta_i y \in V_1$.) Since the covariants $\mathscr{O}(V)^U$ can be identified with the invariants $\mathscr{O}(V \oplus V_1)$ the theorem above has some interesting consequences for covariants.

Example 4.1 (Covariants of V_3^N). The covariants of V_3^N can be identified with the invariants of $V_3^N \oplus V_1$. The case $N = 1$ is well-known and classical: $\mathscr{O}(V_3 \oplus V_1)^{\mathrm{SL}_2} = \mathbb{C}[h, f_{1,3}, f_{2,2}, f_{3,3}]$, where h is the discriminant of V_3 and the $f_{i,j}$ are bihomogeneous invariants of degree (i,j) corresponding to $V_3 \subset \mathscr{O}(V_3)_1$, $V_2 \subset \mathscr{O}(V_3)_2$ and $V_1 \subset \mathscr{O}(V_3)_3$. Recall that an embedding $V_n \subset \mathscr{O}(V_3)_d$ defines a covariant $\varphi \colon V_3 \to V_n$ of degree d and thus an invariant $f_{d,n} \colon (f, \ell) \mapsto [\varphi(f), \ell^n]$ where the bracket $[\cdot,\cdot]$ denotes the invariant bilinear form on $V_n \times V_n$.

It is easy to see that $h, f_{1,3}, f_{2,2}$ form a system of parameters, i.e. define the nullcone of $V_3 \oplus V_1$. Therefore, their polarizations (in the variables of V_3) define the nullcone of $V_3^N \oplus V_1$ for any $N \geq 1$. Therefore, we always have a system of parameters in degree 4 and thus can easily calculate the Hilbert series for small N, e.g.,

$$\mathrm{Hilb}_{V_3^2 \oplus V_1} = \frac{h_2}{(1-t^2)(1-t^4)^6} \quad \text{and} \quad \mathrm{Hilb}_{V_3^3 \oplus V_1} = \frac{h_3}{(1-t^2)^3(1-t^4)^8},$$

where

$$h_2 := 1 + 6t^4 + 13t^6 + 12t^8 + 13t^{10} + 6t^{12} + t^{16}$$

and

$$h_3 := 1 + 24t^4 + 62t^6 + 177t^8 + 300t^{10} + 320t^{12} + 300t^{14} \\ + 177t^{16} + 62t^{18} + 24t^{20} + t^{24}.$$

For the calculation we use the fact (due to Knop [Kn89]) that the degree of the Hilbert series is $\leq -\dim V$ and that the numerator is palindromic since the invariant ring is Gorenstein. The Theorem of Weyl implies that the covariants for V_3^N are obtained from those of V_3^3 by polarization. Since the representation is symplectic they are even obtained from V_3^2 by polarization (see Schwarz [Sch87]).

5 Generators and system of parameters for the invariants of 3-qubits

Lemma 5.1. *Consider the polynomial ring $\mathbb{C}[a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}]$ in the coefficients of a quadratic form in three variables and put*

$$d := \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Then the elements $\{a_{11} - a_{22},\ a_{22} - a_{33},\ a_{12}, a_{13}, a_{23}, d\}$ form a homogeneous system of parameters.

Proof. The proof is easy: one simply shows that the zero set of these functions is the origin.

Let us now consider N copies of the standard representation $\mathbb{C}^n$ of the complex orthogonal group $\mathrm{O}_n = \mathrm{O}_n(\mathbb{C})$: $W := \mathbb{C}^N \otimes \mathbb{C}^n$. The first fundamental theorem for O_n and SO_n tells us that the invariants under O_n are generated by the quadratic invariants $\sum_{\nu=1}^n x_{i\nu} x_{j\nu}$ $(1 \leq i \leq j \leq N)$ and that for SO_n we have to add the $n \times n$ minors of the matrix $(x_{i\nu})$. In terms of representation theory this means the following. We have (by Cauchy's formula)

$$S^2\mathbb{C}^N \otimes \mathbb{C} \subset S^2(\mathbb{C}^N \otimes \mathbb{C}^n) \quad \text{and} \quad \bigwedge^n \mathbb{C}^N \otimes \mathbb{C} \subset S^n(\mathbb{C}^N \otimes \mathbb{C}^n),$$

where $\mathbb{C}$ denotes the trivial representation of SO_n, and these subspaces form a generating system for $S(\mathbb{C}^N \otimes \mathbb{C}^n)^{\mathrm{SO}_n}$.

As before we denote by V_m the irreducible representation of SL_2 of dimension $m+1$. We apply the above first to the the case of three copies of the irreducible three-dimensional representation V_2 of SL_2: $W = \mathbb{C}^3 \otimes V_2$. Then the subspaces $S^2\mathbb{C}^3 \otimes V_0$

and $\bigwedge^3 \mathbb{C}^3 \otimes V_0$ form a minimal generating system for the SL_2-invariants. Thus we get six generators in degree 2 and one generator in degree 3.

Now we consider the space $\mathbb{C}^3 \otimes V_2$ as a representation of $\mathrm{SO}_3 \times \mathrm{SL}_2$ and denote the six quadratic generators by $a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}$ with the obvious meaning. Then the cubic generator q satisfies the relation

$$q^2 = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Moreover, the space $S^2\mathbb{C}^3$ decomposes under SO_3 into the direct sum of two irreducible representations

$$S^2\mathbb{C}^3 = S_0^2\mathbb{C}^3 \oplus \mathbb{C}$$

where $\mathbb{C}$ is the trivial representation. In terms of coordinates, $\mathbb{C}$ is spanned by $a_{11} + a_{22} + a_{33}$ and $S_0^2\mathbb{C}^3$ by $\{a_{11} - a_{22}, a_{22} - a_{33}, a_{12}, a_{13}, a_{23}\}$. With Lemma 6 above we therefore have the following result.

Proposition 5.1. *Consider the representation $W := \mathbb{C}^3 \otimes V_2$ of $\mathrm{SO}_3 \times \mathrm{SL}_2$. Then the 5-dimensional subspace $S_0^2(\mathbb{C}^3) \otimes V_0 \subset S^2(W)$ together with the one-dimensional subspace $\bigwedge^3 \mathbb{C}^3 \otimes V_0 \subset S^3(W)$ forms a homogeneous system of parameters for the invariants $S(W)^{\mathrm{SL}_2}$.*

We want to apply this to the invariants of two copies of 3-qubits, i.e., to the representation

$$V := V_1 \otimes V_1 \otimes V_1 \oplus V_1 \otimes V_1 \otimes V_1 = \mathbb{C}^2 \otimes V_1 \otimes V_1 \otimes V_1$$

of $G := \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$. We consider this as a representation of $\mathrm{SL}_2 \times \mathrm{SO}_4$:

$$V = \mathbb{C}^2 \otimes V_1 \otimes \mathbb{C}^4,$$

where $\mathbb{C}^4$ is the standard representation of SO_4. As a representation of SO_4 this is the direct sum of four copies of the standard representation. Therefore, the SO_4 invariants are generated by $S^2(\mathbb{C}^2 \otimes V_1) \otimes V_0 \subset S^2(V)$ and $\bigwedge^4(\mathbb{C}^2 \otimes V_1) \otimes V_0 \subset S^4(V)$, i.e., we have ten generators in degree 2 and one generator q_4 in degree 4. Moreover, the induced morphism

$$\pi_1 \colon V \to S^2(\mathbb{C}^2 \otimes V_1)$$

is surjective (and homogeneous of degree 2), and π_1 is the quotient map under O_4. The generator q_4 is invariant under the full group G. The 10-dimensional representation $S^2(\mathbb{C}^2 \otimes V_1)$ decomposes under SL_2 in the form

$$S^2(\mathbb{C}^2 \otimes V_1) = S^2(\mathbb{C}^2) \otimes V_2 \oplus \bigwedge^2 \mathbb{C}^2 \otimes V_0 = \mathbb{C}^3 \otimes V_2 \oplus \mathbb{C} \otimes V_0.$$

Thus there is G-invariant q_2 in degree 2 given by the second summand. We have seen above that the SL_2-invariants of $\mathbb{C}^3 \otimes V_2$ are generated by six invariants in degree 2 and one in degree 3, represented by the subspaces $S^2(\mathbb{C}^3)\otimes V_0 \subset S^2(\mathbb{C}^3\otimes V_2)$ and $\bigwedge^3\mathbb{C}^3\otimes V_0 \subset S^3(\mathbb{C}^3\otimes V_2)$. This proves the first part of the following theorem. The second part is an immediate consequence of Proposition 5.1 above.

Theorem 5.1. *The* $\mathrm{SL}_2\times\mathrm{SL}_2\times\mathrm{SL}_2$*-invariants of* $(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)^{\oplus 2}$ *are generated by one invariant* q_2 *in degree 2, seven invariants* $p_1,\dots,p_6,q_4$ *in degree 4, and one invariant* q_6 *in degree 6. A homogeneous system of parameters for the invariant ring is given by* $q_2,p_1,\dots,p_5,q_6$ *where* $p_1,\dots,p_5$ *span the subspace* $S_0^2(\mathbb{C}^3)\otimes V_0$ *stable under* SO_3 *acting on* $\mathbb{C}^3$.

Remark 5.1. The generating invariants have the following bidegrees: $\deg q_2=(1,1)$, $\deg q_4=(2,2)$, $\deg q_6=(3,3)$, and the bidegrees of the p_is are $(4,0)$, $(3,1)$, $(2,2)$, $(2,2)$, $(1,3)$, $(0,4)$.

Remark 5.2. The invariants of $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ under $G=\mathrm{SL}_2\times\mathrm{SL}_2\times\mathrm{SL}_2$ are generated by one invariant p of degree 4. It is given by the consecutive quotient maps

$$p\colon \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2=\mathbb{C}^2\otimes\mathbb{C}^4 \xrightarrow{/\mathrm{SO}_4} S^2\mathbb{C}^2=V_2 \xrightarrow{/\mathrm{SL}_2} \mathbb{C}.$$

The nullcone $p^{-1}(0)$ is irreducible of dimension 7 and contains a dense orbit, namely the orbit of $v_0:=e_1\otimes e_1\otimes e_0+e_1\otimes e_0\otimes e_1+e_0\otimes e_1\otimes e_1$. (In fact, it easy to see, by Hilbert's criterion, that v_0 is in the nullcone; moreover, the annihilator of v_0 in $\mathrm{Lie}\,G$ has dimension 2, hence Gv_0 is an orbit of dimension 7.) Therefore, all fibres of p are irreducible (of dimension 7) and contain a dense orbit. More precisely, we have the following result. (We use the notation $e_{ijk}:=e_i\otimes e_j\otimes e_k$.)

Proposition 5.2. *The nullcone of* $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ *contains six orbits, the origin* $\{0\}$*, the orbit* Ge_{111} *of the highest weight vector which is of dimension 4, the dense orbit* $G(e_{110}+e_{101}+e_{011})$ *of dimension 7, and the three orbits of the elements* $e_{100}+e_{010}$*,* $e_{010}+e_{001}$*,* $e_{001}+e_{100}$ *which are of dimension 5 and which are permuted under the symmetric group* $\mathscr{S}_3$ *permuting the three factors in the tensor product.*

Proof. The weight vector e_{ijk} has weight

$$\varepsilon_{ijk}:=((-1)^{i+1},(-1)^{j+1},(-1)^{k+1})\in\mathbb{Z}^3=X(\mathbb{C}^*\times\mathbb{C}^*\times\mathbb{C}^*)$$

and so the set X_V of weights of $V:=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ consists of the vertices of a cube in $\mathbb{R}^3$ centered in the origin. There are four maximal unstable subsets of X_V in the sense of [KrW06, Definition 1.1], up to the action of the Weyl group, namely the set of vertices of the three faces of the cube containing the highest weight ε_{111}, and the set $\{\varepsilon_{111},\varepsilon_{011},\varepsilon_{101},\varepsilon_{110}\}$. The corresponding maximal unstable subspaces of V are (see [KrW06, Definition 1.2]):

$$\begin{aligned}
W_1 &:= \langle e_{111},e_{110},e_{101},e_{100}\rangle\\
W_2 &:= \langle e_{111},e_{110},e_{011},e_{010}\rangle\\
W_3 &:= \langle e_{111},e_{011},e_{101},e_{001}\rangle\\
U &:= \langle e_{111},e_{011},e_{101},e_{110}\rangle.
\end{aligned}$$

It follows that the nullcone is given as a union

$$\mathcal{N}_V = GU \cup GW_1 \cup GW_2 \cup GW_3.$$

The subspace U is stabilized by $B \times B \times B$ whereas $W_1 = e_1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is stable under $B \times \mathrm{SL}_2 \times \mathrm{SL}_2$, and similarly for W_2 and W_3. Since the spaces W_i are not stable under G we get $\dim GW_i = \dim W_i + 1 = 5$, and so $GU = \mathcal{N}_V$.

The group $\mathrm{SL}_2 \times \mathrm{SL}_2$ has three orbits in $\mathbb{C}^2 \otimes \mathbb{C}^2$, the dense orbit of $e_1 \otimes e_0 + e_0 \otimes e_1$, the highest weight orbit of $e_1 \otimes e_1$, and $\{0\}$. This shows that $\overline{G(e_{110}+e_{101})} = GW_1$ and that $GW_1 \setminus G(e_{110}+e_{101}) \subset \overline{Ge_{111}}$, and similarly for W_2 and W_3. One also sees that the elements $e_{110}+e_{101}$, $e_{110}+e_{011}$, and $e_{101}+e_{011}$ represent three different orbits of dimension 5, all containing the highest weight orbit in their closure. In fact, $GW_1 = \{ge_1 \otimes v \mid g \in G \text{ and } v \in \mathbb{C}^2 \otimes \mathbb{C}^2\}$, and so $ge_1 \otimes v$ is not in W_2 except if v is a multiple of $ge_1 \otimes ge_1$. In particular, $GW_1 \cap GW_2 \cap GW_3 = \overline{Ge_{111}}$.

Finally, it is easy to see that $(B \otimes B \otimes B)v_0 = \mathbb{C}^* e_{110} \times \mathbb{C}^* e_{101} \times \mathbb{C}^* e_{011} \times \mathbb{C} e_{111}$. Hence, $\overline{Gv_0} = GU = \mathcal{N}_V$ and $\mathcal{N}_V \setminus Gv_0 \subset GW_1 \cup GW_2 \cup GW_3$.

Proposition 5.3. *The invariants in degree* 4 *of any number of copies of* Q_3 *define the nullcone. In particular, for any* $N \geq 1$ *there is a system of parameters of* $Q_3^{\oplus N}$ *in degree 4.*

Proof. We can identify $\mathbb{C}^N \otimes Q_3$, as a representation of ${\mathrm{SL}_2}^3$, with $\mathbb{C}^N \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$, as a representation of $\mathrm{SL}_2 \times \mathrm{SO}_4$. The quotient of $\mathbb{C}^N \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$ by O_4 is given by

$$\pi \colon \mathbb{C}^N \otimes \mathbb{C}^2 \otimes \mathbb{C}^4 \to S^2(\mathbb{C}^N \otimes \mathbb{C}^2),$$

where the image of π is the closed cone of symmetric matrices of rank ≤ 4 (First Fundamental Theorem for the orthogonal group; see [GoW98, Theorem 4.2.2] or [Pro07, 11.2.1]). This means that the O_4-invariants are generated by the obvious quadratic invariants. Moreover, the morphism π is SL_2-equivariant.

As a representation of SL_2 we have

$$S^2(\mathbb{C}^N \otimes \mathbb{C}^2) = S^2(\mathbb{C}^N) \otimes V_2 \oplus \textstyle\bigwedge^2 \mathbb{C}^N \otimes \mathbb{C},$$

where V_2 is the three-dimensional irreducible representation of SL_2 corresponding to the standard representation of SO_3, and $\mathbb{C}$ denotes the trivial representation. Again, consider this as a representation of O_3. Then the O_3-invariants are generated by the quadratic (and the linear) invariants. Summing up we see that the invariant ring

$$(\mathcal{O}(\mathbb{C}^n \otimes Q_3)^{\mathrm{O}_4})^{\mathrm{O}_3}$$

is generated by the elements of degree 2 and 4. By construction,

$$(\mathcal{O}(\mathbb{C}^n \otimes Q_3)^{\mathrm{O}_4})^{\mathrm{O}_3} \subset (\mathcal{O}(\mathbb{C}^n \otimes Q_3)^{\mathrm{SL}_2 \times \mathrm{SL}_2})^{\mathrm{SL}_2} = \mathcal{O}(\mathbb{C}^n \otimes Q_3)^{\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2}$$

and the latter is a finite module over the former. Therefore, both quotients have the same nullcone and so the nullcone is defined by invariants in degree 2 and 4.

Remark 5.3. The representation $Q_3 \oplus Q_3$ has one invariant of degree 2 and eight invariants of degree 4. Since the dimension of the quotient is 7 it follows that there is a system of parameters for the invariant ring consisting of seven invariants of degree 4. A priori it is not clear that there is also a system of parameters consisting of one invariant of degree 2 and six invariants of degree 4 as suggested by the Hilbert series which has the form

$$\mathrm{Hilb}_{Q_3 \oplus Q_3} = \frac{1+t^4+t^6+t^{10}}{(1-t^2)(1-t^4)^6}.$$

However, the analysis above shows that in the case of two copies of Q_3 we obtain the following composition of quotient maps

$$\pi\colon Q_3 \oplus Q_3 \xrightarrow{\pi_1} S^2\mathbb{C}^2 \otimes V_2 \oplus \mathbb{C} \xrightarrow{\pi_2} S^2S^2\mathbb{C}^2 \oplus \mathbb{C},$$

where π_1 is the quotient by O_4 and π_2 the quotient by O_3. Since both morphisms π_1 and π_2 are surjective in this case it follows that the zero fiber $\mathcal{N}$ of π is defined by the quadratic invariant and six invariants of degree 4. As we remarked above the (reduced) zero fiber of π is the nullcone of $Q_3 \oplus Q_3$ with respect to $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, hence these seven invariants form a homogeneous system of parameters for the ring of invariants.

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Decomposing Symmetric Powers of Certain Modular Representations of Cyclic Groups

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Summary. For a prime number p, we construct a generating set for the ring of invariants for the $p+1$ dimensional indecomposable modular representation of a cyclic group of order p^2, and show that the Noether number for the representation is p^2+p-3. We then use the constructed invariants to explicitly describe the decomposition of the symmetric algebra as a module over the group ring, confirming the Periodicity Conjecture of Ian Hughes and Gregor Kemper for this case. In the final section, we use our results to compute the Hilbert series for the corresponding ring of invariants together with some other related generating functions.

Key words: Modular representation theory, invariant theory, cyclic groups, symmetric powers

Mathematics Subject Classification (2000): 13A50

This paper is dedicated to Gerry Schwarz on the occasion of his sixtieth birthday.

