Séminaire BOURBAKI Juin 1995
47ème année, 1994-95, n° 802

CHALLENGING PROBLEMS ON AFFINE n-SPACE

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1. INTRODUCTION

There is no doubt that complex affine n-space $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{C}}$ is one of the basic objects in algebraic geometry. It is therefore surprising how little is known about its geometry and its symmetries. Although there has been some remarkable progress in the last few years, many basic problems remain open. The new interest in the subject came mainly from questions related to algebraic transformation groups and invariant theory. In fact, a number of exciting new examples (and counterexamples) were discovered in studying general group actions on affine spaces. They gave us important new insight. However, our knowledge of these spaces is still very incomplete.

Following is a list of eight basic problems in this context. Some of them—like the *Cancellation Problem* and the *Embedding Problem*—are rather well-known and have been studied by several authors. Others—such as the *Linearization Problem* and the *Complement Problem*—might appear new in this setting although they have been around since a long time. The *Jacobian Problem* is certainly the most famous one. Although it is strongly related to some of the others, we will not have the time to discuss it here in detail.

- **Characterization Problem.** Find an algebraic-geometric characterization of $\mathbb{A}^n$.
- **Cancellation Problem.** Does an isomorphism $Y \times \mathbb{A}^k \simeq \mathbb{A}^{n+k}$ imply that Y is isomorphic to $\mathbb{A}^n$?
- **Embedding Problem.** Is every closed embedding $\mathbb{A}^k \hookrightarrow \mathbb{A}^n$ equivalent to the standard embedding?
- **Automorphism Problem.** Give an algebraic description of the group of (polynomial) automorphisms of $\mathbb{A}^n$.
- **Linearization Problem.** Is every automorphism of $\mathbb{A}^n$ of finite order linearizable?

- **Complement Problem.** Given two irreducible hypersurfaces $E, F \subset \mathbb{A}^n$ and an isomorphism of their complements, does it follow that E and F are isomorphic?
- **Fixed Point Problem.** Does every reductive group action on $\mathbb{A}^n$ have fixed points?
- **Jacobian Problem.** Is every polynomial morphism $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ of maximal rank an isomorphism?

There are some obvious relations between these problems. For instance, a positive solution of the *Linearization Problem* would imply a positive solution of the *Cancellation Problem*. In fact, if $Y \times \mathbb{A}^k$ is isomorphic to an affine space $\mathbb{A}^n$ consider the automorphism of $\mathbb{A}^n \simeq Y \times \mathbb{A}^k$ given by $(y, z) \mapsto (y, -z)$. Then $Y \times \{0\}$ is the fixed point set, hence is isomorphic to $\mathbb{A}^{n-k}$ in case the automorphism is linearizable. Some more hidden connections will appear during our discussion in the next paragraphs.

All problems above can be formulated in more algebraic terms as questions about the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ considered as the algebra of regular functions on $\mathbb{A}^n$. For instance, the *Embedding Problem* for the line into the plane has the following equivalent formulation (see §4):

If $a(t), b(t) \in \mathbb{C}[t]$ are two polynomials generating the polynomial ring $\mathbb{C}[t]$ where $\deg a \leq \deg b$, then the degree of b is a multiple of the degree of a.

The complex affine line $\mathbb{A}^1 = \mathbb{C}$ is, of course, well understood. It has many nice properties which characterize it among all curves. For instance

- $\mathbb{A}^1$ *is the only affine normal rational curve without non-constant invertible functions;*
- $\mathbb{A}^1$ *is the only affine factorial curve without non-constant invertible functions;*
- $\mathbb{A}^1$ *is the only acyclic (or contractible) normal curve.*

Also we know that

- *Every non-constant morphism $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$ is a finite (ramified) covering;*
- *The automorphism group of $\mathbb{A}^1$ is the (algebraic) group of affine transformations.*

It is easy to see that with these well-known facts about $\mathbb{A}^1$ the problems formulated above have nice and satisfactory solutions. On the other hand, these properties do not hold in higher dimension and the situation becomes much more complicated. For the affine plane $\mathbb{A}^2$ there are some fundamental theorems which give a rather clear picture and an almost complete understanding of its algebraic and geometric properties. However, for larger dimension we have only very few general results. But there are several exciting new examples which shed some light into the mystery.

In the next paragraphs we give a short account on the present situation of the different problems. Some of them are already discussed in our report [Kr89a] from 1989. Since then there was some interesting development, mainly in connection with the *general Linearization Problem*. Nevertheless, the problems formulated above are still far from being solved. We are convinced that they will finally have a negative solution, at least for large dimension. Recent work on unipotent group actions even suggests possible ways to construct counterexamples.

Acknowlegment. I thank Peter Russell and Mikhail Zaidenberg for their help in preparing this report.

2. CONTRACTIBLE VARIETIES AND CHARACTERIZATION OF $\mathbb{A}^n$

Based on fundamental work by Fujita, Miyanishi and Sugie there is an important and simple *algebraic* characterization of the affine plane ([MiS80], cf. [Su89]).