1 Introduction

Suppose that V is a finite-dimensional representation of a finite group G over a field $\mathbf{F}$, i.e., V is a finitely generated module over the group ring $\mathbf{F}G$. The action of G on V induces an action on the dual V^* which extends to an action by algebra automorphisms on the symmetric algebra $\mathbf{F}[V] := S(V^*)$. The elements of V^*, and thus

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also the elements of $\mathbf{F}[V]$, represent $\mathbf{F}$-valued functions on V. If $\{x_1, x_2, \ldots, x_n\}$ is a basis for V^* then $\mathbf{F}[V]$ can be identified with the ring of polynomials $\mathbf{F}[x_1, x_2, \ldots, x_n]$. Let $\mathbf{F}[V]_d$ denote the subspace of homogeneous polynomials of degree d. Since the action of G preserves degree, $\mathbf{F}[V]_d$ is a module over $\mathbf{F}G$ and

$$\mathbf{F}[V] = \bigoplus_{d=0}^{\infty} \mathbf{F}[V]_d$$

is a decomposition into a direct sum of finite-dimensional $\mathbf{F}G$-modules. Of course $\mathbf{F}[V]_d$ is precisely the dth symmetric power of V^*. Understanding the action of G on $\mathbf{F}[V]_d$, and hence the action on $\mathbf{F}[V]$, is an important problem in representation theory. The primary goal is to write $\mathbf{F}[V]_d$ as a direct sum of indecomposable $\mathbf{F}G$-modules, refining the given decomposition of $\mathbf{F}[V]$. This means decomposing $\mathbf{F}[V]_d$ for infinitely many d. An important aspect of the group action is the *ring of invariants*

$$\mathbf{F}[V]^G := \{f \in \mathbf{F}[V] \mid g(f) = f, \ \forall g \in G\},$$

a finitely generated subalgebra of $\mathbf{F}[V]$. A fundamental problem in invariant theory is the construction of a finite generating set for $\mathbf{F}[V]^G$. Since G is finite, $\mathbf{F}[V]$ is a finite module over $\mathbf{F}[V]^G$. Thus $\mathbf{F}[V]$ is a module over both $\mathbf{F}[V]^G$ and $\mathbf{F}G$. Perhaps the right approach is to study $\mathbf{F}[V]$ as a finitely generated module over the extended group ring $\mathbf{F}[V]^G G$. Certainly, in the work of both Karagueuzian and Symonds [11] and Hughes and Kemper [10], the finite $\mathbf{F}[V]^G$-module structure of $\mathbf{F}[V]$ has been used to reduce decomposing $\mathbf{F}[V]$ over $\mathbf{F}G$ to a finite problem. Here we will develop and exploit an explicit description of the ring of invariants to obtain a similarly explicit description of the $\mathbf{F}G$-module $\mathbf{F}[V]_d$ for a specific representation V of $G = \mathbf{Z}/p^2$.

For the remainder of the chapter, we assume that $\mathbf{F}$ has characteristic p for a prime number p, and that $G \cong \mathbf{Z}/p^r$ is a cyclic group of order p^r. Choose a generator σ for G. The isomorphism type of a representation of G is determined by the Jordan canonical form of σ. Since the order of σ is a power of p, and since a field of characteristic p has no nontrivial pth roots of unity, all the eigenvalues of σ must be 1. If $m \leq p^r$, then the $m \times m$ matrix over $\mathbf{F}$ consisting of a single Jordan block with eigenvalue 1 determines an indecomposable $\mathbf{F}G$-module which we denote by V_m. Note that if $m > p^r$, then the matrix has order greater than p^r and does not determine a representation of G. It follows from the form of the matrix that V_m is faithful if and only if $p^{r-1} < m \leq p^r$, and that V_m is a cyclic $\mathbf{F}G$-module. It is clear that if the Jordan canonical form of σ consists of more than one Jordan block then the representation will be decomposable. Thus the complete set of inequivalent indecomposable $\mathbf{F}G$-modules are, up to isomorphism, $V_1, V_2, \ldots, V_{p^r}$. Furthermore, from the Jordan canonical form it is easy to see that these modules are naturally embedded into one another: $V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_{p^r}$. Note that the one-dimensional space of G-fixed points, $V_m^G \cong V_1$, is the *socle* of V_m. Moreover, V_1 is the unique irreducible module, $V_{p^r} \cong \mathbf{F}G$ is the unique projective indecomposable, and an $\mathbf{F}G$-module is projective if and only if it is injective (see, e.g., [1, Chapter II]). Also, it is easy to see that the representation V_m is induced from a representation of a proper subgroup of G if and only if p divides m.

For $f \in \mathbf{F}[V_n]$, we define the *norm* of f, denoted by $N^G(f)$, to be the product over the G-orbit of f. Clearly $N^G(f) \in \mathbf{F}[V_n]^G$. For a subgroup $L = \langle \sigma^{p^t} \rangle$, we define the *relative transfer* $\mathrm{Tr}_L^G := \sum_{i=0}^{p^t-1} \sigma^i \in \mathbf{F}G$. For a graded vector space $A = \oplus_{d=0}^{\infty} A_d$, the *Hilbert series* of A is given by $\mathscr{H}(A,t) := \sum_{d=0}^{\infty} \dim(A_d) t^d$.

Our two main results concern the representation V_{p+1} and are stated as Theorem 1.1 and Theorem 1.2 below. The following example illustrates these theorems and some of our other results.

Example 1.1. Let $\mathbf{F}$ be any field of characteristic $p = 3$. We consider $V := V_4$, the indecomposable four-dimensional representation of the cyclic group $G = \mathbf{Z}/9$ of order 9. The group G contains the subgroup L of order 3.

The first few homogeneous components of $\mathbf{F}[V]$ decompose into indecomposable $\mathbf{F}G$-modules as follows.

$$
\begin{array}{rlrrrrrrr}
\mathbf{F}[V]_0 & \cong V_1, & & & & & & & \\
\mathbf{F}[V]_1 & \cong V_4, & & & & & & & \\
\mathbf{F}[V]_2 & \cong V_7 & \oplus & V_3, & & & & & \\
\mathbf{F}[V]_3 & \cong V_2 & \oplus & V_3 & \oplus & V_6 & \oplus & V_9, & \\
\mathbf{F}[V]_4 & \cong V_5 & \oplus & 2V_3 & \oplus & V_6 & \oplus & 2V_9, & \\
\mathbf{F}[V]_5 & \cong V_8 & \oplus & 3V_3 & \oplus & 2V_6 & \oplus & 3V_9, & \\
\mathbf{F}[V]_6 & \cong & & 4V_3 & \oplus & 3V_6 & \oplus & 6V_9, & \\
\mathbf{F}[V]_7 & \cong & & 5V_3 & \oplus & 4V_6 & \oplus & 9V_9, & \\
\mathbf{F}[V]_8 & \cong & & 6V_3 & \oplus & 5V_6 & \oplus & 13V_9 & \\
\mathbf{F}[V]_9 & \cong V_1 & \oplus & 7V_3 & \oplus & 6V_6 & \oplus & 18V_9 & \\
\mathbf{F}[V]_{10} & \cong V_4 & \oplus & 8V_3 & \oplus & 7V_6 & \oplus & 24V_9 & \\
\mathbf{F}[V]_{11} & \cong V_7 & \oplus & 10V_3 & \oplus & 8V_6 & \oplus & 31V_9 & \\
\mathbf{F}[V]_{12} & \cong V_2 & \oplus & 11V_3 & \oplus & 10V_6 & \oplus & 40V_9, & \\
\mathbf{F}[V]_{13} & \cong V_5 & \oplus & 13V_3 & \oplus & 11V_6 & \oplus & 50V_9, & \\
\mathbf{F}[V]_{14} & \cong V_8 & \oplus & 15V_3 & \oplus & 13V_6 & \oplus & 61V_9, & \\
\mathbf{F}[V]_{15} & \cong & & 17V_3 & \oplus & 15V_6 & \oplus & 75V_9, & \\
\mathbf{F}[V]_{16} & \cong & & 19V_3 & \oplus & 17V_6 & \oplus & 90V_9, & \\
\mathbf{F}[V]_{17} & \cong & & 21V_3 & \oplus & 19V_6 & \oplus & 107V_9. &
\end{array}
$$

Theorem 1.1 asserts that $\mathbf{F}[V]^G$ is generated by

$$
\begin{aligned}
M &:= N^G(x_3) = x_3^3 - x_3x_2^2 + x_3^2x_1 + x_3x_2x_1, \\
N &:= N^G(x_4) = x_4^9 - x_4^3x_3^6 + \cdots + 2x_4x_3x_2x_1^6,
\end{aligned}
$$

and the image of the relative transfer, $\mathrm{Tr}_L^G(\mathbf{F}[x_4^3 - x_4x_1^2, x_3, x_2, x_1])$.

In fact, a Magma [3] computation shows that $\mathbf{F}[V]^G$ is minimally generated by M and N together with the following nine invariants.

$$\begin{aligned}
\mathrm{Tr}^G_L(x_3) &= x_1,\\
\mathrm{Tr}^G_L(-x_3^2) &= x_2^2 + x_1x_3 - x_1x_2 - x_1^2,\\
\mathrm{Tr}^G_L(-x_2x_3^2) &= x_2^3 - x_1x_2^2 - x_1^2x_3 + x_1^3,\\
\mathrm{Tr}^G_L(-x_3(x_4^3 - x_4x_1^2)) &= x_2x_3^3 + \cdots,\\
\mathrm{Tr}^G_L(x_3^2(x_4^3 - x_4x_1^2)) &= x_2x_3^4 + \cdots,\\
\mathrm{Tr}^G_L(x_2x_3^2(x_4^3 - x_4x_1^2)) &= x_2^2x_3^4 + \cdots,\\
\mathrm{Tr}^G_L(-x_3(x_4^3 - x_4x_1^2)^2) &= x_3^7 + \cdots,\\
\mathrm{Tr}^G_L(-x_3^2(x_4^3 - x_4x_1^2)^2) &= x_3^8 + \cdots,\\
\mathrm{Tr}^G_L(x_2x_3^2(x_4^3 - x_4x_1^2)^2) &= x_2x_3^8 + \cdots.
\end{aligned}$$

The (one dimensional) socle of the noninduced indecomposable summand in $\mathbf{F}[V]_i$ for $i = 0,1,2,\ldots,5$ may be chosen to contain 1, x_1, x_1^2, M, x_1M, and x_1^2M respectively. For degrees $d \geq 9$, write $d = 9a + c$ where $0 \leq c \leq 8$. Then $\mathbf{F}[V]_d \cong \mathbf{F}[V]_c \oplus \alpha V_3 \oplus \beta V_6 \oplus \gamma V_9$ for some nonnegative integers α, β, and γ. Furthermore if $0 \leq c \leq 5$ and if we denote by f a nonzero element of the socle of the noninduced summand in $\mathbf{F}[V]_c$, then the noninduced summand in $\mathbf{F}[V]_d$ may be chosen such that its socle is spanned by $N^a f$. If we let $b_i(d)$ denote the number of copies of the summand V_{3i} in $\mathbf{F}[V]_d$ and define $B_i(t) := \sum_{d=0}^{\infty} b_i(d)t^d$ then the results of Section 6 show that

$$B_i(t) = \frac{t^{5+i}}{1-t^9} + \frac{-t^i}{1-t^3} + \frac{t^{i-1}-t^i}{3(1-t^3)}\left(\frac{1}{(1-t)^3} - \frac{1-4t^9+3t^{12}}{(1-t^3)(1-t^9)}\right)$$

for $i = 1,2,3$.

Corollary 3.2 shows that the Hilbert Series for $\mathbf{F}[V_4]^{\mathbf{Z}/9}$ is given by

$$\mathscr{H}(\mathbf{F}[V_4]^{\mathbf{Z}/9}, t) = \frac{1}{3(1-t^3)}\left(\frac{1}{(1-t)^3} + \frac{2-3t^3+3t^9}{(1-t^3)(1-t^9)}\right).$$

In Section 2 we develop tools for decomposing $\mathbf{F}[V_n]$ as an $\mathbf{F}G$-module. We then specialise to $r=2$ and $n=p+1$. In Section 3 we construct generators for $\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}$. We apply the "ladder technique" described in [15, §7], using group cohomology and a spectral sequence argument, to prove the following.

Theorem 1.1. *Suppose $G \cong \mathbf{Z}/p^2$ and let $L \cong \mathbf{Z}/p$ denote its nontrivial proper subgroup. The ring of invariants $\mathbf{F}[V_{p+1}]^G$ is generated by $N^G(x_p)$, $N^G(x_{p+1})$ and elements from the image of the relative transfer,* $\mathrm{Tr}^G_L\left(\mathbf{F}[N^L(x_{p+1}), x_p, \ldots, x_1]\right)$.

We also use the spectral sequence to derive a formula for the Hilbert series of the ring of invariants $\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}$.

Recall that the Noether number of a representation is the largest degree of an element in a minimal homogeneous generating set for the corresponding ring of invariants. In Section 4 we use the generating set given by Theorem 1.1 to show that, for $p > 2$, the Noether number for V_{p+1} is $p^2 + p - 3$. In Section 5 we use the constructed generating set to describe the $\mathbf{F}\mathbf{Z}/p^2$-module structure of $\mathbf{F}[V_{p+1}]$, confirming the Periodicity Conjecture of Hughes and Kemper [10, Conjecture 4.6] in this case and proving the following.

Theorem 1.2. *Let $G \cong \mathbf{Z}/p^2$ and let d be any nonnegative integer. In the decomposition of $\mathbf{F}[V_{p+1}]_d$ into a direct sum of indecomposable $\mathbf{F}G$-modules there is at most one indecomposable summand V_m which is not induced from a representation of a proper subgroup. In particular, writing $d = ap^2 + bp + c$ where $0 \le b, c < p$, there is exactly one noninduced indecomposable summand when $b \le p-2$ and $\mathbf{F}[V_{p+1}]_d$ is an induced module when $b = p-1$. Moreover, if $b \le p-2$ then the noninduced indecomposable summand is isomorphic to V_{cp+b+1} and we may choose the decomposition of $\mathbf{F}[V_{p+1}]_d$ such that the socle of this summand, V_{cp+b+1}^G, is spanned by the invariant $N^G(x_{p+1})^a N^G(x_p)^b x_1^c$.*

We note that Symonds, in a recent paper [16] based on his joint work with Karagueuzian [11], has proven the Periodicity Conjecture of Hughes and Kemper. He goes on to prove that for $p^{r-1} < n < p^r$ and $d < p^r$, the $\mathbf{F}\mathbf{Z}/p^r$-module $\mathbf{F}[V_n]_d$ is isomorphic to $\Omega^{-d}\Lambda^d(V_{p^r-n})$ modulo induced modules [16, Corollary 3.11]. Here Λ^d denotes the dth exterior power and Ω^{-d} denotes the dth cokernel of a minimal injective resolution (see [2, page 30]). It is instructive to compare this with Theorem 1.2 and Example 1.1.

In Section 6 we give a second computation of the Hilbert series of $\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}$. We also compute generating functions encoding the number of summands of each isomorphism type in $\mathbf{F}[V_{p+1}]_d$.

2 Preliminaries

Let $G = \langle\sigma\rangle \cong \mathbf{Z}/p^r$. It is convenient to define $\Delta := \sigma - 1 \in \mathbf{F}G$. It is easy to see that Δ acts as a twisted derivation on $\mathbf{F}[V_n]$, i.e., $\Delta(a \cdot b) = a\Delta(b) + \Delta(a)\sigma(b)$. We denote the full transfer, $\mathrm{Tr}^G_{\langle 1\rangle}$, by Tr^G and the image of the relative transfer, $\mathrm{Tr}^G_L(\mathbf{F}[V_n]^L)$, by $\mathrm{Im}\,\mathrm{Tr}^G_L$, for any subgroup L. Clearly $\mathrm{Im}\,\mathrm{Tr}^G_L$ is an ideal in $\mathbf{F}[V_n]^G$. A simple calculation with binomial coefficients shows that $\Delta^{p^t} = \sigma^{p^t} - 1$ and $\Delta^{p^t-1} = (\sigma^{p^t} - 1)/(\sigma - 1) = \mathrm{Tr}^G_L$. We denote the group cohomology of G with coefficients in the $\mathbf{F}G$-module W by $H^*(G,W)$. Note that $H^0(G,W)$ is just the fixed subspace W^G. Furthermore, since G is cyclic, $H^{2i-1}(G,W) = \ker(\mathrm{Tr}^G|_W)/\mathrm{Im}(\Delta|_W)$ and $H^{2i}(G,W) = \ker(\Delta|_W)/\mathrm{Im}(\mathrm{Tr}^G|_W)$ for $i > 0$ (see, e.g., [6, §2.1]). It is clear from the definition of group cohomology that $H^i(G,P) = 0$ if $i > 0$ and P is projective. Thus $H^1(G,V_{p^r}) = H^2(G,V_{p^r}) = 0$. Furthermore, if V_m is generated as an $\mathbf{F}G$-module by e, then $H^0(G,V_m) = V_m^G = \mathrm{span}_{\mathbf{F}}(\Delta^{m-1}(e))$ and $\{e, \Delta(e), \ldots, \Delta^{m-1}(e)\}$ is a vector space basis for V_m. If we identify V_{m-1} with the submodule $\Delta(V_m)$, then, for $m < p^r$, $H^1(G,V_m)$ is the one-dimensional vector space V_m/V_{m-1} and $H^2(G,V_m)$ is the one dimensional vector space V_m^G.

Let W be any finite dimensional $\mathbf{F}G$-module. Define $\mathscr{L}_t(W) := \Delta^{t-1}(W)$. Clearly $\mathscr{L}_{i+1}(W) \subseteq \mathscr{L}_i(W)$. Furthermore, since σ has order p^r, $\mathscr{L}_{p^r+1}(W) = 0$. Thus we have the following filtration of W by $\mathbf{F}G$-modules,

$$W = \mathscr{L}_1(W) \supseteq \mathscr{L}_2(W) \supseteq \mathscr{L}_3(W) \supseteq \cdots \supseteq \mathscr{L}_{p^r}(W) \supseteq \mathscr{L}_{p^r+1}(W) = 0.$$

This filtration of W obviously induces a filtration of the subspace W^G:

$$W^G = \mathscr{L}_1^G(W) \supseteq \mathscr{L}_2^G(W) \supseteq \mathscr{L}_3^G(W) \supseteq \cdots \supseteq \mathscr{L}_{p^r}^G(W) \supseteq \mathscr{L}_{p^r+1}^G(W) = 0,$$

where $\mathscr{L}_t^G(W) := \mathscr{L}_t(W) \cap W^G$.

Definition 2.1. For a nonzero $f \in W$, we define the *length* of f, denoted by $\ell(f)$, by $\ell(f) \geq t \iff f \in \mathscr{L}_t(W)$. Note that $1 \leq \ell(f) \leq p^r$. We refer to the above filtration of W as the *length filtration* and say that a basis $\mathscr{B}$ for W^G is *compatible* with the length filtration if $\mathscr{L}_t^G(W) \cap \mathscr{B}$ is a basis for $\mathscr{L}_t^G(W)$ for all t (using the convention that the empty set is a basis for the zero vector space).