Theorem 1. *Let Y be a smooth affine surface. Assume that Y is factorial and that there is a dominant morphism $\mathbb{A}^N \to Y$ for some N. Then Y is isomorphic to $\mathbb{A}^2$.*(1)

The second condition can be replaced by the two assumptions that Y has no non-constant invertible functions and that the *logarithmic Kodaira-dimension* $\bar{\kappa}(Y)$ of Y is $-\infty$. (We refer to the literature for the definition of the logarithmic Kodaira dimension; see [Ii82] Chap. 11.) The essential step in the proof is to show that Y contains a "cylinderlike" open set $U \simeq C \times \mathbb{A}^1$, C a curve (see [Su89]).

The theorem has a number of important consequences. In particular, it solves the *Cancellation Problem* in dimension 2 (see §3), even in the following slightly stronger form:

If Y is of dimension 2 and $Y \times Z \simeq \mathbb{A}^n$ for some n then $Y \simeq \mathbb{A}^2$.

But there are some other interesting applications as well. For example, a two-dimensional quotient of an action of a semisimple group G on $\mathbb{A}^n$ is isomorphic to $\mathbb{A}^2$, and the same holds for an action of a unipotent group.

We will see in Example 1 below that a similar theorem does not hold in dimension > 2. On the other hand, Miyanishi has given an interesting characterization of $\mathbb{A}^3$ [Mi87], but it is not strong enough to solve the *Cancellation Problem*. We refer to the report [Su89] of Sugie for a discussion of these results and for further references.

(1) This result holds for any algebraically closed field k if we assume that the dominant morphism is separable [Ru81].

In an important paper RAMANUJAM [Ra71] has given a beautiful *topological* characterization of $\mathbb{A}^2$ (as an algebraic surface).

Theorem 2. *An affine smooth contractible surface which is simply connected at infinity is isomorphic to $\mathbb{A}^2$. In particular, every normal affine surface which is homeomorphic to $\mathbb{A}^2$ is isomorphic to $\mathbb{A}^2$.*

This result, too, does not hold in dimension > 2. RAMANUJAM has constructed a contractible smooth affine surface R which is not isomorphic to $\mathbb{A}^2$. It follows now from h-cobordism theory that the threefold $Y := R \times \mathbb{A}^1$ is homeomorphic to $\mathbb{A}^3$ (cf. Proposition 1 below). But Y is not isomorphic to $\mathbb{A}^3$, because of Theorem 1 above. Thus there exists an *exotic algebraic structur* on $\mathbb{C}^3$.

Contractible smooth surfaces have been studied extensively by GURJAR and MIYANISHI, by TOM DIECK and PETRIE and by ZAIDENBERG (see [GuM87], [tDP89], [tDP90]). Using the examples in [GuM87] the second authors were able to describe all those of logarithmic Kodaira dimension $\bar{\kappa} = 1$, and they showed that many appear as hypersurfaces in $\mathbb{A}^3$(2). Moreover, they produced infinitely many examples with logarithmic Kodaira dimension $\bar{\kappa} = 2$. ZAIDENBERG pointed out that this leads to new exotic structures (cf. Proposition 2 of §3) and produced an infinite series of non-isomorphic exotic $\mathbb{C}^n$'s [Za91].

ZAIDENBERG also discovered the first *exotic analytic structures* on $\mathbb{C}^3$ [Za93]. He proves an analytic cancellation theorem which implies that for the RAMANUJAM surfaces R mentioned above the threefold $R\times\mathbb{C}$ is not biholomorphic to $\mathbb{A}^3$ (cf. [Ka94]). We refer to his recent report [Za95] for a thorough discussion of these results.

In this context, we should mention the following general result which seems to be due to DIMCA ([Di90]).

Proposition 1. *A smooth contractible affine variety X of dimension $d \geq 3$ is diffeomorphic to $\mathbb{C}^d$.*

The proof is based on the famous h-cobordism theorem of SMALE and uses a result of HAMM showing that the link at infinity of X is simply connected (cf. [Di92] Chap. 5, 4.25 (ii) and Chap. 1, 6.12).

An important input to the study of contractible varieties came from the *Linearization Problem* for $\mathbb{C}^*$-actions on $\mathbb{A}^3$ (see §6). It had a major influence on the

(2) It was shown recently by KALIMAN and MAKAR-LIMANOV [KaM95a] that all smooth contractible surfaces of logarithmic Kodaira dimension 1 can be realized as hypersurfaces in $\mathbb{A}^3$.

further development. It turned out that a certain class of contractible threefolds have to be understood in detail: They all appear in an explicit way as hypersurfaces in four-dimensional representation spaces of $\mathbb{C}^*$ and were the candidates for possible counterexamples. The first examples of contractible smooth hypersurfaces in $\mathbb{A}^n$ were discovered by LIBGOBER in a different context [Li77]. His list was generalized by DIMCA [Di90], by KALIMAN [Ka93] and finally completed by RUSSELL [Ru92]. Following is the simplest example from RUSSELL's list:

Example 1. Let Y be the hypersurface in $\mathbb{A}^4$ defined by the equation

$$x + x^2y + z^3 + t^2 = 0.$$

Then we have:

(a) *Y is smooth and diffeomorphic to $\mathbb{C}^3$.*

(b) *The closed subset $Z := \{x = 0\}$ is isomorphic to $C \times \mathbb{A}^1$ where C is the cusp given by $z^3 + t^2 = 0$, and the complement $Y_x := Y \setminus \{x = 0\}$ is isomorphic to $\mathbb{C}^* \times \mathbb{A}^2$.*

(c) *Y admits a $\mathbb{C}^*$-action given by $s \cdot (x, y, t, z) = (s^6x, s^{-6}y, s^2z, s^3t)$ with quotient morphism $\pi: Y \to \mathbb{A}^2$, $(x, y, t, z) \mapsto (xy, yz^3)$.*

(d) *Y is factorial and there is a dominant morphism $\mathbb{A}^3 \to Y$.*

The last assertion indicates that Y might be isomorphic to $\mathbb{A}^3$ and thus provides us with a counterexamples to the *Linearization Conjecture for $\mathbb{C}^*$-actions on $\mathbb{A}^3$* (see §6), to the *Complement Problem* and to the ABHYANKAR-SATAYE *Conjecture* about embeddings of $\mathbb{A}^2$ into $\mathbb{A}^3$ (see §4). In fact, the zero set of the function y on Y is isomorphic to $\mathbb{A}^2$ whereas the subvariety given by $y = 1$ has Euler characteristic 3. However, MAKAR-LIMANOV recently showed in a remarkable paper [Ma94] that

(e) *Y is not isomorphic to $\mathbb{A}^3$.*

Thus, the example implies that the characterization of $\mathbb{A}^2$ given in Theorem 1 does not extend to higher dimension even if we assume, in addition, that the variety is contractible.

The basic idea of MAKAR-LIMANOV was to study *locally nilpotent vector fields* on the variety Y (i.e., locally nilpotent derivations of the coordinate ring $\mathcal{O}(Y)$ of Y) and to show that they have a common kernel different from the constants $\mathbb{C}$. This is obviously impossible for $\mathbb{A}^n$. (Recall that locally nilpotent vector fields on Y are in bijective correspondence with actions of the additive group $\mathbb{C}^+$.) Generalizing this idea KALIMAN and MAKAR-LIMANOV introduced a new invariant for affine varieties, namely the subalgebra

$$\mathrm{ML}(Y) := \bigcap_{\delta} \ker \delta \subset \mathcal{O}(Y)$$

where δ runs through all locally nilpotent derivations of $\mathcal{O}(Y)$ [KaM95b]. It turned out that this invariant can be calculated in many cases. For instance, we have $\mathrm{ML}(Y) = \mathbb{C}[x]$ in Example 1 above. In fact, they show that in all examples of RUSSELL's list the invariant $\mathrm{ML}(Y)$ is strictly larger than $\mathbb{C}$. Thus none of these threefolds is isomorphic to $\mathbb{A}^3$.

In a second paper [KaM95a] KALIMAN and MAKAR-LIMANOV give another criterion to decide whether a hypersurface $H \subset \mathbb{A}^n$ is isomorphic to $\mathbb{A}^{n-1}$. It is based on the remark that the existence of a dominant morphism $\mathbb{A}^{n-1} \to H \subset \mathbb{A}^n$ implies certain restrictions on the degrees of the monomials occuring in the equation of H. The application of this criterion is rather easy. It follows again that most of the examples from RUSSELL's list are not isomorphic to $\mathbb{A}^3$.

Remark 1. It is an open problem if all vector bundles on these contractible threefolds are trivial. For the examples from RUSSELL's list there is a $\mathbb{C}^*$-action with an isolated fixed point which implies that the "zero fiber" contains an embedded line $L \simeq \mathbb{A}^1$ (see §4; in Example 1 above the line is given by $x = z = t = 0$). If every rank 2 vector bundle is trivial then, by a famous result of SERRE [Se61], the line L has to be complete intersection. i.e., defined by 2 equations. So far we have not been able to verify this.

3. CANCELLATION OF VARIETIES

The general *Cancellation Problem* was already discussed in the early 70's. It is sometimes refered to as *Zariski's Problem* although ZARISKI's question was different (see [Na67]). The problem at that time was to decide for which rings A, B an isomorphism $A[x] \simeq B[x]$ implies that A and B are isomorphic ("uniqueness of coefficient rings", see [EaH73]). It was shown by HOCHSTER [Ho72] that this fails in general. In his counterexample, he takes the coordinate ring A of the tangent bundle over the 2-sphere which is a finitely generated $\mathbb{R}$-algebra and uses the geometric fact that the tangent bundle is stably trivial, but not trivial.

A geometric formulation of the *Cancellation Problem* in dimension 2 can be found in RAMANUJAM's paper [Ra71], but his topological characterization of $\mathbb{A}^2$ (§2 Theorem 2) does not solve the problem. Only the algebraic characterization of $\mathbb{A}^2$ given later by FUJITA, MIYANISHI and SUGIE was sufficient as already mentioned in the previous paragraph (see §2 Theorem 1).

Theorem 3. ([Fu79]) *If $Y \times \mathbb{A}^k \simeq \mathbb{A}^{2+k}$ then Y is isomorphic to $\mathbb{A}^2$.*(3)

Earlier, under additional assumptions FUJITA and IITAKA [IiF77] have proved the following more general result.(4) (Since RAMANUJAM's surface R has logarithmic Kodaira dimension 2 it already follows from this proposition that $R \times \mathbb{A}^1$ is not isomorphic to $\mathbb{A}^3$.)

Proposition 2. *Let X, Y be two varieties and assume that the $\bar{\kappa}(Y) \geq 0$. Then any isomorphism $\varphi: X \times \mathbb{A}^k \xrightarrow{\sim} Y \times \mathbb{A}^k$ induces an isomorphism of X and Y.*

However, the following example which is due to DANIELEWSKI [Da89] shows that the additional assumption here is essential.

Example 2. Consider the smooth surfaces $Y_n \subset \mathbb{C}^3$ defined by the equations

$$x^n y + z^2 = 1, \quad n \in \mathbb{N}.$$

(a) *The varieties $Y_n \times \mathbb{A}^1$ are all isomorphic.*
(b) (FIESELER) *The topological spaces Y_n are not homeomorphic. In fact, $\pi_1^\infty(Y_n) \simeq \mathbb{Z}/2n$.* ($\pi_1^\infty$ denotes the fundamental group at infinity.)

In the construction of DANIELEWSKI the varieties Y_n appear as total spaces of principal $\mathbb{C}^+$-bundles over the prevariety $\tilde{\mathbb{A}}$—the affine line with a point doubled, obtained by identifying two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$. The interesting point is that these total spaces are all *affine* varieties. This immediately implies assertion (a) by forming the fiber product of two such bundles over the base $\tilde{\mathbb{A}}$ and using the fact that every principal $\mathbb{C}^+$-bundle over an affine variety is trivial (Hilbert's Theorem 90).

There is a nice geometric description of these examples given by TOM DIECK and KRAFT. Consider the subgroups $\mathbb{C}^*, \mathbb{C}^+ \subset \mathrm{SL}_2$ embedded in the usual way as diagonal and upper triangular unipotent matrices, respectively. Then the quotient $Y := \mathrm{SL}_2 / \mathbb{C}^*$ by right multiplication with $\mathbb{C}^*$ is an affine quadric ($\simeq Y_2$) on which $\mathbb{C}^+$ acts from the left. This action is locally free and determines a $\mathbb{C}^+$-bundle over $\tilde{\mathbb{A}}$. (This can be seen by first forming the quotient $\mathbb{C}^+ \backslash \mathrm{SL}_2 \simeq \mathbb{C}^2 \setminus \{0\}$ by left multiplication with $\mathbb{C}^+$ and then studying the action of $\mathbb{C}^*$.) Moreover, considering the bundle $Y \to \mathrm{SL}_2 / B \simeq \mathbb{P}^1$ where B is the subgroup of all upper triangular matrices it follows easily that Y is diffeomorphic to the line bundle $\mathcal{O}(-2)$ on $\mathbb{P}^1$ and so its fundamental group at infinity is $\mathbb{Z}/2$. (With a slight modification we also obtain the other examples Y_n of DANIELEWSKI.)

(3) This result holds for any perfect base field k (cf. [Kam80]).

(4) An interesting cancellation result for complete varieties can be found in [Fu81].

In the paper [Fi94] FIESELER studies (and classifies) $\mathbb{C}^+$-actions on normal affine surfaces. If the action is free (and the surface therefore smooth) then the geometric quotient exists as a smooth affine curve $\tilde{C}$ with several multiple points. If, in addition, all fibers are reduced then it is a principal $\mathbb{C}^+$-bundle over $\tilde{C}$. Conversely, a principal $\mathbb{C}^+$-bundle on such a non-separated curve is an affine surface if and only if the total space is separated. As before, all these surfaces become isomorphic when crossed with $\mathbb{A}^1$. In this way FIESELER obtains many new examples of the same kind as in DANIELEWSKI's example above.

Remark 2. There is an interesting example of a $\mathbb{C}^+$-action on $\mathbb{A}^5$ given by WINKELMANN [Wi90] which is a principal bundle over a non-affine variety. Following an idea of POPOV we have constructed, starting from this example, an infinite series of affine varieties Z_n of dimension 5, all total spaces of non-equivalent principal $\mathbb{C}^+$-bundles with the property that $Z_n \times \mathbb{A}^1 \simeq \mathbb{A}^6$ for all n. So far we have not been able to show that the Z_n are not isomorphic to $\mathbb{A}^5$!
Another interesting feature of WINKELMANN's example is that the quotient $\mathbb{A}^5/\mathbb{C}^+$ is a *non-affine* variety which is diffeomorphic to $\mathbb{A}^4$. It follows from this that all varieties Z_n are diffeomorphic to $\mathbb{A}^5$.