Lemma 2.1. *If W is a finite-dimensional $\mathbf{F}G$-module, then*

$$\dim(W) = \sum_{t=1}^{p^r} t\left(\dim(\mathscr{L}_t^G(W)) - \dim(\mathscr{L}_{t+1}^G(W))\right).$$

Proof. Choose a decomposition of W into indecomposable $\mathbf{F}G$-modules. For each indecomposable summand, choose a basis in which σ is in Jordan canonical form. The union of these bases gives a basis for W. Intersecting this basis with W^G gives a basis for W^G, say $\mathscr{B}$, which is compatible with the length filtration. It is clear that the number of elements in $\mathscr{B} \cap (\mathscr{L}_t^G(W) \setminus \mathscr{L}_{t+1}^G(W))$ coincides with the number of indecomposable modules in the decomposition which are isomorphic to V_t, giving the required formula.

Suppose that W is a finite-dimensional $\mathbf{F}G$-module and $\mathscr{B}$ is a basis for W^G which is compatible with the length filtration. For each $\alpha \in \mathscr{B}$ choose $\gamma \in W$ with $\Delta^{\ell(\alpha)-1}(\gamma) = \alpha$. (The existence of a suitable γ follows from the definition of length.) Define $V(\alpha)$ to be the $\mathbf{F}G$-module generated by γ. Note that α spans the socle of $V(\alpha)$ and that $\dim(V(\alpha)) = \ell(\alpha)$.

Proposition 2.1.

$$W = \bigoplus_{\alpha \in \mathscr{B}} V(\alpha).$$

Proof. The natural homomorphism of the external direct sum of the $V(\alpha)$ to W is injective on the socle and is therefore injective. Thus the internal sum of the $V(\alpha)$ is direct. It follows from Lemma 2.1 that the dimension of W coincides with the dimension of $\oplus_{\alpha \in \mathscr{B}} V(\alpha)$, giving equality.

The above shows how we may obtain a direct sum decomposition of W into indecomposable submodules from any basis of W^G which is compatible with the length filtration of W^G. Clearly every such decomposition arises in this way. Furthermore, an element $f \in W^G$ has length t if and only if there is an $\mathbf{F}G$ decomposition $W = W' \oplus V_t$ with f spanning V_t^G.

Note that if $f,h \in \mathbf{F}[V]^G$ then $\ell(fh) \geq \ell(f)$. To see this write $f = \Delta^{\ell(f)-1}(F)$. Then $fh = \Delta^{\ell(f)-1}(Fh)$. In general it may happen that $\ell(fh) > \max\{\ell(f), \ell(h)\}$. Computer computations together with various results, such as Proposition 5.2, lead us to make the following conjecture.

Conjecture 2.1. Suppose $f,h \in \mathbf{F}[V]^G$ with $\ell(f) \equiv 0 \pmod p$. Then $\ell(fh) \equiv 0 \pmod p$.

For $n \leq p^r$, choose an $\mathbf{F}G$-module generator x_n for V_n^* and define $x_i = \Delta^{n-i}(x_n)$ for $i = 1, \ldots, n-1$. Then $\{x_1, x_2, \ldots, x_n\}$ is a basis of V_n^*. Let $\bar{\mathbf{F}}$ denote the algebraic closure of $\mathbf{F}$ and define $\overline{V_n} := \bar{\mathbf{F}} \otimes_{\mathbf{F}} V_n$. Let $\{e_1, e_2, e_3, \ldots, e_n\}$ denote the basis for $\overline{V_n}$ dual to $\{1 \otimes x_1, \ldots, 1 \otimes x_n\}$. Note that e_1 generates $\overline{V_n}$ as an $\bar{\mathbf{F}}G$-module and that $\Delta(e_n) = 0$. Using the inclusion $\mathbf{F} \subseteq \bar{\mathbf{F}}$, allows us to interpret elements of $\mathbf{F}[V_n]$ as regular functions on $\overline{V_n}$, i.e., we identify $\mathbf{F}[V_n]$ in a natural way with a subset of $\bar{\mathbf{F}}[\overline{V_n}]$. For a subset $X \subseteq \bar{\mathbf{F}}[\overline{V_n}]$, define $\mathscr{V}(X) = \{v \in \overline{V_n} \mid f(v) = 0 \; \forall f \in X\}$.

Lemma 2.2. *Suppose $p^{r-1} < n \leq p^r$ and let H denote the subgroup $\langle \sigma^{p^{t+1}} \rangle \cong \mathbf{Z}/p^{r-t-1}$ of $G = \langle \sigma \rangle \cong \mathbf{Z}/p^r$, where $0 \leq t \leq r-1$.*

(a) $\mathscr{V}(\operatorname{Im}\operatorname{Tr}_H^G) = \overline{V_n}^{\mathbf{Z}/p^{r-t}} = \operatorname{span}_{\bar{\mathbf{F}}}\{e_{n-p^t+1}, e_{n-p^t+2}, \ldots, e_{n-1}, e_n\}$.
(b) For $f \in \mathbf{F}[V_n]^G$, if $\ell(f) \geq p^t + 1$ then

$$f \in \sqrt{\operatorname{Im}\operatorname{Tr}_H^G} = \left((x_1, x_2, \ldots, x_{n-p^t})\mathbf{F}[V_n]\right) \cap \mathbf{F}[V_n]^G.$$

Proof. The equality $V_n^{\mathbf{Z}/p^{r-t}} = \operatorname{span}_{\mathbf{F}}\{e_{n-p^t+1}, e_{n-p^t+2}, \ldots, e_{n-1}, e_n\}$ is easily verified. The equality $\mathscr{V}(\operatorname{Im}\operatorname{Tr}_H^G) = \overline{V_n}^{\mathbf{Z}/p^{r-t}}$ follows from [8, Proposition 12.5] (see also [5, Theorem 12]). This equality of sets may be expressed equivalently as the equality of ideals $\sqrt{\operatorname{Im}\operatorname{Tr}_H^G} = \left((x_1, x_2, \ldots, x_{n-p^t})\mathbf{F}[V_n]\right) \cap \mathbf{F}[V_n]^G$ (see, e.g., [5, Proposition 11]). Thus it only remains to show that if $\ell(f) \geq p^t + 1$ then $f \in \sqrt{\operatorname{Im}\operatorname{Tr}_H^G}$.

To see this suppose that $\ell(f) \geq p^t + 1$. Then $f = \Delta^{p^t}(F)$ for some $F \in \mathbf{F}[V_n]$. Therefore $f(v) = (\Delta^{p^t} F)(v) = ((\sigma - 1)^{p^t} F)(v) = (\sigma^{p^t}(F) - F)(v) = F(\sigma^{-p^t}(v)) - F(v)$. Thus $f(v) = 0$ if v is fixed by σ^{p^t}, i.e., if $v \in \overline{V_n}^{\mathbf{Z}/p^{r-t}}$. Hence if $\ell(f) \geq p^t + 1$ then $f \in \sqrt{\operatorname{Im}\operatorname{Tr}_H^G}$.

Proposition 2.2. *Suppose that f is a non-zero homogeneous element of $\mathbf{F}[V_n]^G$. Then $\ell(fN^G(x_n)) = \ell(f)$.*

Proof. Denote $N^G(x_n)$ by N. Define t such that $p^{t-1} < n \leq p^t$ (the case $n = 1$ is trivial). Then the leading term of N is $x_n^{p^t}$. Let $\mathbf{F}[V_n]^\flat$ denote the span of the monomials in $\mathbf{F}[V_n]$ which, as polynomials in x_n, have degree less than p^t. The fact that

$x_n \notin \Delta(\mathbf{F}[V_n])$ means that $\mathbf{F}[V_n]^\flat$ is an $\mathbf{F}G$-submodule of $\mathbf{F}[V_n]$. For an arbitrary polynomial $h \in \mathbf{F}[V_n]$, viewing h as a polynomial in x_n and dividing by $N^G(x_n)$ gives $h = qN + r$ for unique $r \in \mathbf{F}[V_n]^\flat$ and $q \in \mathbf{F}[V_n]$. This gives the $\mathbf{F}G$-module decomposition $\mathbf{F}[V_n] = N\mathbf{F}[V_n] \oplus \mathbf{F}[V_n]^\flat$ (compare with [10, Lemma 2.9] and [14, § 2]). As noted above $\ell(Nf) \geq \ell(f)$. Suppose $Nf = \Delta^s(F)$ and write $F = NF_1 + F_0$ with $F_0 \in \mathbf{F}[V_n]^\flat$. Then $Nf = \Delta^s(NF_1 + F_0) = N\Delta^s(F_1) + \Delta^s(F_0)$ with $\Delta^s(F_0) \in \mathbf{F}[V_n]^\flat$. Since the division algorithm produces a unique quotient and remainder, we have $f = \Delta^s(F_1)$. This shows that $\ell(f) \geq \ell(Nf)$.

3 Computing $\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}$

In this section we use the ladder technique described in [15, §7] to prove Theorem 1.1. We use the notation

$$G := \langle \sigma \rangle \cong \mathbf{Z}/p^2,\ L := \langle \sigma^p \rangle \cong \mathbf{Z}/p, \text{ and } Q := G/L = \langle \overline{\sigma} \rangle \cong \mathbf{Z}/p.$$

Note that $\deg\left(N^G(x_p)\right) = p$, $\deg\left(N^G(x_{p+1})\right) = p^2$, and $Tr_L^G(x_p) = x_1$.

The action of L on V_{p+1}^* is given by $\sigma^p(x_{p+1}) = x_{p+1} + x_1$ and $\sigma^p(x_i) = x_i$ for $i \leq p$. Thus as L-modules, $\mathbf{F}[V_{p+1}] \cong \mathbf{F}[V_2 \oplus (p-1)V_1]$. Therefore $\mathbf{F}[V_{p+1}]^L \cong \mathbf{F}[N^L(x_{p+1}), x_p, \ldots, x_1]$ with $N^L(x_{p+1}) = x_{p+1}^p - x_1^{p-1}x_{p+1}$. The action of Q on $\mathbf{F}[V_{p+1}]^L$ is given by $\overline{\sigma}(N^L(x_{p+1})) = N^L(x_{p+1}) + x_p^p - x_1^{p-1}x_p$ and $\overline{\sigma}(x_i) = \sigma(x_i)$ for $i = 1, 2, \ldots, p$. Define

$$A := \mathbf{F}[z_p, \ldots, z_1, X_p, \ldots, X_1]$$

with $\deg(z_i) = p$ and $\deg(X_i) = 1$. Further define an algebra homomorphism $\pi : A \to \mathbf{F}[V_{p+1}]^L$ by $\pi(z_i) = x_{i+1}^p - x_1^{p-1}x_{i+1}$ and $\pi(X_i) = x_i$. Note that π is a degree preserving surjection with $\pi(z_p) = N^L(x_{p+1})$. Further note that the kernel of π is the ideal

$$I := \left(z_{p-1} - \left(X_p^p - X_1^{p-1}X_p\right), \ldots, z_1 - \left(X_2^p - X_1^{p-1}X_2\right)\right)A.$$

Define an action, by algebra automorphisms, of Q on A by taking $\overline{\sigma}(z_i) = z_i + z_{i-1}$ and $\overline{\sigma}(X_i) = X_i + X_{i-1}$ for $i > 1$, $\overline{\sigma}(z_1) = z_1$, and $\overline{\sigma}(X_1) = X_1$. Thus as $\mathbf{F}Q$-modules, $A \cong \mathbf{F}[2V_p]$ and π is a map of $\mathbf{F}Q$-modules.

The short exact sequence of $\mathbf{F}Q$-modules, $0 \to I \to A \xrightarrow{\pi} \mathbf{F}[V_{p+1}]^L \to 0$, gives a long exact sequence on group cohomology

$$0 \to I^Q \to A^Q \to \left(\mathbf{F}[V_{p+1}]^L\right)^Q \to H^1(Q,I) \to H^1(Q,A) \to \cdots.$$

We show that the inclusion of I into A induces an injection of $H^1(Q,I)$ into $H^1(Q,A)$. Thus π restricts to a surjection from A^Q to $\left(\mathbf{F}[V_{p+1}]^L\right)^Q = \mathbf{F}[V_{p+1}]^G$. Since $2V_p$ is a permutation representation of Q, after a suitable change of basis, Q acts on A by permuting the variables. Using the permutation basis, the orbit sums of monomials form a vector space basis for A^Q. Since $Q \cong \mathbf{Z}/p$, these orbits are of size p or size 1. The orbits of size p span projective $\mathbf{F}Q$-module summands of $\mathbf{F}[V_{p+1}]^L$ and the orbits of size 1 span trivial summands. It is easy to see that, in the original basis, the orbits of size 1 are polynomials in $N^Q(z_p)$ and $N^Q(X_p)$, whereas the orbit sums coming from orbits of size p are elements in the image of the transfer. Thus A^Q is generated by $N^Q(z_p)$, $N^Q(X_p)$ and elements from $\mathrm{Tr}^Q(A)$, giving Theorem 1.1.

We now prove a number of results with the goal of completing the proof of Theorem 1.1 by showing that the inclusion of I into A induces an injection of $H^1(Q,I)$ into $H^1(Q,A)$ (see Theorem 3.2 (b)). We start by describing $H^*(Q,A)$.

Proposition 3.1. *(a)* $H^2(Q,A) = A^Q/\,\mathrm{Tr}^Q(A) \cong \mathbf{F}[N^Q(X_p), N^Q(z_p)]$.
(b) $H^1(Q,A)$ *is a principal* A^Q*-module with annihilator given by* $\mathrm{Tr}^Q(A)$.

Proof. It follows from the discussion above that, as an $\mathbf{F}Q$-module, A consists of projective summands and trivial summands with the trivial summands spanned by the monomials in $N^Q(X_p)$ and $N^Q(z_p)$. The projective summands do not contribute to the cohomology. The trivial summands contribute nonzero classes to both the first and second cohomology.

Remark 3.1. Since $A^Q/\,\mathrm{Tr}^Q(A)$ is a domain, $\mathrm{Tr}^Q(A)$ is a prime ideal. Also, although $H^1(Q,A)$ does not have a multiplicative structure, it is isomorphic to $\mathbf{F}[N^Q(X_p), N^Q(z_p)]$ as an A^Q-module.

To compute $H^*(Q,I)$, we start by resolving I as an $A-\mathbf{F}Q$-module using a Koszul resolution (see, e.g., [4, §1.6]). Observe that I is generated by an A-regular sequence of length $p-1$. Furthermore, these generators span the degree p homogeneous component, I_p, of I and, as an $\mathbf{F}Q$-module, $I_p \cong V_{p-1}$. Let μ denote the Q-equivariant map from $V_{p-1} \otimes A$ to A given by identifying elements of V_{p-1} with elements of I_p and then using the multiplication in A. Let $\Lambda^i(V_{p-1})$ denote the ith exterior power of V_{p-1}. Define $\zeta^i : \Lambda^i(V_{p-1}) \to \Lambda^{i-1}(V_{p-1}) \otimes V_{p-1}$ by

$$\zeta^i(v_1 \wedge v_2 \wedge \cdots \wedge v_i) = \sum_{j=1}^{i} (-1)^{i-j} (v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_i) \otimes v_j$$

for all $v_1, v_2, \ldots, v_i \in V_{p-1}$. Define $F^{-i} := \Lambda^i(V_{p-1}) \otimes A$ for $i = 0,1,2,\ldots,p-1$ and define $\rho^{-i} : F^{-i} \to F^{-i+1}$ to be $(1_{\Lambda^{i-1}(V_{p-1})} \otimes \mu) \circ (\zeta^i \otimes 1_A)$. This gives the following sequence of $A-\mathbf{F}Q$-modules,

$$0 \to F^{1-p} \xrightarrow{\rho^{1-p}} F^{2-p} \xrightarrow{\rho^{2-p}} \cdots \xrightarrow{\rho^{-3}} F^{-2} \xrightarrow{\rho^{-2}} F^{-1} \xrightarrow{\mu} I \to 0.$$

Since the generators of I form a regular A-sequence, it follows from [4, Corollary 1.6.14] that this sequence is exact. For $i > 0$, define K^{-i} to be the kernel of the map $\rho^{-i} : F^{-i} \to F^{-i+1}$. For convenience, we define $K^0 := I$, $K^1 := A/I$, $K^a := 0$ for $a > 1$, $F^1 := A/I$ and $F^a := 0$ for $a > 1$. Using the exactness of the resolution, we get a series of short exact sequences $0 \to K^{-i} \to F^{-i} \to K^{-i+1} \to 0$.

For each of these short exact sequences, we apply group cohomology to get a long exact sequence:

$$\begin{aligned}0 \to H^0(Q,K^{-i}) \to H^0(Q,F^{-i}) \to H^0(Q,K^{-i+1}) \to H^1(Q,K^{-i})\\ \to H^1(Q,F^{-i}) \to H^1(Q,K^{-i+1}) \to \cdots .\end{aligned}$$

Defining $D^{a,b} := H^b(Q,K^a)$ and $E^{a,b} := H^b(Q,F^a)$ gives a bigraded exact couple which leads to a spectral sequence. This is essentially the construction given at the end of [12, Chapter XI, §5]. We will use this spectral sequence to describe $H^1(Q,I)$. The following series of lemmas lead to a description of $H^b(Q,F^a)$.

Lemma 3.1. *The map ζ^{i+1} is an isomorphism of* $\mathbf{F}Q$*-modules from $\Lambda^{i+1}(V_{p-1})$ to a direct summand of $\Lambda^i(V_{p-1}) \otimes V_{p-1}$.*

Proof. It is clear that ζ^{i+1} followed by the natural projection from $\Lambda^i(V_{p-1}) \otimes V_{p-1}$ to $\Lambda^{i+1}(V_{p-1})$ is $i+1$ times the identity map on $\Lambda^{i+1}(V_{p-1})$. Since $i+1 \leq p-1$, this is an isomorphism.

Lemma 3.2. $V_{p-1} \otimes V_{p-1} \cong V_1 \oplus (p-2)V_p$.

Proof. If the representation ring $R_{\mathbf{FZ}/p}$ is extended by adjoining an element α satisfying $V_2 = \alpha + \alpha^{-1}$, then by [10, Lemma 2.3], $V_n = (\alpha^n - \alpha^{-n})/(\alpha - \alpha^{-1})$. Thus, in the augmented representation ring,

$$\begin{aligned}V_{p-1}^2 - V_p \cdot V_{p-2} &= \frac{(\alpha^{p-1} - \alpha^{-(p-1)})^2 - (\alpha^p - \alpha^{-p})(\alpha^{p-2} - \alpha^{-(p-2)})}{(\alpha - \alpha^{-1})^2}\\ &= \frac{\alpha^{2p-2} - 2 + \alpha^{-(2p-2)} - (\alpha^{2p-2} - \alpha^2 - \alpha^{-2} + \alpha^{-(2p-2)})}{\alpha^2 - 2 + \alpha^{-2}}\\ &= 1 = V_1.\end{aligned}$$

Therefore, $V_{p-1} \otimes V_{p-1} \cong V_p \otimes V_{p-2} \oplus V_1$. It follows from [1, Chapter II §7 Lemma 4] that $V_p \otimes V_i \cong iV_p$. Thus $V_{p-1} \otimes V_{p-1} \cong (p-2)V_p \oplus V_1$, as required.