4. EMBEDDINGS OF VARIETIES AND COMPLEMENTS

The starting point of this problem was the following famous result by ABHYANKAR-MOH and SUZUKI [AbM75], [Suz74]. It first appeared with a faulty proof in a paper of SEGRE [Seg57] and was later corrected by CANALS and LLUIS [CaL70] who made a similar mistake in their arguments!

Theorem 4. *All embeddings of the line $\mathbb{A}^1$ into the plane $\mathbb{A}^2$ are equivalent.*(5)

(Here *embedding* of a variety Y means an isomorphism of Y with a closed subvariety of $\mathbb{A}^n$, and two embeddings $\alpha, \beta\colon Y \hookrightarrow \mathbb{A}^n$ are called *equivalent* if there is a (polynomial) automorphism φ of $\mathbb{A}^n$ such that $\varphi \circ \alpha = \beta$.)

The original proofs given by ABHYANKAR-MOH and SUZUKI are rather different. The first uses approximate roots and Tschirnhausen transformation whereas the second is based on subharmonic partitions. There are several new proofs of this result, using quite different methods, e.g. by RUDOLPH [Rud82] using knot theory, by MIYANISHI and SUZUKI using resolution of singularities, by GURJAR-MIYANISHI [GuM95] using

(5) The result does not hold in positive characteristic $p > 0$ as one can see from the following example of an embedded line: $L : y^{p^2} - x - x^{2p} = 0$ [Sa76].

the classification of acyclic surfaces and by A'CAMPO-OKA [AO95] using Tschirnhausen resolution towers. Before we proceed to possible generalizations of this result we give two equivalent formulations of the theorem.

(i) *If* $f\colon \mathbb{A}^2 \to \mathbb{A}^1$ *is a polynomial map such that the fiber* $f^{-1}(0)$ *is isomorphic to* $\mathbb{A}^1$, *then* f *is a trivial fibration.*

(ii) *If* $a(t), b(t) \in \mathbb{C}[t]$ *are two polynomials generating* $\mathbb{C}[t]$, *then the degree of one is a multiple of the degree of the other.*

The equivalence of Theorem 4 with (i) is easy. A priori, formulation (ii) is slightly stronger because it implies the equivalence of any two embeddings under a *tame* automorphism, i.e., an automorphism from the subgroup generated by the affine transformations and the Jonquière transformations $(x, y) \mapsto (x, y + ax^n)$ $(a \in \mathbb{C}, n \in \mathbb{N})$. But it is known since JUNG [Ju42] that in dimension 2 every automorphism is tame (see §5).

A first generalization would be to replace the line $\mathbb{A}^1$ by any other curve. This does not work in general: It is easy to see that the two embeddings $t \mapsto (t, t^{-1})$ and $t \mapsto (t^2, t^{-1})$ of $\mathbb{A}^1 \setminus \{0\}$ are not equivalent. However, it is known that the theorem generalizes to smooth curves of genus 1 or 2 with only one place at infinity, but again this fails for higher genus. An example is given by the isomorphic smooth curves $C\colon y^4 + x^3 + 1 = 0$ and $D\colon y^7 + x^2 + 1 = 0$ of genus 3 with one place at infinity which are not equivalently embedded (cf. [AO95]). We should remark here that NEUMANN used the link at infinity as a tool to classify affine plane curves [Ne89b] (cf. [Ne89a]).

It is an easy exercise to show, using Theorem 4, that all embeddings of the cross $xy = 0$ into $\mathbb{A}^2$ are equivalent. JELONEK has remarked that a negative solution of the *Embedding Problem* for the n-cross $x_1 x_2 \cdots x_n = 0$ would imply that the *Jacobian Conjecture* for $\mathbb{A}^n$ is false [Je92].

Another interesting generalization is due to LIN and ZAIDENBERG [ZaL83] (see also [GuM95]). It concerns all curves homeomorphic to $\mathbb{A}^1$ (i.e., irreducible and of Euler characteristic 1).

Theorem 5. *Let* $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^2$ *be an injective morphism. Then* φ *is equivalent to one of the maps* $t \mapsto (t^p, t^q)$ *where* p *and* q *are relatively prime. In particular, an irreducible affine plane curve of Euler characteristic 1 has at most one cusp and is given by an equation of the form* $x^q + y^p = 0$, *up to equivalence.*[6]

(6) Actually, the result is more general and includes *reducible* curves of Euler characteristic 1.

It is well-known that every smooth affine variety of dimension d can be embedded into $\mathbb{A}^{2d+1}$ and that the bound $2d+1$ is optimal (see [Sr91]. The question whether these embeddings are all equivalent was settled by a very general embedding theorem due to KALIMAN [Ka91] and NORI (unpublished, cf. [Sr91]). A weaker statement was proved independently by JELONEK [Je87].

Theorem 6. *Let Z be a smooth affine variety of dimension d. If $n \geq 2d+2$ then all embeddings of Z into $\mathbb{A}^n$ are equivalent.*(7)

As an example we see that all embeddings of the line $\mathbb{A}^1$ into $\mathbb{A}^n$ are equivalent for $n > 3$. Thus the only open case are the embeddings of the line into affine 3-space $\mathbb{A}^3$.

An important case of the *Embedding Problem* is given by the hypersurface embedding $\mathbb{A}^{n-1} \hookrightarrow \mathbb{A}^n$ (ABHYANKAR-SATHAYE Conjecture(8)). This problem occurs in the study of $\mathbb{C}^*$-actions on affine n-space in the following way (cf. §6). Let us assume that there is an isolated fixed point x_0 and consider the corresponding "zero fiber" $F_0 := \{x \in \mathbb{A}^n \mid \overline{\mathbb{C}^* x} \ni x_0\}$. (Here $\overline{\mathbb{C}^* x}$ denotes the closure of the $\mathbb{C}^*$-orbit of x.) It follows from BIAŁYNICKI-BIRULA's theorem [Bi76] (or from the slice theorem of LUNA [Lu73]) that F_0 contains a unique irreducible component H of codimension 1 which is isomorphic to $\mathbb{A}^{n-1}$ and defined by a semi-invariant polynomial f, i.e., $f(t \cdot x) = t^m f(x)$ for some $m \in \mathbb{Z}$ ($t \in \mathbb{C}^*$). If the action is *linearizable* then f is necessarily equivalent to a coordinate function, i.e., the embedding is equivalent to the standard embedding. A typical situation is described in Example 1 from §2: The threefold Y carries a $\mathbb{C}^*$-action whose zero fiber contains the 2-dimensional component $H \simeq \mathbb{A}^2$ given by the semi-invariant y of weight -6. But the action cannot be linearized because the hypersurface given by $y = 1$ has Euler characteristic 3.

Remark 3. In this context we should mention two related problems, the *Extension Problem* and the *Identity Problem*. The first one asks if every automorphism of a closed subvariety $Z \subset \mathbb{A}^n$ extends to an automorphism of $\mathbb{A}^n$. The second asks for subvarieties Z with the following property: Every automorphism of $\mathbb{A}^n$ fixing pointwise Z is the identity. These questions have been studied by JELONEK [Je91,93,94].

The *Complement Problem* is in some way the complementary problem to the embedding problem for hypersurfaces. Here we assume that we know something about

(7) The theorem is true for any infinite base field k, and there is also a formulation for singular varieties; see [Ka91], [Sr91].

(8) SATHAYE did ask a weaker question [Sa76]: Given a polynomial map $f : \mathbb{A}^n \to \mathbb{C}$ such that $f^{-1}(0)$ is isomorphic to $\mathbb{A}^{n-1}$, does it follow that all fibers are isomorphic to $\mathbb{A}^{n-1}$? (SATHAYE Conjecture)

the complement $\mathbb{A}^n \setminus H$ of a hypersurface and want to retrieve information about H itself. Clearly, there are some strong relations between the problems. For example, it is easy to see that a positive solution of the SATHAYE Conjecture would imply a positive answer to the *Complement Problem* if one of the hypersurfaces is isomorphic to $\mathbb{A}^{n-1}$.

In dimension 2 we can prove the following result:

Proposition 3. (KRAFT, VUST) *Let $C_1, C_2 \subset \mathbb{A}^2$ be two irreducible curves and assume that there is an isomorphism $\varphi: \mathbb{A}^2 \setminus C_1 \to \mathbb{A}^2 \setminus C_1$.*

(a) *If the genus of C_1 is ≥ 1, then φ extends to $\mathbb{A}^2$ inducing an isomorphism $C_1 \xrightarrow{\sim} C_2$.*

(b) *If the Euler characteristic of C_1 is one, then $C_1 \simeq C_2$.*

The assumption of irreducibility is essential as shown by the following example given by VUST: Consider the morphism $\psi: \mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (xy, y)$ and let $L := \{y = 0\}$ be the x-axis and $C := \{y^2 = x^3\}$ the cusp. Then $\psi^{-1}(L \cup C) = L \cup C'$ where C' is the smooth curve given by $x^3 y = 1$ which is disjoint from L, and ψ induces an isomorphism $\mathbb{A}^2 \setminus (L \cup C) \xrightarrow{\sim} \mathbb{A}^2 \setminus (L \cup C')$.

Another example was suggested by DERKSEN. Start with the configuration of four projective lines in $\mathbb{P}^2$ given by $xyz(x - y) = 0$. Removing the line $z = 0$ or the line $y = 0$ we obtain two different configurations of three lines in $\mathbb{A}^2$, corresponding to $xy(x - y) = 0$ and $xy(x - 1) = 0$, whose complements are isomorphic.

This problem arose in connection with the study of free actions of the additive group $\mathbb{C}^+$ on $\mathbb{A}^n$. It is conjectured that

> *Every free action of the additive $\mathbb{C}^+$ on $\mathbb{A}^3$ is equivalent to a translation action $(s, (x, y, z)) \mapsto (x + s, y, z)$.*

We can prove this under the additional assumption that the action is separated.