Lemma 3.3. *For i even,*

$$\Lambda^i(V_{p-1}) \cong V_1 \oplus \frac{1}{p}\left(\binom{p-1}{i} - 1\right)V_p$$

and for i odd,

$$\Lambda^i(V_{p-1}) \cong V_{p-1} \oplus \frac{1}{p}\left(\binom{p-1}{i} - (p-1)\right)V_p.$$

Proof. First observe that $\dim_{\mathbf{F}}(\Lambda^i(V_{p-1})) = \binom{p-1}{i} \equiv (-1)^i \pmod{p}$. Therefore the dimensions are correct. Thus it follows from Lemma 3.1 that $\Lambda^i(V_{p-1})$ is a non-projective summand of $\Lambda^{i-1}(V_{p-1}) \otimes V_{p-1}$. Also note that the result is true for $i = 0$

and $i = 1$. We proceed by induction. Suppose the result holds for i. For i even this gives,

$$\begin{aligned}\Lambda^i(V_{p-1}) \otimes V_{p-1} &\cong V_1 \otimes V_{p-1} \oplus \frac{1}{p}\left(\binom{p-1}{i} - 1\right) V_p \otimes V_{p-1}\\ &\cong V_{p-1} \oplus \frac{p-1}{p}\left(\binom{p-1}{i} - 1\right) V_p\end{aligned}$$

and therefore, $\Lambda^i(V_{p-1}) \otimes V_{p-1}$ is isomorphic to V_{p-1} plus projective modules. Hence $\Lambda^{i+1}(V_{p-1})$ is isomorphic to V_{p-1} plus projective modules. For i odd,

$$\begin{aligned}\Lambda^i(V_{p-1}) \otimes V_{p-1} &\cong V_{p-1} \otimes V_{p-1} \oplus \frac{1}{p}\left(\binom{p-1}{i} - (p-1)\right) V_p \otimes V_{p-1}\\ &\cong V_{p-1} \otimes V_{p-1} \oplus \frac{p-1}{p}\left(\binom{p-1}{i} - (p-1)\right) V_p\end{aligned}$$

and, therefore, $\Lambda^i(V_{p-1}) \otimes V_{p-1}$ is isomorphic to $V_{p-1} \otimes V_{p-1}$ plus projective modules. From Lemma 3.2, $V_{p-1} \otimes V_{p-1}$ is isomorphic to V_1 plus projective modules. Hence $\Lambda^{i+1}(V_{p-1})$ is isomorphic to V_1 plus projective modules.

Lemma 3.4. *For $a \leq 0$ and $b > 0$, $H^b(Q, F^a)$ is a principal A^Q-module with annihilator* $\mathrm{Tr}^Q(A)$.

Proof. As an $\mathbf{F}Q$-module, A is a direct sum of projective summands with socles contained in $\mathrm{Tr}^Q(A)$ and one-dimensional summands spanned by monomials in $N^Q(X_p)$ and $N^Q(z_p)$. It follows from Lemma 3.3 that $\Lambda^{-a}(V_{p-1})$ contains a single nonprojective summand. Note that projective summands do not contribute to the cohomology. Further note that for any module M and projective module P, $M \otimes P$ is projective. Thus $H^b(Q, \Lambda^{-a}(V_{p-1}))$ is a one dimensional vector space and $H^b(Q, F^a) \cong H^b(Q, \Lambda^{-a}(V_{p-1}) \otimes A) \cong H^b(Q, A)$. The result follows from Proposition 3.1.

The following lemma is a preliminary step in evaluating $d^{a,b} : E^{a,b} \to E^{a+1,b}$ for $a < 0$ and $b > 0$.

Lemma 3.5. *The inclusion of I_p into A_p induces an injection from $H^1(Q, I_p)$ to $H^1(Q, A_p)$ and the zero map from $H^2(Q, I_p)$ to $H^2(Q, A_p)$.*

Proof. To see that the inclusion induces an injection from $H^1(Q, I_p)$ to $H^1(Q, A_p)$, first note that Δ is a twisted derivation and that $\Delta(f)$, for f a generator of A, lies in $\mathrm{Span}_{\mathbf{F}}(z_{p-1}, \ldots, z_1, X_{p-1}, \ldots, X_1)$. Therefore $\Delta(A)$ is contained in the ideal $(z_{p-1}, \ldots, z_1, X_{p-1}, \ldots, X_1)A$. As an $\mathbf{F}Q$-module, I_p is isomorphic to V_{p-1} with generator $r := z_{p-1} - \left(X_p^p - X_1^{p-1} X_p\right)$. Thus $H^1(Q, I_p)$ is a one dimensional vector space with r representing a nonzero cohomology class. Since r does not lie in the ideal $(z_{p-1}, \ldots, z_1, X_{p-1}, \ldots, X_1)A$, this element does not lie in $\Delta(A)$ and, therefore, represents a nonzero class in $H^1(Q, A_p)$.

To see that inclusion induces the zero map from $H^2(Q,I_p)$ to $H^2(Q,A_p)$, observe that $I_p \cong V_{p-1}$ and A_p is isomorphic to V_1 plus projectives. Thus the map on cohomology is determined by a $\mathbf{Z}/p$-equivariant map from V_{p-1} to V_1, and all such maps induce the zero map in second cohomology.

For $a<0$, the first differential in the spectral sequence is the map on cohomology $d^{a,b} : H^b(Q,F^a) \to H^b(Q,F^{a+1})$ induced by $\rho^a : F^a \to F^{a+1}$

Theorem 3.1. *For $b>0$ and $a<0$, $d^{a,b} : H^b(Q,F^a) \to H^b(Q,F^{a+1})$ is a monomorphism if a and b have the same parity and zero if a and b have different parities.*

Proof. From Lemma 3.4, $H^b(Q,F^a)$ is a principal A^Q-module with annihilator $\mathrm{Tr}^Q(A)$. By Proposition 3.1, $\mathrm{Tr}^Q(A)$ is a prime ideal. Therefore, since $d^{a,b}$ is an A^Q-module map, it is sufficient to evaluate $d^{a,b}$ on a generator. Thus, using the definition of ρ^a, we see that $d^{a,b}$ is determined by the composition

$$\Lambda^{-a}(V_{p-1}) \xrightarrow{\zeta^{-a}} \Lambda^{-a-1}(V_{p-1}) \otimes V_{p-1} \xrightarrow{\cong} \Lambda^{-a-1}(V_{p-1}) \otimes I_p \xrightarrow{\subset} \Lambda^{-a-1}(V_{p-1}) \otimes A.$$

It follows from Lemmas 3.1 and 3.3 that ζ^{-a} induces an isomorphism in cohomology. Thus $d^{a,b}$ is determined by the inclusion of $\Lambda^{-a-1}(V_{p-1}) \otimes I_p$ into $\Lambda^{-a-1}(V_{p-1}) \otimes A$.

For a odd, using Lemma 3.3, $\Lambda^{-a-1}(V_{p-1})$ is isomorphic to V_1 plus projectives. Therefore, in this case, $d^{a,b}$ is induced by the inclusion of I_p into A_p. Thus, using Lemma 3.5, if a and b are both odd then $d^{a,b}$ is injective and if a is odd and b is even then $d^{a,b}$ is zero.

For a even, using Lemma 3.3, $\Lambda^{-a-1}(V_{p-1})$ is isomorphic to V_{p-1} plus projectives. Therefore, in this case, $d^{a,b}$ is induced by the inclusion of $V_{p-1} \otimes I_p$ into $V_{p-1} \otimes A_p \cong V_{p-1} \otimes V_{p-1}$. By Lemma 3.2, $V_{p-1} \otimes V_{p-1}$ is isomorphic to V_1 plus projectives. Since A_p is isomorphic to V_1 plus projectives, $V_{p-1} \otimes A_p$ is isomorphic to V_{p-1} plus projectives. Thus $d^{a,b}$ is determined by the composition

$$V_1 \to V_{p-1} \otimes I_p \to V_{p-1} \otimes A_p \to V_{p-1}.$$

This map clearly induces the zero map from $H^1(Q,V_1)$ to $H^1(Q,V_{p-1})$. Thus for a even and b odd, $d^{a,b}=0$. To show that $d^{a,b}$ is injective for a even and b even, we need to show that the given map from V_1 to V_{p-1} is nonzero. It follows from Lemma 3.5, that for the purposes of computing cohomology, the inclusion of I_p into A_p is the injection of V_{p-1} into $V_1 \oplus V_p$ taking e' to $(e'', \Delta(e))$ where e, e', and e'' denote elements which generate the cyclic G-modules V_p, V_{p-1}, and V_1, respectively. The cokernel of this map is isomorphic to V_2. Tensoring over $\mathbf{F}$ is exact so we have a short exact sequence

$$0 \to V_{p-1} \otimes V_{p-1} \to V_{p-1} \otimes (V_1 \oplus V_p) \to V_{p-1} \otimes V_2 \to 0.$$

This gives rise to a long exact sequence in cohomology. Recall that $V_{p-1} \otimes V_2 \cong V_{p-2} \oplus V_p$ (see, e.g., [1, Chapter II § 7 Lemma 5]). Thus, modulo projectives, the sequence is $V_1 \to V_{p-1} \to V_{p-2}$. This can only give a long exact sequence on cohomology if the map from V_1 to V_{p-1} is nonzero.

Corollary 3.1. *For $b > 0$ and $a < 0$, the spectral sequence satisfies*

$$E_2^{a,b} = \begin{cases} \mathbf{F} & \text{if } a = 1-p \text{ and } b \text{ odd;} \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Theorem 3.1 that ρ^{-1} induces an isomorphism from $H^1(Q,F^{-1})$ to $H^1(Q,A)$. This map factors through $H^1(Q,I)$ with the first map in the factorisation induced by μ and the second induced by inclusion. Thus to complete the proof of Theorem 1.1, it is sufficient to show the following.

Lemma 3.6. *The map μ induces an epimorphism from $H^1(Q,F^{-1})$ to $H^1(Q,I)$.*

Proof. Denote by $\partial^{a,b}$ the connecting homomorphism from $H^b(Q,K^a)$ to $H^{b+1}(Q,K^{a-1})$ and define a filtration on $H^1(Q,I) = H^1(Q,K^0)$ by

$$\mathscr{F}_t := \text{kernel}(\partial^{-t,t+1} \circ \partial^{-t+1,t} \circ \cdots \circ \partial^{-2,3} \circ \partial^{-1,2} \circ \partial^{0,1}).$$

Since $\partial^{1-p,p} = 0$, we have $\mathscr{F}_{p-1} = H^1(Q,I)$. Using the long exact sequence in cohomology coming from $0 \to K^{-1} \to F^{-1} \to K^0 \to 0$, we see that $\mathscr{F}_0$ is the image of $H^1(Q,F^{-1})$ in $H^1(Q,I)$. We prove the lemma by showing $\mathscr{F}_0 = \mathscr{F}_{p-1}$.

Using the definition of a derived couple (see, e.g., [12, Chapter XI, §5]), we have $D_{t+2}^{a-t-1,b+t+1} = \partial^{a-t,b+t} \circ \cdots \circ \partial^{a-1,b+1} \circ \partial^{a,b}(D^{a,b})$. If $x \in \mathscr{F}_t \setminus \mathscr{F}_{t-1}$ then $\partial^{-t+1,t} \circ \cdots \circ \partial^{0,1}(x)$ is a nonzero element of $D_{t+1}^{-t,t+1}$ which lifts to a nonzero element of $E_{t+1}^{-(t+1),t+1}$. However, it follows from Corollary 3.1 that $E_2^{-(t+1),t+1} = 0$ for $t \geq 1$. Thus $E_{t+1}^{-(t+1),t+1} = 0$ for $t \geq 1$. Therefore $\mathscr{F}_t = \mathscr{F}_0$ for all $t \geq 1$.

These calculations give the following.

Theorem 3.2. *(a) $H^1(Q,I)$ is a principal A^Q-module with generator represented by $z_{p-1} - \left(X_p^p - X_1^{p-1}X_p\right)$ and annihilator $\mathrm{Tr}^Q(A)$.*
(b) The inclusion of I into A induces an A^Q-module monomorphism of $H^1(Q,I)$ to $H^1(Q,A)$ taking $[z_{p-1} - \left(X_p^p - X_1^{p-1}X_p\right)]$ to $-[N^Q(X_p)]$.

This completes the proof of Theorem 1.1.

Using the spectral sequence we can now derive a formula for the Hilbert series of $\mathbf{F}[V_{p+1}]^G$. In Section 6 below, we give a different derivation of this formula.

Corollary 3.2. *The Hilbert series of $\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}$ is*

$$\mathscr{H}(\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2},t) = \frac{1}{p(1-t^p)(1-t)^p} + \frac{p-1-pt^p+t^{p^2}}{p(1-t^p)^2(1-t^{p^2})}.$$

Proof. From Theorem 3.2(b), we have a short exact sequence

$$0 \to I^Q \to A^Q \to \mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2} \to 0$$

which gives

$$\mathscr{H}(\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}) = \mathscr{H}(A^Q) - \mathscr{H}(I^Q). \tag{1}$$

Using Theorem 3.1, the long exact sequences used to define the exact couple break into segments. For $a < 0$, the first segment is the exact sequence

$$0 \to D^{a,0} \to E^{a,0} \to D^{a+1,0} \to D^{a,1} \to \cdots \to D^{a,-a} \to 0$$

resulting in a relation,

$$\sum_{i=0}^{-a-1} (-1)^i \mathscr{H}(E^{a,i}) = \sum_{i=0}^{-a} (-1)^i \mathscr{H}(D^{a,i}) + \sum_{i=0}^{-a-1} (-1)^i \mathscr{H}(D^{a+1,i}).$$

Taking the alternating sum gives

$$\sum_{j=1}^{p-1} \sum_{i=0}^{j-1} (-1)^{i+j} \mathscr{H}(E^{-j,i}) = \sum_{j=1}^{p-1} \left(\sum_{i=0}^{j} (-1)^{i+j} \mathscr{H}(D^{-j,i}) + \sum_{i=0}^{j-1} (-1)^{i+j} \mathscr{H}(D^{-j+1,i}) \right).$$

Cancelling terms results in

$$\sum_{j=1}^{p-1} \sum_{i=0}^{j-1} (-1)^{i+j} \mathscr{H}(E^{-j,i}) = -\mathscr{H}(D^{0,0}) + \sum_{i=1}^{p-1} (-1)^{p-1+i} \mathscr{H}(D^{1-p,i}).$$

However, $D^{1-p,i} = 0$ and $D^{0,0} = I^Q$. So substituting into Equation 1 gives

$$\mathscr{H}(\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}) = \mathscr{H}(A^Q) + \sum_{j=1}^{p-1} \sum_{i=0}^{j-1} (-1)^{i+j} \mathscr{H}(E^{-j,i}). \tag{2}$$

Theorem 3.4 of [10] can be used to compute

$$\mathscr{H}(A^Q) = \frac{1}{p} \left(\frac{1}{(1-t)^p (1-t^p)^p} + \frac{p-1}{(1-t^p)(1-t^{p^2})} \right).$$

The Hilbert series of A^Q decomposes into a contribution from the socles of summands isomorphic to V_1, say $\mathscr{H}_{V_1}$, and a contribution from the socles of summands isomorphic to V_p, say $\mathscr{H}_{V_p}$. Furthermore, it is easy to see that $\mathscr{H}_{V_1} = \left((1-t^p)(1-t^{p^2})\right)^{-1}$, giving

$$\mathscr{H}_{V_p} = \mathscr{H}(A^Q) - \mathscr{H}_{V_1} = \frac{1}{p} \left(\frac{1}{(1-t)^p (1-t^p)^p} - \frac{1}{(1-t^p)(1-t^{p^2})} \right).$$

Using Lemma 3.4, we see that for $b > 0$, $\mathscr{H}(E^{a,b}) = t^{-ap}\mathscr{H}_{V_1}$. To compute the Hilbert series for $E^{a,0} = (\Lambda^{-a}(I_p)\otimes A)^Q$ we use Lemma 3.3 and elementary decomposition formulae to identify the dimensions of the socles. For a even this gives

$$\mathscr{H}(E^{a,0}) = t^{-pa}\left(\frac{\mathscr{H}_{V_1}}{p}\left(p-1+\binom{p-1}{-a}\right)+\mathscr{H}_{V_p}\binom{p-1}{-a}\right)$$

and for a odd

$$\mathscr{H}(E^{a,0}) = t^{-pa}\left(\frac{\mathscr{H}_{V_1}}{p}\left(1+\binom{p-1}{-a}\right)+\mathscr{H}_{V_p}\binom{p-1}{-a}\right).$$

Substituting in Equation 2 gives

$$\begin{aligned}\mathscr{H}(\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}) &= \left(\frac{\mathscr{H}_{V_1}}{p}+\mathscr{H}_{V_p}\right)\sum_{j=0}^{p-1}\binom{p-1}{j}(-t^p)^j+\mathscr{H}_{V_1}\left(1-\frac{1}{p}\sum_{j=1}^{p-1}t^{pj}\right)\\ &= \left(\frac{\mathscr{H}_{V_1}}{p}+\mathscr{H}_{V_p}\right)(1-t^p)^{p-1}+\mathscr{H}_{V_1}\left(1-\frac{1-t^{p^2}}{p(1-t^p)}\right).\end{aligned}$$

Substituting for $\mathscr{H}_{V_1}$ and $\mathscr{H}_{V_p}$, and simplifying, gives

$$\mathscr{H}(\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}) = \frac{1}{p(1-t^p)(1-t)^p}+\frac{p-1-pt^p+t^{p^2}}{p(1-t^p)^2(1-t^{p^2})}.$$

4 The Noether number of V_{p+1}

In this section we use the description of $\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}$ given in Theorem 1.1 to prove the following.

Theorem 4.1. *For $p > 2$, the Noether number of V_{p+1} is p^2+p-3.*

Remark 4.1. A Magma [3] calculation shows that for $p = 2$, the Noether number of V_3 is $p^2 = 4$.

For the remainder of this section we assume that $p \geq 3$. We continue to use the notation described at the beginning of Section 3. Define $M := N^G(x_p)$ and $N := N^G(x_{p+1})$. The theorem is an immediate consequence of the following two lemmas.

Lemma 4.1. *The Noether number of V_{p+1} is less than or equal to p^2+p-3.*

Proof. Let $\mathscr{H}$ denote the ideal in $\mathbf{F}[V_{p+1}]^L$ generated by the homogeneous G-invariants of positive degree, that is, $\mathscr{H} = \mathbf{F}[V_{p+1}]^G_+\cdot\mathbf{F}[V_{p+1}]^L$. Thus $\mathbf{F}[V_{p+1}]^L/\mathscr{H}$ is a finite-dimensional graded algebra, the ring of relative coinvariants. Let $\mathscr{B}$

denote the set of elements of $\mathbf{F}[V_{p+1}]^L$ of the form $\gamma \cdot x_p^j \cdot N^L(x_{p+1})^k$, with γ a monomial in $\{x_1,\ldots,x_{p-1}\}$ of degree at most $p-2$ and $j,k < p$. The methods of Section 3 of [9] show that $\mathscr{B}$ projects to a spanning set in $\mathbf{F}[V_{p+1}]^L/\mathscr{H}$. Therefore $(p-1)p+(p-1)+p-2 = p^2+p-3$ is an upper bound on the top degree of the relative coinvariants and $\mathrm{Tr}_L^G(\mathscr{B})$ is a generating set for the ideal $\mathrm{Im}\,\mathrm{Tr}_L^G$. By Theorem 1.1, $\mathbf{F}[V_{p+1}]^G$ is generated by N, M and elements from $\mathrm{Im}\,\mathrm{Tr}_L^G$. Thus p^2+p-3 is an upper bound for the Noether number.

Lemma 4.2. *The polynomial* $\mathrm{Tr}_L^G\left(\left(N^L(x_{p+1})x_p\right)^{p-1}x_{p-1}^{p-2}\right)$ *is indecomposable in* $\mathbf{F}[V_{p+1}]^G$. *In particular the Noether number of* V_{p+1} *is at least* p^2+p-3.

Proof. Define $w := N^L(x_{p+1})$ and $z := \mathrm{Tr}_L^G\left(w^{p-1}x_p^{p-1}x_{p-1}^{p-2}\right)$. Suppose, by way of contradiction, that $z = f_1h_1+\cdots+f_sh_s$ where f_i and h_i are homogeneous positive degree elements of $\mathbf{F}[V_{p+1}]^G$. The degree of z as a polynomial in x_{p+1} is less than p^2. Thus N does not appear in the decomposition.

We use the graded reverse lexicographic term order with $x_1 < x_2 < \cdots < x_{p+1}$ and denote the leading monomial of an element $f \in \mathbf{F}[V_{p+1}]$ by $\mathrm{LM}(f)$. It is easy to see that $\mathrm{LM}(M) = x_p^p$. An elementary calculation gives $\mathrm{LM}(z) = x_p^{p^2-1}x_{p-1}^{p-2}$. By relabelling if necessary, we may assume $\mathrm{LM}(f_ih_i) \geq \mathrm{LM}(f_{i+1}h_{i+1})$. Thus, either $\mathrm{LM}(f_1h_1) = \mathrm{LM}(z)$ or $\mathrm{LM}(f_1h_1) = \mathrm{LM}(f_2h_2) > \mathrm{LM}(z)$. Without loss of generality, we may assume $f_1h_1 = cM^m\alpha$, where $c \in \mathbf{F}$ and α is a (nonconstant) product of elements from $\mathrm{Tr}_L^G(\mathscr{B})$.

Let π denote the projection

$$\pi : \mathbf{F}[V_{p+1}] \to \mathbf{F}[V_{p+1}]/(x_1,\ldots,x_{p-2},x_{p-1}^{p-1})\mathbf{F}[V_{p+1}].$$

For convenience, write $f \equiv h$ if $\pi(f) = \pi(h)$. Observe that $\pi(z) \neq 0$, $\pi(w) \equiv x_{p+1}^p$, and $\pi(M) \equiv x_p^p$. Furthermore, the restriction of π to $\mathbf{F}[V_{p+1}]^L$ commutes with the action of $Q = G/L$. Thus $\pi\,\mathrm{Tr}_L^G(\mathscr{B}) = \mathrm{Tr}^Q\pi(\mathscr{B})$. If $\beta \in \mathrm{Tr}_L^G(\mathscr{B})$ with $\pi(\beta) \neq 0$, then $\beta = \mathrm{Tr}_L^G(w^kx_p^jx_{p-1}^\ell)$ and $\pi(\beta) \equiv x_{p-1}^\ell\,\mathrm{Tr}^Q(w^kx_p^j)$. Summing over the action of Q gives

$$\begin{aligned}
\mathrm{Tr}^Q(w^kx_p^j) &\equiv \sum_{\lambda\in\mathbf{F}_p}(x_{p+1}^p+\lambda x_p^p)^k(x_p+\lambda x_{p-1})^j \\
&\equiv \sum_{\lambda\in\mathbf{F}_p}\left(\sum_{t=0}^k\binom{k}{t}\lambda^t x_{p+1}^{p(k-t)}x_p^{tp}\right)\left(\sum_{r=0}^j\binom{j}{r}\lambda^r x_p^{j-r}x_{p-1}^r\right) \\
&\equiv \sum_{r=0}^j\sum_{t=0}^k\left(\sum_{\lambda\in\mathbf{F}_p}\lambda^{r+t}\right)\binom{k}{t}\binom{j}{r}x_{p+1}^{p(k-t)}x_p^{tp+j-r}x_{p-1}^r.
\end{aligned}$$

Recall that $\sum_{\lambda\in\mathbf{F}_p}\lambda^i = 0$ unless i is a nonzero multiple of $p-1$, in which case the sum is -1. Therefore $\mathrm{Tr}^Q(w^kx_p^j) \equiv 0$ if $j+k < p-1$. Moreover, if $p-1 \leq j+k < 2p-2$, we take $t+r = p-1$ to get

$$
\begin{aligned}
\operatorname{Tr}^{Q}(w^k x_p^j) &\equiv -\sum_{t=1}^{k}\binom{k}{t}\binom{j}{p-1-t} x_{p+1}^{p(k-t)} x_p^{pt+j+t-(p-1)} x_{p-1}^{p-1-t} \\
&\equiv -\sum_{r=p-1-k}^{j}\binom{k}{p-1-r}\binom{j}{r} x_{p+1}^{p(k+r-(p-1))} x_p^{p(p-1-r)+j-r} x_{p-1}^{r} \\
&\equiv -\binom{j}{p-1-k} x_p^{k(p+1)+j-(p-1)} x_{p-1}^{p-1-k} + x_{p-1}^{p-k} F
\end{aligned}
$$

with $F \in \mathbf{F}[x_{p+1}, x_p, x_{p-1}]$. Since $j \le p-1$ and $k \le p-1$, we have $j+k \ge 2p-2$ only when $j = p-1$ and $k = p-1$. In this case, there is one additional term, $-x_p^{p^2-p} x_{p-1}^{p-1} \equiv 0$. Since the monomials of degree $kp+j$ taken to zero by π are less than $x_p^{kp+j-p-2} x_{p-1}^{p-2}$, we have, for $k > 0$,

$$
\operatorname{LM}\left(\operatorname{Tr}_L^G\left(w^k x_p^j\right)\right) = x_p^{k(p+1)+j-(p-1)} x_{p-1}^{p-1-k}.
$$

Assume, by way of contradiction, that α is the product of at least two factors, say $\alpha = \beta_1\beta_2\cdots\beta_d$ with $\beta_i \in \operatorname{Tr}_L^G(\mathscr{B})$. Since $\operatorname{LM}(cM^m\alpha) \ge \operatorname{LM}(z) = x_p^{p^2-1} x_{p-1}^{p-2}$, we have $\operatorname{LM}(\alpha) \ge x_p^{p^2-mp-1} x_{p-1}^{p-2}$. Therefore, since we are using the graded reverse lexicographic order, $\pi\operatorname{LM}(\alpha) \ne 0$. Furthermore, $\operatorname{LM}(\beta_i)$ divides $\operatorname{LM}(\alpha)$. Thus $\pi(\beta_i) \ne 0$ giving $\beta_i = \operatorname{Tr}_L^G(w^{k_i} x_p^{j_i} x_{p-1}^{\ell_i})$ with $j_i + k_i \ge p-1$. Using the formulae above gives

$$
\operatorname{LM}(\beta_1\beta_2) = x_p^{(p+1)(k_1+k_2)+j_1+j_2-2(p-1)} x_{p-1}^{2(p-1)-k_1-k_2+\ell_1+\ell_2}.
$$

Again, using $\operatorname{LM}(cM^m\alpha) \ge \operatorname{LM}(z)$ gives $2(p-1)-k_1-k_2+\ell_1+\ell_2 \le p-2$ which simplifies to $k_1+k_2 \ge p+\ell_1+\ell_2 \ge p$. However, $\deg(\beta_1\beta_2) = p(k_1+k_2)+j_1+j_2+\ell_1+\ell_2 \le p^2+p-3$, giving $k_1+k_2 \le p$. Therefore $k_1+k_2 = p$. Furthermore, adding the inequalities $j_i+k_i \ge p-1$ gives $j_1+j_2+k_1+k_2 = j_1+j_2+p \ge 2(p-1)$ which simplifies to $j_1+j_2 \ge p-2$. Thus $\deg(\beta_1\beta_2) \ge p(k_1+k_2)+j_1+j_2 \ge p^2+p-2 > \deg(cM^m\alpha)$, giving a contradiction. Thus we must have that α is an element of $\operatorname{Tr}_L^G(\mathscr{B})$.

It remains to consider the case $f_1h_1 = cM^m\alpha$ with $\alpha \in \operatorname{Tr}_L^G(\mathscr{B})$ and $m > 0$. As above, $\pi\operatorname{LM}(\alpha) \ne 0$ gives $\alpha = \operatorname{Tr}_L^G(w^k x_p^j x_{p-1}^{\ell})$ with $k+j \ge p-1$ and $\ell+p-1+k \le p-2$. The degree constraint gives $p(m+k)+j+\ell = p^2+p-3$. Since $\alpha \in \operatorname{Tr}_L^G(\mathscr{B})$, we have $j,k \le p-1$ and $\ell \le p-2$. Thus $j+\ell \le 2p-3$. Therefore, either $m+k = p$ and $j+\ell = p-3$ or $m+k = p-1$ and $j+\ell = 2p-3$.

We first consider the case $m+k = p-1$ and $j+\ell = 2p-3$. Since $j \le p-1$ and $\ell \le p-2$, we have $j = p-1$ and $\ell = p-2$. Using the above formula for $\pi\operatorname{Tr}^{Q}(w^k x_p^j)$, with $m > 0$, gives

$$
\pi\left(M^m \operatorname{Tr}_L^G\left(w^{p-1-m} x_p^{p-1} x_{p-2}^{p-2}\right)\right) \equiv \binom{p-1}{m} x_p^{p^2-1-m} x_{p-1}^{p-2+m} \equiv 0.
$$

Thus $\operatorname{LM}\left(M^m \operatorname{Tr}_L^G\left(w^{p-1-m} x_p^{p-1} x_{p-2}^{p-2}\right)\right) < \operatorname{LM}(z)$.

This leaves the case $m+k=p$ and $j+\ell=p-3$. Again using the formula for $\pi\,\mathrm{Tr}^Q(w^k x_p^j)$ gives

$$\begin{aligned}\mathrm{LM}\left(M^m\,\mathrm{Tr}_L^G\left(w^{p-m}x_p^j x_{p-1}^{p-3-j}\right)\right) &= x_p^{mp+kp+j-(p-1-k)}x_{p-1}^{p-1-k+\ell}\\ &= x_p^{p^2+j-m+1}x_{p-1}^{p-4+m-j}.\end{aligned}$$

However, $j+k\geq p-1$ gives $m-j\leq 1$. Therefore $\mathrm{LM}(cM^m\alpha)\neq\mathrm{LM}(z)$. However, it is possible to choose α and m so that $\mathrm{LM}(cM^m\alpha)=x_p^{p^2+p-3-s}x_{p-1}^s$ for $s=0,\ldots,p-3$. Note that $s=p-4+m-j=\ell+m-1$. Since $m\geq 1$, $s=0$ occurs only when $m=1$ and $\ell=0$. In general, we may take $m=1,2,\ldots,s+1$ and $\ell=s+1-m$. Define $T_{m,s}:=M^m\,\mathrm{Tr}_L^G(w^{p-m}x_p^{p-4+m-s}x_{p-1}^{s+1-m})$. To complete the proof of the lemma, it is sufficient to show that no linear combination of elements of $\mathscr{S}:=\{T_{m,s}\mid s=0,\ldots,p-3,\ m=1,2,\ldots,s+1\}$ has lead monomial $\mathrm{LM}(z)=x_p^{p^2-1}x_{p-1}^{p-2}$. Our argument is essentially Gauss–Jordan elimination applied to $\pi\mathscr{S}$.

Using the above formula for $\pi\,\mathrm{Tr}^Q(w^k x_p^j)$ gives

$$T_{m,s}\equiv-\sum_{r=m-1}^{p-4+m-s}\binom{p-m}{p-1-r}\binom{p-4+m-s}{r}x_{p+1}^{p(r-m+1)}x_{p-1}^{r+s+1-m}x_p^*.$$

Reindexing with $i=r+s+1-m$ gives

$$\begin{aligned}T_{m,s}&\equiv-\sum_{i=s}^{p-2}\binom{p-m}{p-i+s-m}\binom{p-4+m-s}{i-s+m-1}x_{p+1}^{p(i-s)}x_{p-1}^i x_p^*\\ &\equiv-\sum_{i=s}^{p-2}\binom{p-m}{i-s}\binom{p-4+m-s}{i-s+m-1}x_{p+1}^{p(i-s)}x_{p-1}^i x_p^*.\end{aligned}$$

Note that in a field of characteristic p, $\binom{p-a}{b}=(-1)^a\binom{a-1+b}{a-1}$. Thus

$$T_{m,s}\equiv(-1)^{s+1}\sum_{i=s}^{p-2}\binom{m-1+i-s}{m-1}\binom{i+2}{s-m+3}x_{p+1}^{p(i-s)}x_{p-1}^i x_p^*.$$

A simple calculation confirms $\binom{a}{c}\binom{a+b}{b}=\binom{b+c}{c}\binom{a+b}{b+c}$, giving

$$\begin{aligned}T_{m,s}&\equiv(-1)^{s+1}\sum_{i=s}^{p-2}\binom{s+2}{m-1}\binom{i+2}{s+2}x_{p+1}^{p(i-s)}x_{p-1}^i x_p^{p^2+p-3-i-p(i-s)}\\ &\equiv(-1)^{s+1}\binom{s+2}{m-1}\sum_{i=s}^{p-2}\binom{i+2}{s+2}x_{p+1}^{p(i-s)}x_{p-1}^i x_p^{p^2+p-3-i-p(i-s)}.\end{aligned}$$

Therefore

$$(-1)^{s+1}T_{m,s}\binom{s+2}{m-1}^{-1}\equiv\sum_{i=s}^{p-2}\binom{i+2}{s+2}x_{p+1}^{p(i-s)}x_{p-1}^i x_p^{p^2+p-3-i-p(i-s)}$$

is independent of m. Thus $\{\pi(T_{1,s}) \mid s = 0,\ldots,p-3\}$ is a basis for $\pi\mathscr{S}$. Since $\{\mathrm{LM}(T_{1,s}) \mid s = 0,\ldots,p-3\} = \{x_p^{p^2+p-3-s}x_{p-1}^s \mid s = 0,\ldots,p-3\}$, no linear combination of elements of $\mathscr{S}$ has lead monomial $x_p^{p^2-1}x_{p-1}^{p-2}$.

The final assertion of the lemma follows from the fact that $\deg z = p^2+p-3$.

5 Decomposing $\mathbf{F}[V_{p+1}]$

The main goal of this section is to describe the $\mathbf{F}G$-module decomposition of $\mathbf{F}[V_{p+1}]$. We do this by considering a basis for $\mathbf{F}[V_{p+1}]_d^G$ which is compatible with the length filtration. By Theorem 1.1, $\mathbf{F}[V_{p+1}]^G$ is generated by $M := N^G(x_p)$ and $N := N^G(x_{p+1})$ together with elements of the relative transfer, $\mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]^L)$. To identify the summands occurring in the decomposition, we need to determine the lengths of the basis elements.

Suppose $f \in \mathbf{F}[V_{p+1}]^G$. It follows from Lemma 2.2 that if $\ell(f) \geq 2$ then $f \in \sqrt{\mathrm{Im}\,\mathrm{Tr}_L^G} = ((x_1,x_2,\ldots,x_p)\mathbf{F}[V_{p+1}]) \cap \mathbf{F}[V_{p+1}]^G$, and if $\ell(f) \geq p+1$ then $f \in \sqrt{\mathrm{Im}\,\mathrm{Tr}^G} = ((x_1)\mathbf{F}[V_{p+1}]) \cap \mathbf{F}[V_{p+1}]^G$. Furthermore, since $\mathrm{Tr}_L^G = \Delta^{p-1}$, if $\ell(f) \geq p$ then $f \in \mathrm{Im}\,\mathrm{Tr}_L^G$. Since $N = N^G(x_{p+1})$ has lead term[1] $x_{p+1}^{p^2}$, we have that $N \notin (x_1,x_2,\ldots,x_p)\mathbf{F}[V_{p+1}]$ and thus $\ell(N) = 1$. Similarly since the lead term of $M = N^G(x_p)$ is x_p^p, we have that $M \notin (x_1)\mathbf{F}[V_{p+1}]$ and thus $\ell(M) \leq p$.

Lemma 5.1. *Let $1 \leq q < p$. Then*

$$\Delta^{qp}(x_{p+1}^i) = \begin{cases} 0, & \text{if } i < q; \\ q!x_1^q, & \text{if } i = q; \\ x_1^q h \quad \text{for some } h \in \mathbf{F}[V_{p+1}], & \text{if } i \geq q+1. \end{cases}$$

In particular, $\Delta^{qp}(x_{p+1}^i) \in (x_1^q)\mathbf{F}[V_{p+1}]$ for all $i \geq 0$.

Proof. We consider $\Delta^{qp}(x_{p+1}^i)$ using induction on q. For $q = 1$ we have

$$\Delta^p(x_{p+1}^i) = (x_{p+1}+x_1)^i - x_{p+1}^i = \sum_{j=0}^{i-1}\binom{i}{j}x_1^{i-j}x_{p+1}^j$$
$$= \begin{cases} 0, & \text{if } i = 0; \\ x_1, & \text{if } i = 1; \\ x_1\sum_{j=0}^{i-1}\binom{i}{j}x_1^{i-j-1}x_{p+1}^j, & \text{if } i \geq 2. \end{cases}$$

[1] Use any monomial order with $x_1 < x_2 < \cdots < x_{p+1}$.