5. AUTOMORPHISMS OF AFFINE n-SPACE

The structure of the automorphism group of $\mathbb{A}^n$—the *affine Cremona group*—still remains a mystery. Only in dimension 2 is there a satisfactory description (besides the trivial case $n = 1$). For the following discussion let us introduce some notation. First recall that a polynomial map $\varphi = (\varphi_1, \ldots, \varphi_n): \mathbb{A}^n \to \mathbb{A}^n$, $\varphi_i \in \mathbb{C}[x_1, \ldots, x_n]$, is an isomorphism (i.e., has a polynomial inverse) if and only if it is bijective. This is equivalent to the condition that the polynomials φ_i generate the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$.

We denote by $\mathcal{G}_n$ the group all polynomial automorphism of $\mathbb{A}^n$ and define the two subgroups $\mathcal{A}_n$ of *affine transformation* and $\mathcal{J}_n$ of *triangular transformation* (also called the *Jonquière subgroup*) in the following way:

$$\mathcal{A}_n := \{\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathcal{G}_n \mid \varphi_i \text{ of degree 1 for all } i\},$$

$$\mathcal{J}_n := \{\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathcal{G}_n \mid \varphi_i \in \mathbb{C}[x_1, \ldots, x_i] \text{ for all } i\}.$$

Clearly, $\mathcal{A}_n$ is the semidirect product of GL_n with the subgroup $\mathcal{T}_n$ of translations. Also, $\mathcal{A}_n$ is an algebraic group whereas $\mathcal{J}_n$ is infinite dimensional (for $n > 1$).

In dimension 2 the structure of $\mathcal{G}_2$ is given by the following theorem which goes back to van der Kulk [Ku53]. Another proof, based on the theory of trees, was given by Danilov-Gizatullin [GiD75].

Theorem 7. *The automorphism group $\mathcal{G}_2$ is the amalgamated product $\mathcal{A}_2 *_{\mathcal{B}_2} \mathcal{J}_2$ where $\mathcal{B}_2 := \mathcal{A}_2 \cap \mathcal{J}_2$.*[9]

The theorem claims that every automorphism φ has a decomposition of the form

$$\varphi = \alpha_1\gamma_1\alpha_2\gamma_2 \cdots \alpha_k\gamma_k, \quad \alpha_i \in \mathcal{A}_2,\ \gamma_i \in \mathcal{J}_2$$

which is unique modulo the obvious relations $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ and $(\gamma\beta)\alpha = \gamma(\beta\alpha)$ for $\alpha \in \mathcal{A}_2$, $\beta \in \mathcal{B}_2$, $\gamma \in \mathcal{J}_2$. In particular, every element has a well defined *length*, namely the minimal number of elements from $\mathcal{A}_2 \cup \mathcal{J}_2$ needed to express it as a product.

As an immediate consequence we get a positive answer to the *Linearization Problem* in dimension 2. In fact, if a power of an element $\varphi = \alpha_1\gamma_1\alpha_2\gamma_2 \cdots \alpha_k\gamma_k$ is the identity then there must be some cancellation in the product

$$\varphi\varphi = \alpha_1\gamma_1 \cdots \alpha_k\gamma_k\alpha_1\gamma_1 \cdots \alpha_k\gamma_k.$$

This is possible only if either $\beta := \alpha_k\gamma_k\alpha_1 \in \mathcal{B}_2$ or $\beta' := \gamma_k\alpha_1\gamma_1 \in \mathcal{B}_2$. In the first case the conjugate element $\alpha_1^{-1}\varphi\alpha = \gamma_1\alpha_2 \cdots \alpha_{k-1}\gamma_{k-1}(\alpha_k\beta)$ has shorter length than φ and similarly in the second case. Now it follows easily by induction that φ is conjugate to an element of GL_2.

More generally, one knows by a result due to Serre (see [Se80]) that every subgroup of bounded length in an amalgamated product is conjugate to a subgroup of one of the factors. It was shown by Wright [Wr79] that an *algebraic* subgroup of $\mathcal{G}_2$ (i.e., the image of an algebraic group G acting algebraically on $\mathbb{A}^2$) is of bounded length. Hence we obtain the following corollary (see [Kam79]):

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(9) This holds for any base field k.

Corollary 1. *Every algebraic subgroup G of $\mathcal{G}_2$ is conjugate to a subgroup of $\mathcal{A}_2$ or of $\mathcal{J}_2$. In particular, every reductive subgroup of $\mathcal{G}_2$ is conjugate to a subgroup of* GL_2.

A second application is the following result which goes back to RENTSCHLER [Re68].

Corollary 2. *Every locally nilpotent vector field on $\mathbb{A}^2$ can be triangularized, i.e., is equivalent to one of the form $f(x)\dfrac{\partial}{\partial y}$ where $f(x)$ is a polynomial.*

Remark 4. Another consequence of the structure theorem is the non-existence of non-trivial forms of $\mathbb{A}^2 = \mathbb{A}^2_{\mathbb{C}}$ (see [Sh66]). More precisely, if $k \subset \mathbb{C}$ is any subfield and A a k-algebra such that $\mathbb{C} \otimes_k A \simeq \mathbb{C}[x, y]$, then $A \simeq k[x, y]$ as a k-algebra.(10) Again, this problem is completely open in dimension ≥ 3. We even do not know if $\mathbb{A}^3_{\mathbb{R}}$ is the only real form of $\mathbb{A}^3_{\mathbb{C}}$.