Now take $q+1 \geq 2$. Then

$$\begin{aligned}
\Delta^{(q+1)p}(x_{p+1}^i) &= \Delta^{qp}(\Delta^p(x_{p+1}^i)) \\
&= \Delta^{qp}(\sum_{j=0}^{i-1} \binom{i}{j} x_1^{i-j} x_{p+1}^j) \\
&= \sum_{j=0}^{i-1} \binom{i}{j} x_1^{i-j} \Delta^{qp}(x_{p+1}^j).
\end{aligned}$$

By induction this gives

$$\begin{aligned}
\Delta^{(q+1)p}(x_{p+1}^i) &= \sum_{j=q}^{i-1} \binom{i}{j} x_1^{i-j} \Delta^{qp}(x_{p+1}^j) \\
&= \begin{cases} 0, & \text{if } i-1 < q; \\ \binom{q+1}{q} x_1 \Delta^{qp}(x_{p+1}^q), & \text{if } i-1 = q; \\ x_1 \sum_{j=q}^{i-1} \binom{i}{j} x_1^{i-j-1} \Delta^{qp}(x_{p+1}^j), & \text{if } i-1 > q. \end{cases} \\
&= \begin{cases} 0, & \text{if } i < q+1; \\ (q+1)x_1 q! x_1^q, & \text{if } i = q+1; \\ x_1 \sum_{j=q}^{i-1} x_1^q h_j \quad \text{where } h_j \in \mathbf{F}[V_{p+1}], & \text{if } i \geq q+1. \end{cases} \\
&= \begin{cases} 0, & \text{if } i < q+1; \\ (q+1)! x_1^{q+1}, & \text{if } i = q+1; \\ x_1^{q+1} h \quad \text{for some } h \in \mathbf{F}[V_{p+1}], & \text{if } i \geq q+1. \end{cases}
\end{aligned}$$

Proposition 5.1. *Let f be a nonzero element of $\mathbf{F}[V_{p+1}]^G$ and let $1 \leq q < p$. Then $\ell(f) \geq qp+1 \iff f \in \mathrm{Im}\Delta^{qp} \iff f \in (x_1^q)\mathbf{F}[V_{p+1}]^G$.*

Proof. The first equivalence is just the definition of length. For the second equivalence, first suppose that $f = \Delta^{qp}(F) \in \mathrm{Im}\Delta^{qp}$. Write $F = \sum_{i=0}^r f_i x_{p+1}^i$ where each $f_i \in \mathbf{F}[x_1, x_2, \ldots, x_p]$. Then $f = \Delta^{qp}(F) = \sum_{i=0}^r f_i \Delta^{qp}(x_{p+1}^i) \in (x_1^q)\mathbf{F}[V_{p+1}]^G$ by the previous lemma. Conversely, suppose that $f \in (x_1^q)\mathbf{F}[V_{p+1}]^G$ and write $f = x_1^q f'$ where $f' \in \mathbf{F}[V_{p+1}]^G$. Then

$$\Delta^{qp}\left(\frac{x_{p+1}^q f'}{q!}\right) = \frac{f'}{q!}\Delta^{qp}(x_{p+1}^q) = \frac{f'}{q!} q! x_1^q = f.$$

Proposition 5.2. *Let f be a nonzero element of $\mathbf{F}[V_{p+1}]^G$ and write $f = x_1^q f'$ where x_1 does not divide f'. If $q \geq p$ then $\ell(f) = p^2$. Otherwise $\ell(f) = qp + \ell(f')$.*

Proof. Applying Lemma 5.1 gives $\Delta^{(p-1)p}(x_{p+1}^{p-1}) = (p-1)! x_1^{p-1} = -x_1^p$. Furthermore, $\Delta^p(x_p) = 0$. Thus

$$\Delta^{p^2-1}(x_{p+1}^{p-1}x_p) = \Delta^{p-1}\left((\Delta^p)^{p-1}(x_{p+1}^{p-1}x_p)\right) = \Delta^{p-1}(x_p\Delta^{(p-1)p}(x_{p+1}^{p-1}))$$
$$= -\Delta^{p-1}(x_px_1^{p-1}) = -x_1^{p-1}\Delta^{p-1}(x_p) = -x_1^p.$$

Therefore $\Delta^{p^2-1}(-x_{p+1}^{p-1}x_pf') = x_1^pf'$. This implies that if $q \geq p$ then $\ell(f) = p^2$. Suppose then that $q < p$. By Proposition 5.1, we have $qp \leq \ell(f)-1$. Since x_1^{q+1} does not divide f, Proposition 5.1 also implies that $\ell(f)-1 < (q+1)p$. Write $\ell(f)-1 = qp+r$ where $0 \leq r \leq p-1$ and define $s := \ell(f')-1$. Since x_1 does not divide f', Lemma 2.2 implies that $0 \leq s \leq p-1$. We show that $r = s$.

Clearly there exists $F \in \mathbf{F}[V_{p+1}]$ such that $f = \Delta^{qp+r}(F)$. Therefore $f = \Delta^r(\Delta^{qp}(F)) = \Delta^r(x_1^qF')$ for some $F' \in \mathbf{F}[V_{p+1}]$. Hence $x_1^qf' = f = x_1^q\Delta^r(F')$ and therefore $f' = \Delta^r(F')$. Hence $s+1 = \ell(f') \geq r+1$.

Conversely we may write $f' = \Delta^s(F'')$ for some $F'' \in \mathbf{F}[V_{p+1}]$. Since $s \leq p-1$ we have $\Delta^p(F'') = \Delta^{p-s}(\Delta^s(F'')) = \Delta^{p-s}(f') = 0$. This shows that $F'' \in \mathbf{F}[V_{p+1}]^L$. Thus $\Delta^{qp+s}(x_{p+1}^qF'') = (\Delta^s(\Delta^p)^q)(x_{p+1}^qF'') = \Delta^s(q!x_1^qF'') = q!x_1^q\Delta^s(F'') = q!x_1^qf' = q!f$ where $q! \neq 0$ since $q < p$. This shows that $f \in \mathrm{Im}\Delta^{qp+s}$ and thus $qp+r+1 = \ell(f) \geq qp+s+1$. Therefore $r = s$ as required.

Proposition 5.3. *Let f be a nonzero element in the image of the relative transfer,* $\mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]^L)$. *Suppose that x_1 does not divide f. Then $\ell(f) = p$.*

Proof. Since x_1 does not divide f, Lemma 2.2 implies that $\ell(f) \leq p$. Conversely $\mathrm{Tr}_L^G = 1+\sigma+\sigma^2+\cdots+\sigma^{p-1} = \Delta^{p-1}$. Thus the hypothesis that $f \in \mathrm{Im}\,\mathrm{Tr}_L^G$ implies that $\ell(f) \geq p$.

Remark 5.1. Since elements in $\mathrm{Tr}^G(\mathbf{F}[V_{p+1}])$ have length p^2, it is clear that $\mathbf{F}[V_{p+1}]^G$ is generated by N, M, and elements from $\mathrm{Im}\,\mathrm{Tr}_L^G \setminus \mathrm{Im}\,\mathrm{Tr}^G$.

Proposition 5.4. $\ell(M^j) = j+1$ *for all* $j = 0,1,\ldots,p-1$. *In particular M^j lies in the image of the relative transfer,* $\mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]^L)$, *if and only if $j \geq p-1$.*

Proof. From Theorem 1.1, $\mathbf{F}[V_{p+1}]^G$ is generated by M, N and elements from $\mathrm{Im}\,\mathrm{Tr}_L^G$. Note that $\deg(M) = p$ and $\deg(N) = p^2$. Thus for $d < p^2$, if p does not divide d, we have $\mathbf{F}[V_{p+1}]_d^G = \mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]_d^L)$ and, if $d = ip$ with $i < p$, $\mathbf{F}[V_{p+1}]_d^G = \mathbf{F}\cdot M^i + \mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]_{ip}^L)$. Fix $j \in \{1,2,\ldots,p-1\}$. Choose a basis, $\mathscr{B}$, for $\mathbf{F}[V_{p+1}]_{jp}^G$ so that $\mathscr{B}$ is compatible with the length filtration. Applying Proposition 2.1 gives a decomposition

$$\mathbf{F}[V_{p+1}]_{jp} = \bigoplus_{\alpha\in\mathscr{B}} V(\alpha).$$

Suppose $f \in \mathscr{L}_p^G(\mathbf{F}[V_{p+1}]) \cap \mathscr{B}$. Then $f \in \mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]^L)$. If x_1 does not divide f, then by Proposition 5.3, $\ell(f) = p$. Suppose x_1 does divide f. Write $f = x_1^qf'$ where x_1 does not divide f'. If $f' \in \mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]^L)$, then by Proposition 5.2 and Proposition 5.3, $\ell(f)$ is a multiple of p. If $f' \notin \mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]^L)$, then $f' = cM^i + f''$ for some non-zero $c \in \mathbf{F}$ and some $f'' \in \mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]^L)$. Thus $\deg(f') = ip$ and $q = (j-i)p$. Thus $q > p$ and by Proposition 5.2, $\ell(f)$ is p^2. Hence, for any $f \in \mathscr{L}_p^G(\mathbf{F}[V_{p+1}]) \cap \mathscr{B}$, $\ell(f)$ is a multiple of p. Therefore p divides the dimension of

$$\bigoplus_{\alpha\in\mathscr{L}_p^G(\mathbf{F}[V_{p+1}])\cap\mathscr{B}} V(\alpha).$$

Since

$$\dim\mathbf{F}[V_{p+1}]_{pj} = \binom{p+pj}{pj},$$

Lucas' Lemma (see, e.g., [7]) implies that

$$\dim\mathbf{F}[V_{p+1}]_{pj} \equiv \binom{j+1}{j}\binom{0}{0} \pmod{p}.$$

Thus $\dim\mathbf{F}[V_{p+1}]_{pj} \equiv j+1 \pmod{p}$. This shows that $M^j \notin \mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]^L)$ for all $j \le p-2$ and that $M^{p-1} \in \mathrm{Tr}_L^G(\mathbf{F}[V_{p+1}]^L)$. Furthermore, for $1 \le j < p-1$ we have $\ell(M^j) < p$ and $\ell(M^j) \equiv j+1 \pmod{p}$ and thus $\ell(M^j) = j+1$. Since $\mathbf{F}[V_{p+1}]_0 \cong \mathbf{F}$, it is clear that $\ell(M^0) = 1$.

Define $\mathbf{F}[V_{p+1}]^\flat$ to be the span of the monomials in $\mathbf{F}[V_{p+1}]$ which, as polynomials in x_{p+1}, have degree less than p^2. It follows from the proof of Proposition 2.2 that, as $\mathbf{F}G$-modules, $\mathbf{F}[V_{p+1}] = N\mathbf{F}[V_{p+1}] \oplus \mathbf{F}[V_{p+1}]^\flat$. Thus a decomposition of $\mathbf{F}[V_{p+1}]^\flat$ gives a decomposition of $\mathbf{F}[V_{p+1}]$. Therefore the following theorem implies Theorem 1.2.

Theorem 5.1. *(i) For $d < p^2 - p$, an $\mathbf{F}G$-module decomposition of $\mathbf{F}[V_{p+1}]_d^\flat$ includes precisely one noninduced indecomposable summand. Divide p into d to get $d = bp + c$ with $0 \le c < p$. The noninduced summand is isomorphic to V_{cp+b+1} and the decomposition may be chosen so that the socle of the noninduced indecomposable is spanned by $x_1^c M^b$.*
(ii) For $d \ge p^2 - p$, $\mathbf{F}[V_{p+1}]_d^\flat$ is a direct sum of indecomposable induced modules.

Proof. Fix d and choose a basis, $\mathscr{B}$, for $(\mathbf{F}[V_{p+1}]_d^\flat)^G$, so that $\mathscr{B}$ is compatible with the length filtration. Applying Proposition 2.1 gives a decomposition

$$\mathbf{F}[V_{p+1}]_d^\flat = \bigoplus_{\alpha\in\mathscr{B}} V(\alpha)$$

with $V(\alpha) \cong V_{\ell(\alpha)}$. Write $\alpha = x_1^i \alpha'$ where x_1 does not divide α'. If $i \ge p$, then by Proposition 5.2, $\ell(\alpha) = p^2$ and $V(\alpha)$ is projective. Suppose $i < p$. If $\alpha' \in \mathrm{Im}\,\mathrm{Tr}_L^G$, then by Proposition 5.3, $\ell(\alpha') = p$. Thus, using Proposition 5.2, $\ell(\alpha) = ip + p$ and $V(\alpha)$ is an induced module. Suppose $\alpha' \notin \mathrm{Im}\,\mathrm{Tr}_L^G$. Then $\alpha' = kM^j + h$ where $j < p-1$, k is a nonzero element of $\mathbf{F}$ and $h \in \mathrm{Im}\,\mathrm{Tr}_L^G$. It follows from Proposition 5.4 that $\ell(\alpha') = j+1$. Applying Proposition 5.2 gives $\ell(\alpha) = pi + j + 1$. This last case is the only way in which a noninduced summand can appear in the decomposition. Note that in this case, $d = pj + i$ with $0 \le i < p$ and $j < p-1$, giving $d \le (p-2)p + (p-1) = p^2 - p - 1$, $i = c$ and $j = b$. Suppose, by way of contradiction, that $\alpha_1, \alpha_2 \in \mathscr{B}$ are distinct elements both having length $cp + b + 1$. Then $\alpha_1 = x_1^c(k_1 M^b + h_1)$ and $\alpha_2 = x_1^c(k_2 M^b + h_2)$ with $k_i \in \mathbf{F} \setminus \{0\}$ and $h_i \in \mathrm{Im}\,\mathrm{Tr}_L^G$. From

Proposition 5.2, $\ell(k_2\alpha_1 - k_1\alpha_2) \geq cp + p > cp + b + 1$ contradicting the fact that $\mathscr{B}$ is compatible with the length filtration.

Remark 5.2. The strong form of the Hughes–Kemper Periodicity Conjecture [10, Conjecture 4.6] states that for $p^{m-1} < n \leq p^m$ and $d > p^m - n$, $\mathbf{F}[V_n]_d^\flat$ is induced. The preceding theorem verifies the conjecture for $n = p+1$.

6 Generating functions

In Section 3 we derived the Hilbert series for the ring of invariants $\mathbf{F}[V_{p+1}]^G$ using a spectral sequence. Here we rederive this Hilbert series using the module decomposition. The method of this section has the advantage that we also obtain generating functions which give the multiplicities for each of the indecomposable $\mathbf{F}G$-modules as summands in $\mathbf{F}[V_{p+1}]_n$.

Throughout the section we write $n = \alpha p^2 + \beta p + \gamma$ where $0 \leq \beta, \gamma \leq p-1$. If $\beta \neq p-1$ then by Theorem 1.2 we know that $\mathbf{F}[V_{p+1}]_n$ contains exactly one noninduced summand, $V_{d(n)}$, where $d(n) = \gamma p + \beta + 1$. For convenience we also define $d(n) = \gamma p + \beta + 1 = (\gamma+1)p$ when $\beta = p-1$.

Define integer-valued functions $a_1(n), a_2(n), \ldots, a_p(n)$ by

$$\mathbf{F}[V_{p+1}]_n \cong V_{d(n)} \oplus a_1(n)V_p \oplus a_2(n)V_{2p} \oplus \cdots \oplus a_p(n)V_{p^2}. \tag{3}$$

By Propositions 5.1 and 5.2, an invariant f spans the socle of a copy of V_{ip} where $p > i \geq 2$, if and only if $f = x_1 h$ where the invariant h spans the socle of a copy of $V_{(i-1)p}$. Clearly if $n = \deg(f)$ then $\deg(h) = n - 1 \geq 0$. This means that for all $2 \leq i \leq p-1$ and $\beta \neq p-1$, we have

$$a_i(n) = \begin{cases} 0, & \text{if } n = 0, 1; \\ a_{i-1}(n-1), & \text{if } n \geq 1. \end{cases} \tag{4}$$

When $\beta = p-1$, we have $V_{d(n)} = V_{(\gamma+1)p}$. For $1 \leq \gamma < p-1$, we can choose the decompositions so that the socle of $V_{d(n)}$ is x_1 times the socle of $V_{d(n-1)}$. Therefore Equation 4 also holds when $\beta = p-1$.

Similarly, Proposition 5.2 with $q = p-1$ and $q = p$ combined with Theorem 1.2 implies that

$$a_p(n) = \begin{cases} 0, & \text{if } n = 0, 1; \\ a_p(n-1) + a_{p-1}(n-1), & \text{if } p \text{ does not divide } n; \\ a_p(n-1) + a_{p-1}(n-1) + 1, & \text{if } p \text{ divides } n \text{ and } n \neq 0. \end{cases} \tag{5}$$

Note that in the case where p divides n and $n \neq 0$, we have $x_1 V_{d(n-1)} \cong V_{p^2}$.

Comparing dimensions in the decomposition (3) yields the equation:

$$\binom{n+p}{p} = d(n) + pa_1(n) + 2pa_2(n) + \cdots + p^2 a_p(n). \tag{6}$$

Define generating functions:

$$D(t) = \sum_{n=0}^{\infty} d(n)t^n$$
$$A_i(t) = \sum_{n=0}^{\infty} a_i(n)t^n \quad \text{for } i = 1, 2, \ldots, p.$$

In terms of these generating functions, the above recursive conditions, (4) and (5), become:

$$\begin{aligned} A_i(t) &= \sum_{n=0}^{\infty} a_i(n)t^n \\ &= a_i(0) + t\sum_{n=1}^{\infty} a_{i-1}(n-1)t^{n-1} \\ &= tA_{i-1} \quad (\text{for } i = 2, 3, \ldots, p-1) \end{aligned}$$

and

$$\begin{aligned} A_p(t) &= \sum_{n=0}^{\infty} a_p(n)t^n \\ &= a_p(0) + t\sum_{n=1}^{\infty} \left(a_p(n-1) + a_{p-1}(n-1) + \delta_n^0\right)t^{n-1} \\ &\qquad \text{where } \delta_n^0 = \begin{cases} 1, & \text{if } n \equiv 0 \pmod{p}; \\ 0, & \text{otherwise.} \end{cases} \\ &= tA_p(t) + tA_{p-1}(t) + \sum_{n=1}^{\infty} t^{np} \\ &= tA_p(t) + tA_{p-1}(t) + \frac{t^p}{1-t^p}. \end{aligned} \tag{7}$$

Again using the generating functions, the dimension equation (6) becomes

$$\frac{1}{(1-t)^{p+1}} = D(t) + pA_1(t) + 2pA_2(t) + \cdots + p^2 A_p(t).$$

Substituting $A_2(t) = tA_1(t)$, $A_3(t) = t^2 A_1(t)$, $\ldots$, $A_{p-1}(t) = t^{p-2}A_1(t)$, we are left with the following two equations in A_1 and A_p.

$$A_p(t) = tA_p(t) + t^{p-1}A_1(t) + \frac{t^p}{1-t^p}$$
$$pA_1(t) + 2ptA_1(t) + \cdots + (p^2-p)t^{p-2}A_1(t) + p^2A_p(t) + D(t) = \frac{1}{(1-t)^{p+1}}.$$

Collecting terms this system becomes:

$$\begin{aligned} -t^{p-1}A_1(t) + (1-t)A_p(t) &= \frac{t^p}{1-t^p} \\ p(1+2t+3t^2+\cdots+(p-1)t^{p-2})A_1(t) + p^2A_p(t) &= -D(t) + \frac{1}{(1-t)^{p+1}}. \end{aligned}$$

Note that, as is easily verified by integration, we have

$$1 + 2t + 3t^2 + \cdots + mt^{m-1} = \frac{1-(m+1)t^m + mt^{m+1}}{(1-t)^2}. \tag{8}$$

Thus the above system of equations becomes

$$\begin{aligned} -t^{p-1}A_1(t) + (1-t)A_p(t) &= \frac{t^p}{1-t^p} \qquad (9) \\ p\left(\frac{1-pt^{p-1}+(p-1)t^p}{(1-t)^2}\right)A_1(t) + p^2A_p(t) &= -D(t) + \frac{1}{(1-t)^{p+1}}. \end{aligned}$$

Solving for A_1 and A_p yields

$$A_1(t) = \left(\frac{-p^2t^p}{1-t^p} + \frac{1}{(1-t)^p} - (1-t)D(t)\right)\left(\frac{1-t}{p(1-t^p)}\right)$$
$$A_p(t) = \left(\frac{1-pt^{p-1}+(p-1)t^p}{(1-t)^2}\left(\frac{pt^p}{1-t^p}\right) + \frac{t^{p-1}}{(1-t)^{p+1}} - t^{p-1}D(t)\right)\left(\frac{1-t}{p(1-t^p)}\right).$$

Thus a closed form for $D(t)$ will yield closed forms for $A_1(t)$ and $A_p(t)$. To obtain a closed expression for $D(t)$ we observe that the sequence $\{d(n)\}_{n=0}^{\infty}$ is the sum of two periodic sequences, one of period p and one of period p^2. From this using Equation (8) twice we get

$$\begin{aligned} D(t) &= \left(\sum_{\gamma=0}^{p-1} p\gamma t^\gamma\right)\frac{1}{1-t^p} + \left(\sum_{\beta=0}^{p-1}\sum_{\gamma=0}^{p-1}(\beta+1)t^{p\beta+\gamma}\right)\frac{1}{1-t^{p^2}} \\ &= pt\left(\sum_{\gamma=0}^{p-1}\gamma t^{\gamma-1}\right)\frac{1}{1-t^p} + \left(\sum_{\beta=0}^{p-1}(\beta+1)t^{p\beta}\right)\left(\sum_{\gamma=0}^{p-1}t^\gamma\right)\frac{1}{1-t^{p^2}} \\ &= pt\left(\frac{1-pt^{p-1}+(p-1)t^p}{(1-t)^2}\right)\frac{1}{1-t^p} \\ &\quad + \left(\frac{1-(p+1)(t^p)^p+p(t^p)^{p+1}}{(1-t^p)^2}\right)\left(\frac{1-t^p}{1-t}\right)\frac{1}{1-t^{p^2}}. \end{aligned}$$

Substituting this expression into the expression for A_p given above and simplifying yields the following.

$$A_p(t) = \frac{1}{p(1-t^p)}\left(\frac{t^{p-1}}{(1-t)^p} - \frac{t^{p-1}-(p+1)t^{p^2+p-1}+pt^{p^2+2p-1}}{(1-t^p)(1-t^{p^2})}\right).$$

Using this expression for $A_p(t)$ in (9) gives

$$A_1(t) = \frac{-t}{1-t^p} + \frac{1-t}{p(1-t^p)}\left(\frac{1}{(1-t)^p} - \frac{1-(p+1)t^{p^2}+pt^{p^2+p}}{(1-t^p)(1-t^{p^2})}\right).$$

Repeatedly using the recursive equation for $A_p(n)$ (7) we obtain

$$\begin{aligned}
A_p &= tA_p + tA_{p-1} + \frac{t^p}{1-t^p}\\
&= t(tA_p + tA_{p-1} + \frac{t^p}{1-t^p}) + tA_{p-1} + \frac{t^p}{1-t^p}\\
&= t^2A_p + (t^2+t)A_{p-1} + (t+1)\frac{t^p}{1-t^p}\\
&= t^2(tA_p + tA_{p-1} + \frac{t^p}{1-t^p}) + (t^2+t)A_{p-1} + (t+1)\frac{t^p}{1-t^p}\\
&= t^3A_p + (t^3+t^2+t)A_{p-1} + (t^2+t+1)\frac{t^p}{1-t^p}\\
&\ \ \vdots\\
&= t^{p-1}A_p + (t^{p-1}+t^{p-2}+\cdots+t)A_{p-1} + (t^{p-2}+t^{p-3}+\cdots+1)\frac{t^p}{1-t^p}\\
&= t^{p-1}\left(A_p + (1+t^{-1}+\cdots+t^{-(p-2)})A_{p-1}\right) + \left(\frac{1-t^{p-1}}{1-t}\right)\frac{t^p}{1-t^p}\\
&= t^{p-1}\left(A_p + (A_{p-1}+A_{p-2}+\cdots+A_1) + \left(\frac{1}{1-t} - \frac{1}{1-t^p}\right)\right).
\end{aligned}$$

Note that the Hilbert series of the ring of invariants $\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}$ is given by $\mathscr{H}(\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2},t) = 1/(1-t) + \sum_{i=1}^{p} A_i(t)$. Thus we have

$$A_p = t^{p-1}\left(\mathscr{H}(\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2},t) - \frac{1}{1-t^p}\right).$$

Therefore

$$\mathscr{H}(\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2},t) = \frac{1}{t^{p-1}}A_p(t) + \frac{1}{1-t^p}$$
$$= \frac{1}{p(1-t^p)}\left(\frac{1}{(1-t)^p} + \frac{(p-1)-pt^p+t^{p^2}}{(1-t^p)(1-t^{p^2})}\right),$$

providing a second proof of Corollary 3.2.

In the above description, the summand $V_{d(n)}$ is sometimes an induced summand. More precisely, this happens exactly when $\beta = p-1$. Thus if we decompose the induced component

$$(\mathbf{F}[V_{p+1}]_n)_{\text{induced}} \cong b_1(n)V_p \oplus b_2(n)V_{2p} \oplus \cdots \oplus b_p(n)V_{p^2}$$

we have

$$b_i(n) = \begin{cases} a_i(n)+1, & \text{if } \gamma = i-1 \text{ and } \beta = p-1; \\ a_i(n), & \text{otherwise.} \end{cases}$$

Thus the generating function $B_i(t) = \sum_{n=0}^{\infty} b_i(n)t^n$ is given by

$$B_i(t) = A_i(t) + \frac{t^{p^2-p+i-1}}{1-t^{p^2}} = t^{i-1}A_1(t) + \frac{t^{p^2-p+i-1}}{1-t^{p^2}}.$$

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Differential Operators on Grassmann Varieties

Will Traves[1]

Summary. Following Weyl's account in *The Classical Groups* we develop an analogue of the (first and second) Fundamental Theorems of Invariant Theory for rings of differential operators: when V is a k-dimensional complex vector space with the standard $SL_k\mathbb{C}$ action, we give a presentation of the ring of invariant differential operators $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ and a description of the ring of differential operators on the G.I.T. quotient, $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$, which is the ring of differential operators on the (affine cone over the) Grassmann variety of k-planes in n-dimensional space. We also compute the Hilbert series of the associated graded rings $GrD(\mathbb{C}[V^n])^{SL_k}$ and $Gr(D(\mathbb{C}[V^n]^{SL_k\mathbb{C}}))$. This computation shows that earlier claims that the kernel of the map from $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ to $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ is generated by the Casimir operator are incorrect. Something can be gleaned from these earlier incorrect computations though: the kernel meets the universal enveloping algebra of $\mathfrak{sl}_k\mathbb{C}$ precisely in the central elements of $U(\mathfrak{sl}_k\mathbb{C})$.

Key words: Invariant theory, Weyl algebra, differential operators, Grassmann variety, Hilbert series

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If $R = \mathbb{C}[V] = \mathbb{C}[x_1,\dots,x_n]$ is a polynomial ring then the ring $D(R)$ of $\mathbb{C}$-linear differential operators [7, 12] on R is just the Weyl algebra on V. That is, $D(R) = \mathbb{C}\langle x_1,\dots,x_n,\partial_1,\dots,\partial_n\rangle$, where $\partial_i = \partial/\partial x_i$ should be interpreted as the derivation on x_i and most of the variables commute, but we impose the relations

$$[\partial_i, x_i] = \partial_i x_i - x_i\partial_i = 1,$$

which just encode the usual product rule from calculus. More generally, if X is an affine variety with coordinate ring $R = \mathbb{C}[X] = \mathbb{C}[V]/I$ then the ring of differential

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operators on X can be described as

$$D(R) = D(\mathbb{C}[V]/I) = \frac{\{\theta \in D(\mathbb{C}[V]) : \ \theta(I) \subset I\}}{ID(\mathbb{C}[V])}. \quad (1)$$

In either setting, the ring of differential operators is a filtered algebra, with filtration given by the order of the operator. The associated graded ring $GrD(R)$ is a commutative ring and there is a map, the symbol map, from $D(R)$ to $GrD(R)$. We write ξ_i for the symbol of ∂_i.

When a group G acts on V, it not only induces an operator on the coordinate ring $\mathbb{C}[V]$, but also on the Weyl algebra $D(\mathbb{C}[V])$: if $\theta \in D(\mathbb{C}[V])$ and $g \in G$ then for all $f \in \mathbb{C}[V]$,

$$(g\theta)(f) = g(\theta(g^{-1}f)).$$

Our aim in the first part of this chapter is to describe the ring of invariant differential operators $D(R)^G$ and the ring of differential operators $D(R^G)$ for certain classical rings of invariants R^G (with $R = \mathbb{C}[V^n]$ for a k-dimensional complex vector space V equipped with the usual $G = SL_k\mathbb{C}$ action). Although we might try to express R^G in the form $\mathbb{C}[V]/I$ and use equation (1), it is difficult to describe the set $\{\theta \in D(\mathbb{C}[V]) : \ \theta(I) \subset I\}$, so this is impractical. Rather, we exploit the natural map $\pi_* : D(R)^G \to D(R^G)$ given by restricting invariant operators to the invariant ring. The properties of this map are quite subtle – the reader is referred to Schwarz's detailed exposition [15] – but in general π_* need not be surjective and is almost never injective. Fortunately, Schwarz showed that in the cases we are interested in, π_* is surjective with kernel equal to $D(R)\mathfrak{sl}_k\mathbb{C} \cap D(R)^G$. We exploit this to describe generators for $D(R^G)$. This is particularly interesting since $R^G = \mathbb{C}[V^n]^{SL_k\mathbb{C}}$ is the coordinate ring of the Grassmann variety $G(k,n)$ of k-planes in n-space (which is a cone over the projective Grassmann variety).

The Fundamental Theorem of Invariant Theory gives a presentation of the coordinate ring $\mathbb{C}[G(k,n)] = R^G$. Following a path strongly advocated by Weyl [19][1] we extend the Fundamental Theorem to $D(R)^G$, giving a presentation of the invariant differential operators on the affine variety $G(k,n)$. Because the action of G on R preserves the filtration, we also have graded rings $Gr(D(R))^G = Gr(D(R)^G)$ and $GrD(R^G)$. Applying the Fundamental Theorem to $GrD(R)^G$, we obtain generators and relations of the graded algebra. These lift to generators of $D(R)^G$ and each of the relations on the graded algebra extends to a relation on $D(R)^G$.

In conference talks based on two earlier papers [17, 18] I claimed that $ker(\pi_*)$ was generated by the Casimir operator. This is false. In the second part of this paper, we show that this cannot be true by computing the Hilbert series of $GrD(R)^G$, $GrD(R^G)$ and the graded image of $ker(\pi_*)$. However, the earlier computations were not entirely without merit: they predict that $ker(\pi_*) \cap U(\mathfrak{sl}_k\mathbb{C}) = Z(U(\mathfrak{sl}_k\mathbb{C}))$, a result that we prove using infinitesimal methods. The Hilbert series computations suggest that $GrD(R^G)$ may be Gorenstein; however, this remains an open question.

[1] Also see the references in Olver [13], particularly those in Chapter 6.

The invariant-theoretic methods of this chapter are described in Derksen and Kemper's very nice book [3]. As previously mentioned, this chapter also makes crucial use of Schwarz's results in [15]. It is a pleasure to dedicate this paper to Gerry Schwarz on the occasion of his 60th birthday.

1 A fundamental theorem

Let V be a k-dimensional complex vector space and let V^* be the dual space of V. Then $\mathbb{C}[V^r \oplus (V^*)^s]$ is generated by the coordinates x_{ij} $(1 \leq i \leq k, 1 \leq j \leq r)$ and ξ_{ij} $(1 \leq i \leq k, 1 \leq j \leq s)$. Moreover, we have a natural $SL_k\mathbb{C}$ action on $\mathbb{C}[V^r \oplus (V^*)^s]$: $SL_k\mathbb{C}$ acts diagonally, on the V's by the standard representation and on the V^*'s by the contragredient representation. To be explicit, if $\mathbf{e}_i$ is the ith standard basis vector then $A \in SL_k\mathbb{C}$ acts via:

$$\begin{aligned} A \cdot x_{ij} &= (x_{1j}, x_{2j}, \ldots, x_{kj})A\mathbf{e}_i \\ \text{and} \quad A \cdot \xi_{ij} &= (\xi_{1j}, \xi_{2j}, \ldots, \xi_{kj})(A^{-1})^T \mathbf{e}_i, \end{aligned}$$

where B^T is the usual transpose of B, $B^T_{ij} = B_{ji}$. To clarify with a simple example, if $r = s = 1$ and $k = 2$, then the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{C}$$

acts on the variables $x_{11}, x_{21}, \xi_{11}, \xi_{21}$ to give

$$\begin{aligned} &A \cdot x_{11} = ax_{11} + cx_{21},\ A \cdot x_{21} = bx_{11} + dx_{21}, \\ &A \cdot \xi_{11} = d\xi_{11} - b\xi_{21},\ A \cdot \xi_{21} = -c\xi_{11} + a\xi_{21}. \end{aligned}$$

If $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{C}$ is the canonical pairing, for each $i \leq r$ and $j \leq s$ we have an invariant $\langle ij \rangle : V^r \oplus (V^*)^s \to \mathbb{C}$ that sends $(v_1, \ldots, v_r, w_1, \ldots, w_s)$ to $\langle v_i, w_j \rangle$. In coordinates $\langle ij \rangle = \sum_{\ell=1}^k x_{\ell i}\xi_{\ell j}$.

There are other invariants too. If $I = (I_1, I_2, \ldots, I_k)$ is a sequence with $1 \leq I_1 < I_2 < \cdots < I_k \leq r$, then we have a bracket invariant $[I] = [I_1 I_2 \cdots I_k] : V^r \oplus (V^*)^s \to \mathbb{C}$ given by

$$(v_1, \ldots, v_r, w_1, \ldots, w_s) \to \det(v_{I_1} v_{I_2} \cdots v_{I_k}).$$

This is a polynomial of degree k that only involves the x_{ij}. As well, if $J = (J_1, \ldots, J_k)$ is a sequence with $1 \leq J_1 < J_2 < \cdots < J_k \leq s$, then we have an invariant $|J| = |J_1 J_2 \cdots J_k| : V^r \oplus (V^*)^s \to \mathbb{C}$ given by

$$(v_1, \ldots, v_r, w_1, \ldots, w_s) \to \det(w_{J_1} w_{J_2} \cdots w_{J_k}).$$

This is a polynomial of degree k that only involves the ξ_{ij}.

The Fundamental Theorem of Invariant Theory describes the $SL_k\mathbb{C}$-invariants in the ring $\mathbb{C}[V^r \oplus (V^*)^s]$ and the relations among them (see [14, Sections 9.3 and 9.4]).

Theorem 1.1 (Fundamental Theorem of Invariant Theory). *Let V be a k-dimensional complex vector space. The invariant ring*

$$\mathbb{C}[V^r \oplus (V^*)^s]^{SL_k\mathbb{C}}$$

is generated by all $\langle ij \rangle$ $(1 \le i \le r, 1 \le j \le s)$, *all* $[I] = [I_1 I_2 \cdots I_k]$ $(1 \le I_1 < I_2 < \cdots < I_k \le r)$ *and all* $|J| = |J_1 J_2 \cdots J_k|$ $(1 \le J_1 < J_2 < \cdots < J_k \le s)$. *The relations among these generators are of five types:*
(a) For $1 \le I_1 < I_2 < \cdots < I_k \le r$ *and* $1 \le J_1 < J_2 < \cdots < J_k \le s$*:*
$\det(\langle IJ \rangle) = \det(\langle I_a J_b \rangle)_{a,b=1}^k = [I_1 I_2 \cdots I_k]|J_1 J_2 \ldots J_k|$
(b) For $1 \le I_1 < I_2 < \cdots < I_{k+1} \le r$ *and* $1 \le j \le s$*:*
$\sum_{\ell=1}^{k+1} (-1)^{\ell-1} [I_1 I_2 \cdots \hat{I}_\ell \cdots I_{k+1}] \langle I_\ell j \rangle = 0$
(c) For $1 \le J_1 < J_2 < \cdots < J_{k+1} \le s$ *and* $1 \le i \le r$*:*
$\sum_{\ell=1}^{k+1} (-1)^{\ell-1} \langle i J_\ell \rangle |J_1 J_2 \cdots \hat{J}_\ell \cdots J_{k+1}| = 0$
(d) For $1 \le I_1 < I_2 < \cdots < I_{k-1} \le r$ *and* $1 \le J_1 < J_2 < \cdots < J_{k+1} \le r$*:*
$\sum_{\ell=1}^{k+1} (-1)^{\ell-1} [I_1 I_2 \cdots I_{k-1} J_\ell][J_1 J_2 \cdots \hat{J}_\ell \cdots J_{k+1}] = 0$
(e) For $1 \le I_1 < I_2 < \cdots < I_{k-1} \le s$ *and* $1 \le J_1 < J_2 < \cdots < J_{k+1} \le s$*:*
$\sum_{\ell=1}^{k+1} (-1)^{\ell-1} |I_1 I_2 \cdots I_{k-1} J_\ell| \; |J_1 J_2 \cdots \hat{J}_\ell \cdots J_{k+1}| = 0.$

Example 1.1. When $r = n$ and $s = 0$, Theorem 1.1 shows that $\mathbb{C}[V^n]^{SL_k\mathbb{C}}$ is generated by brackets $[I]$ satisfying the relations in (d). This invariant ring is the *bracket algebra*, the coordinate ring $\mathbb{C}[G(k,n)]$ of the Grassmann variety of k-planes in n-space. In this context, the generators are called the Plücker coordinates and the relations in part (d) are called the Grassmann–Plücker relations.

Now if V is a k-dimensional vector space, $(GrD(\mathbb{C}[V^n]))^{SL_k\mathbb{C}} = \mathbb{C}[V^n \oplus (V^*)^n]^{SL_k\mathbb{C}}$ so we can apply Theorem 1.1 in the case $r = s = n$ to compute $(GrD(\mathbb{C}[V^n]))^{SL_k\mathbb{C}}$. By [17, Theorem 1], $Gr(D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}) = (GrD(\mathbb{C}[V^n]))^{SL_k\mathbb{C}}$, so the lifts of the generators for $\mathbb{C}[V^n \oplus (V^*)^n]^{SL_k\mathbb{C}}$ generate $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. Lifting the generators is easy since we only need to replace ξ_{ij} with ∂_{ij}.

Together with some very subtle work by Schwarz [15], this remark is also sufficient to determine the generators of the ring of differential operators $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}}) = D(G(k,n))$ on the affine cone $\mathrm{Spec}(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ over the Grassmann variety $G(k,n)$.

Theorem 1.2. *The ring of differential operators* $D(\mathbb{C}[G(k,n)])$ *on the affine cone over* $G(k,n)$ $(0 < k < n)$ *is generated by the images under* π_* *of the lifts to* $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ *of the operators* $\langle ij \rangle$, $[I_1 I_2 \cdots I_k]$ *and* $|J_1 J_2 \cdots J_k|$.

Proof. Using Theorem 1.1 we see that these are the operators (with ξ_{ij} replacing ∂_{ij}) that generate $(GrD\mathbb{C}[V^n])^{SL_k\mathbb{C}} = \mathbb{C}[V^n \oplus (V^*)^n]^{SL_k\mathbb{C}}$. These lift to generators of $(D(\mathbb{C}[V^n]))^{SL_k\mathbb{C}}$. Although $SL_k\mathbb{C}$ does not in general satisfy the LS-alternative, this representation of $SL_k\mathbb{C}$ does satisfy the LS-alternative (see Schwarz [15, 11.6]) and so $\pi_* : (D(\mathbb{C}[V^n]))^{SL_k\mathbb{C}} \to D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ is surjective or $\mathbb{C}[V^n]^{SL_k\mathbb{C}}$ is smooth. Since the affine cone over $G(k,n)$ is singular for $0 < k < n$, π_* must be surjective. Thus these generators restrict to generators of $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}}) = D(\mathbb{C}[G(k,n)])$.

Now we turn to the relations among the generators of $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. Each of the relations in Theorem 1.1 extends to an ordered relation on $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. The relations in (b), (c), (d), and (e) apply unchanged in the noncommutative ring $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. However, the relations in part (a) need to be modified: the meaning of the determinant in a noncommutative ring needs to be clarified[2] and there are lower-order terms that need to be added to extend the relations to the non-graded ring. Fortunately, Capelli [1] found a beautiful way to incorporate the lower order terms by changing some of the terms in the determinant.[3] For each $k \in I \cap J$, the term $\langle kk \rangle$ in the matrix $\langle IJ \rangle$ should be replaced by

$$\langle kk \rangle + (\tau_I(k) - 1),$$

where $\tau_I(k)$ is the index of k in the sequence I; that is, $I_{\tau_I(k)} = k$.

Example 1.2. When $n = 4$ and $k = 3$, the extension of identity (a) for $[I] = [123]$ and $|J| = |124|$ is

$$\det \begin{bmatrix} \langle 11 \rangle + 0 & \langle 12 \rangle & \langle 14 \rangle \\ \langle 21 \rangle & \langle 22 \rangle + 1 & \langle 24 \rangle \\ \langle 31 \rangle & \langle 32 \rangle & \langle 34 \rangle \end{bmatrix} = [123]|124|.$$

Example 1.3. When $n = 4$ and $k = 3$, the extension of identity (a) for $[I] = [123]$ and $|J| = |234|$ is

$$\det \begin{bmatrix} \langle 12 \rangle & \langle 13 \rangle & \langle 14 \rangle \\ \langle 22 \rangle + 1 & \langle 23 \rangle & \langle 24 \rangle \\ \langle 32 \rangle & \langle 33 \rangle + 2 & \langle 34 \rangle \end{bmatrix} = [123]|234|.$$

To describe a presentation of the ring $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ in terms of generators and relations we also need to incorporate the commutator relations. The ring $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ is generated by the same operators that generate $\mathbb{C}[V^n \oplus (V^*)^n]^{SL_k\mathbb{C}}$. To present such a ring, take a noncommutative free algebra F in generators with the same names as the generators of $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. The kernel K of the map $F \to D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ given by the natural identification of generators is a two-sided ideal of F.

Theorem 1.3. *The ideal K of relations is generated by the commutator relations and the extensions of each of the relations in Theorem 1.1.*

---

[2] If S is a noncommutative ring and $A \in M_{k\times k}(S)$ is a square matrix with entries in S then by the determinant of A we mean

$$\det(A) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{k\sigma(k)}.$$

[3] I'm grateful to Minoru Itoh for pointing out Capelli's identity to me at a workshop held at Hokkaido University. In my original research, I discovered a less elegant extension for the relations through experiments with the D-modules package of Macaulay2 [11, 4] and the Plural package of Singular [5, 6].

Proof. Let C be the two-sided ideal of F generated by the commutator relations among the generators of $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$. Then F/C is a filtered ring in which each of the generators $\langle ij\rangle$ ($[I]$, $|J|$, resp.) has order 1 (0, 2, resp.). The associated graded ring $Gr(F/C)$ is a commutative polynomial ring whose variables have the same names as those in F. Because the filtrations on F/C and $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ are compatible, there is a surjective map $\delta : Gr(F/C) \to GrD(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ and the kernel of δ is the image of the ideal K in the associated graded ring $Gr(F/C)$. However, the generators of $\ker(\delta)$ are given by the relations in Theorem 1.1. The lifts of these generators to F/C generate the kernel of the map $F/C \to D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ and so these lifts, together with the commutator relations, generate the kernel of the map $F \to D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$.

The generators and relations described so far are enough to give a presentation of $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$; that is, to determine the invariant differential operators. The map π_* from $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ to $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$, the differential operators on the coordinate ring of the geometric quotient $\mathbb{C}[V^n]^{SL_k\mathbb{C}}$, is not injective. Its kernel is a two-sided ideal in $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. Schwarz [15] showed that this kernel consists of the $SL_k\mathbb{C}$-stable part of the left ideal of $D(\mathbb{C}[V^n])$ generated by the operators in the Lie algebra $\mathfrak{sl}_k\mathbb{C}$. In several conference talks based on two previous papers [17, 18] I described an elimination computation in the case $k = 2$, $n = 4$ that seemed to show that this kernel is generated by a single element, the Casimir operator. However, I interpreted the results of this computation incorrectly: in fact $\ker(\pi_*)$ requires many generators. We show this by computing the Hilbert series for the graded image of $\ker(\pi_*)$ in the next section.

The elimination computation mentioned earlier actually shows something interesting that holds more generally: the center of the universal enveloping algebra of $\mathfrak{sl}_2\mathbb{C}$ is the part of the kernel of π_* that lies in $U(\mathfrak{sl}_2\mathbb{C})$.

Theorem 1.4. $\ker(\pi_*) \cap U(\mathfrak{sl}_k\mathbb{C}) = Z(U(\mathfrak{sl}_k\mathbb{C}))$.

Proof. Note that $\ker(\pi_*) \cap U(\mathfrak{sl}_k\mathbb{C}) = U(\mathfrak{sl}_k\mathbb{C})^{SL_k\mathbb{C}}$ since $\ker(\pi_*) = (D(\mathbb{C}[V^n])\mathfrak{sl}_k\mathbb{C})^{SL_k\mathbb{C}}$. An element of $U(\mathfrak{sl}_k\mathbb{C})$ is $SL_k\mathbb{C}$-stable if and only if it is annihilated by the adjoint action of the Lie algebra $\mathfrak{sl}_k\mathbb{C}$ (see, e.g., Sturmfels [16, Lemma 4.5.1]). But if $\delta \in U(\mathfrak{sl}_k\mathbb{C})$ then

$$ad(\sigma)(\delta) = [\sigma, \delta] = 0 \quad \text{for all } \sigma \in \mathfrak{sl}_k\mathbb{C} \iff \delta \in Z(U(\mathfrak{sl}_k\mathbb{C})).$$

This shows that $U(\mathfrak{sl}_k\mathbb{C})^{SL_k\mathbb{C}} = Z(U(\mathfrak{sl}_k\mathbb{C}))$, as desired.

To close this section we emphasize that both $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ and $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ are noncommutative rings. This distinguishes these rings from the rings of operators considered in [8], which are commutative. For instance, when $k = 2$ and $n = 4$, let $\theta \bullet f$ represent the result of applying the operator θ to f and compute

$$(\langle 12\rangle[23]) \bullet [23] = (x_{11}\partial_{12} + x_{21}\partial_{22}) \bullet (x_{12}x_{23} - x_{22}x_{13})^2 = 2[23][13]$$

and $[23]\langle 12\rangle \bullet [23] = [23][13]$. It follows that the operators $[23]\langle 12\rangle$ and $\langle 12\rangle[23]$ are not equal in $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ and $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. So both rings are noncommutative.

2 The Hilbert series of $GrD(R)^G$ and $GrD(R^G)$

We use the method described in Sections 4.6.2–4.6.4 of Derksen and Kemper's book [3] to compute the Hilbert series of $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$. We sketch the method here, simplified to the $SL_2\mathbb{C}$ case, but the reader is referred to their book for details. Let T be a one-dimensional torus acting on an n-dimensional vector space W. The set of characters $X(T)$ is a free rank 1 group; let z be a generator of $X(T)$, which we write in multiplicative notation. After a convenient choice of basis, the action of T on W is diagonal, given by the matrix

$$\rho = \begin{pmatrix} z^{m_1} & 0 & \cdots & 0 \\ 0 & z^{m_2} & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & z^{m_n} \end{pmatrix},$$

where the m_i are integers. The character of W is defined to be the trace of this representation, $\chi^W = z^{m_1} + z^{m_2} + \cdots + z^{m_n}$, and it follows that $\dim W^T$ is the coefficient of $z^0 = 1$ in χ^W. If $W = \oplus_{d=0}^{\infty} W_d$ is a graded vector space and W_d is a rational representation of T for each d, then define the T-Hilbert series of W to be

$$H_T(W,z,t) = \sum_{d=0}^{\infty} \chi^{W_d} t^d.$$

It follows that the Hilbert series $H(W^T,t) = \sum_{d=0}^{\infty} \dim(W_d^T)t^d$ is just the coefficient of $z^0 = 1$ in $H_T(W,z,t)$.

Now fix a maximal torus T in $SL_2\mathbb{C}$ and a Borel subgroup B of $SL_2\mathbb{C}$ containing T. Because the simple positive root of $SL_2\mathbb{C}$ is just twice the fundamental weight, we find that if $W = \oplus_{d=0}^{\infty} W_d$ is a graded vector space and W_d is a rational representation of T for each d, then the Hilbert series of the invariant space $H(W^G,t)$ is just the coefficient of $z^0 = 1$ in $(1-z^2)H_T(W,z,t)$. We are omitting a significant amount of detail here, the interested reader is referred to the argument on pages 186 and 187 of [3].

Example 2.1. Let V be a two-dimensional complex vector space. If $W = GrD(\mathbb{C}[V^4])$ is made into a graded vector space using the total degree order then we can compute the Hilbert series of $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$ as follows. First, note that $GrD(\mathbb{C}[V^4]) = \mathbb{C}[V^4 \oplus (V^*)^4]$ is a polynomial ring in 16 variables $x_{11},\dots,x_{24},\xi_{11},\dots,\xi_{24}$ and that $SL_2\mathbb{C}$ contains a torus that acts on these variables with weight 1 (for x_{1i} and ξ_{2i}) or weight -1 (for x_{2i} and ξ_{1i}). Choosing the variables as a basis for the degree one part of $GrD(\mathbb{C}[V^4])$, we see that a maximal torus T acts diagonally on the degree-one part of $GrD(\mathbb{C}[V^4])$ via a diagonal matrix with eight zs and eight z^{-1}s along the diagonal. Since $W = GrD([V^4])$ is a polynomial ring generated by its degree-one variables, we find that

$$H_T(W,z,t)=\sum_{d=0}^{\infty}\chi^{W_d}t^d=\frac{1}{(1-zt)^8(1-z^{-1}t)^8}.$$

Now we see that the Hilbert series of $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$ is the coefficient of $z^0=1$ in the series expansion of

$$\frac{1-z^2}{(1-zt)^8(1-z^{-1}t)^8}.$$

To compute this, we note that the series converges if $|z^{-1}t|<1$ and $|zt|<1$. We assume that $|z|=1$ and $|t|<1$. To find the coefficient of z^0, we divide by $2\pi iz$ and integrate over the unit circle C in $\mathbb{C}$ (in the positive orientation). So

$$\begin{aligned}H(GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}},t)&=\frac{1}{2\pi i}\int_C\frac{1-z^2}{z(1-zt)^8(1-z^{-1}t)^8}\,dz\\&=\frac{1}{2\pi i}\int_C\frac{(1-z^2)z^7}{(1-zt)^8(z-t)^8}dz.\end{aligned}$$

This is the same as the sum of the residues inside C by the Residue Theorem. Since C has radius 1, and $|t|<1$, there is only one singularity of

$$\frac{(1-z^2)z^7}{(1-zt)^8(z-t)^8}$$

inside C, namely at $z=t$. The residue there is the seventh coefficient in the Taylor series expansion of

$$g(z)=\frac{(1-z^2)z^7}{(1-zt)^8}$$

about $z=t$. Computing $g^{(7)}(t)/7!$ gives

$$H(GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}},t)=\frac{1+15t^2+50t^4+50t^6+15t^8+t^{10}}{(1-t^2)^{13}}.$$

This has expansion $1+28t^2+\cdots$. The coefficient 28 refers to the 28 generators for $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$: the sixteen $\langle ij\rangle$, the six brackets $[ij]$, and the six graded versions of the $|ij|$, $\xi_{1i}\xi_{2j}-\xi_{1j}\xi_{2i}$.

Now we compute the Hilbert series of the ideal $Gr(\ker(\pi_*))$, the image of $\ker(\pi_*)$ in the graded ring $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$. This ideal is generated by the elements of $GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C}$ that are invariant under the action of $SL_2\mathbb{C}$. The left ideal $GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C}$ is generated by three operators

$$\begin{aligned}g_{12}&=x_{11}\xi_{21}+x_{12}\xi_{22}+x_{13}\xi_{23}+x_{14}\xi_{24},\\g_{21}&=x_{21}\xi_{11}+x_{22}\xi_{12}+x_{23}\xi_{13}+x_{24}\xi_{14},\\g_{11}-g_{22}&=x_{11}\xi_{11}+x_{12}\xi_{12}+x_{13}\xi_{13}+x_{14}\xi_{14}-x_{21}\xi_{21}-x_{22}\xi_{22}-x_{32}\xi_{32}-x_{42}\xi_{42}.\end{aligned}$$

These are all eigenvectors under the torus action and the torus acts with weights $2, -2$ and 0 on g_{12}, g_{21} and $g_{11} - g_{22}$, respectively. These three polynomials form a regular sequence in $GrD(\mathbb{C}[V^4])$ and we have an $SL_2\mathbb{C}$-equivariant resolution

$$\begin{aligned}0 &\to GrD(\mathbb{C}[V^4])(-6) \to GrD(\mathbb{C}[V^4])(-4)^3\\ &\to GrD(\mathbb{C}[V^4])(-2)^3 \to GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C} \to 0.\end{aligned}$$

The rightmost map sends the generators of $GrD(\mathbb{C}[V^4])(-2)^3$ to the three generators of $GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C}$ and so the three generators of $GrD(\mathbb{C}[V^4])(-2)^3$ are equipped with torus weights -2, 2, and 0. Similarly, the three generators of $GrD(\mathbb{C}[V^4])(-4)^3$ have torus weights -2, 2, and 0, and the generator of the leftmost module has torus weight 0. The T-Hilbert series of these modules are:

$$H_T(GrD(\mathbb{C}[V^4])(-6), z, t) = \frac{t^6}{(1-zt)^8(1-z^{-1}t)^8}$$

$$H_T(GrD(\mathbb{C}[V^4])(-4)^3, z, t) = \frac{t^4(z^2+1+z^{-2})}{(1-zt)^8(1-z^{-1}t)^8}$$

$$H_T(GrD(\mathbb{C}[V^4])(-2)^3, z, t) = \frac{t^2(z^2+1+z^{-2})}{(1-zt)^8(1-z^{-1}t)^8}.$$

As in Example 2.1, to find the Hilbert series of the $SL_2\mathbb{C}$-invariants, we multiply by $(1-z^2)$ and find the $z^0 = 1$ coefficient in the resulting expression. This produces the Hilbert series:

$$H(GrD(\mathbb{C}[V^4])(-6)^{SL_2\mathbb{C}}, t) = \frac{t^6+15t^8+50t^{10}+50t^{12}+15t^{14}+t^{16}}{(1-t^2)^{13}}$$

$$H([GrD(\mathbb{C}[V^4])(-4)^3]^{SL_2\mathbb{C}}, t) = \frac{36t^6+162t^8+162t^{10}+36t^{12}}{(1-t^2)^{13}}$$

$$H([GrD(\mathbb{C}[V^4])(-2)^3]^{SL_2\mathbb{C}}, t) = \frac{36t^4+162t^6+162t^8+36t^{10}}{(1-t^2)^{13}}.$$

Since the resolution was $SL_2\mathbb{C}$-equivariant, we get a resolution of the invariant modules:

$$\begin{aligned}0 &\to GrD(\mathbb{C}[V^4])(-6)^{SL_2\mathbb{C}} \to \left(GrD(\mathbb{C}[V^4])(-4)^3\right)^{SL_2\mathbb{C}}\\ &\to \left(GrD(\mathbb{C}[V^4])(-2)^3\right)^{SL_2\mathbb{C}} \to \left(GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C}\right)^{SL_2\mathbb{C}} \to 0.\end{aligned}$$

Since the alternating sum of the Hilbert series over an exact sequence is zero, we can determine the Hilbert series for $Gr\ker(\pi_*) = \left(GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C}\right)^{SL_2\mathbb{C}}$,

$$H(Gr\ker(\pi_*),t) = \frac{36t^4+162t^6+162t^8+36t^{10}}{(1-t^2)^{13}} - \frac{36t^6+162t^8+162t^{10}+36t^{12}}{(1-t^2)^{13}} + \frac{t^6+15t^8+50t^{10}+50t^{12}+15t^{14}+t^{16}}{(1-t^2)^{13}} = \frac{36t^4+127t^6+15t^8-76t^{10}+14t^{12}+15t^{14}+t^{16}}{(1-t^2)^{13}}.$$

In two previous papers [17, 18] I claimed that $\ker(\pi_*)$ was generated by the Casimir operator, an operator of total degree 4. However, the degree 4 part of this ideal is a 36-dimensional vector space (rather than a one-dimensional space), so this claim is false. In fact $\ker(\pi_*)$ requires many generators.

Finally, by taking the difference of the Hilbert series for $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$ and $Gr\ker(\pi_*)$, we find the Hilbert series for $GrD(\mathbb{C}[V^4]^{SL_2\mathbb{C}}) = GrD(G(2,4))$ in the total degree order,

$$H(GrD(G(2,4)),t) = \frac{1+18t^2+65t^4+65t^6+18t^8+t^{10}}{(1-t^2)^{10}}.$$

The form of this Hilbert series prompts us to ask whether $GrD(\mathbb{C}[V^4]^{SL_2\mathbb{C}})$ is Gorenstein. By a result due to R. Stanley (see [2, Corollary 4.4.6]) it is enough to show that $GrD(\mathbb{C}[V^4]^{SL_2\mathbb{C}})$ is Cohen–Macaulay. By the Hochster–Roberts theorem [9] (or see [10, Theorem 3.6]), $GrD(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ is always Cohen–Macaulay, but I see no particular reason for the same to be true of $GrD(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$.

Open Question. When are the graded rings $GrD(\mathbb{C}[V^n]^{SL_k\mathbb{C}}) = GrD(G(k,n))$ Gorenstein?

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