It is well-known that a similar amalgamated product structure as in Theorem 7 does not exist in dimension $n \geq 3$. For example, consider the following two automophisms of $\mathbb{A}^3$:

$$\sigma(x, y, z) = (y, x, z) \quad \text{and} \quad \tau(x, y, z) = (x, y, z + x^2).$$

Then $\sigma \in \mathcal{A}_3$, $\tau \in \mathcal{J}_3$ and $\sigma, \tau \notin \mathcal{A}_3 \cap \mathcal{J}_3$. The composition $\sigma \circ \tau \circ \sigma$ maps (x, y, z) to $(x, y, z + y^2)$, hence $\sigma \circ \tau \circ \sigma \in \mathcal{J}_3$ which contradicts the uniqueness of the decomposition. It is also known that Corollary 2 does not generalize to higher dimension: Consider the locally nilpotent vector filed $D := (xz + y^2)(x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z})$ given by BASS [Ba84]. Its zero set is given by $xz + y^2 = 0$ and has an isolated singularity at 0. Hence D cannot be put into triangular form.

The subgroup of $\mathcal{G}_n$ generated by $\mathcal{A}_n$ and $\mathcal{J}_n$ is called the group of *tame* automorphisms. We do not know if every automorphism of $\mathbb{A}^n$ is tame. But it is interesting to remark here that the main result about embeddings (§4, Theorem 6) holds for the subgroup of tame automorphism.

6. LINEARIZATION OF ALGEBRAIC GROUP ACTIONS

The *Linearization Problem* was originally formulated for reductive group actions on affine space (see [Kam79]):

> *Given a reductive algebraic group G acting on affine n-space $\mathbb{A}^n$, can one always find a polynomial change of coordinates such that the action becomes linear?*

(10) This holds for any perfect field k, cf. [Kam75].

Of course, a necessary condition is the existence of a fixed point. More precisely, the fixed point set has to be an affine space $\mathbb{A}^d$ whose embedding into $\mathbb{A}^n$ must be equivalent to a linear one. It is not difficult to see that every non-linearly reductive group admits an action on some affine space without fixed points (see [KrP85]).

The first results here looked very promising. Every such action on $\mathbb{A}^2$ is linearizable as a consequence of the structure theorem (§5, Corollary 1 of Theorem 7), any torus action with an orbit of codimension one is linearizable by BIAŁYNICKI-BIRULA [Bi66/67] (see [KaR82] for related results), and for $\mathbb{A}^3$ and $\mathbb{A}^4$ it was shown by KRAFT-POPOV and PANYUSHEV that every semisimple group action is linearizable [KrP85], [Pa84]. We refer to [Kr89a, §5] for more details and further references.

In 1989 SCHWARZ discovered the first counterexamples, namely non-linearizable actions of the orthogonal group O_2 on $\mathbb{A}^4$ and of SL_2 on $\mathbb{A}^7$ ([Sc89], see [KrS89/92]). Using these results KNOP showed that every connected reductive group which is not a torus admits a faithful non-linearizable action on some affine space $\mathbb{A}^n$ [Kn91]. Using a different approach, MASUDA, MOSER-JAUSLIN and PETRIE produced more examples and discovered the first non-linearizable actions of finite groups, e.g., for dihedral groups of order ≥ 10 on $\mathbb{A}^4$ (see [MaP91] and [MMP91]).

So far, all examples of non-linearizable actions have been obtained from non-trivial G-vector bundles on representation spaces V of G by using an idea of BASS and HABOUSH ([BaH87], see [Kr89b]). As usual, a *G-vector bundle* on a G-variety Y is an algebraic vector bundle $p: \mathcal{V} \to Y$ together with an action of G such that the projection p is G-equivariant and the action is linear on the fibers. A G-vector bundle is called *trivial* if it is isomorphic to a bundle of the form $Y \oplus W \xrightarrow{\text{pr}} Y$ where W is a G-representation.

Since every algebraic vector bundle on affine n-space is trivial by the famous theorem of QUILLEN and SUSLIN, and hence has an affine space as its total space, non-trivial G-vector bundles on representation spaces V of G provide us with interesting G-actions on affine space. In fact, many of these turned out to be non-linearizable! Recently, MEDERER constructed non-trivial equivariant vector bundles for the symmetric group S_3 over $\mathbb{A}^2$. Moreover, he shows that the "moduli space" is infinite dimensional [Me95].

So far there are no counterexamples to the *Linearization Problem* for commutative reductive groups, in particular for tori and for automorphisms of finite order.(11) Moreover, the following result of MASUDA, MOSER-JAUSLIN and PETRIE [MMP95]

(11) In positive characteristic there are counterexamples by ASANUMA [As94].

shows that our previous approach via G-vector bundles cannot produce counterexamples here. (See [KrS95] for a more geometric proof.)

Theorem 8. *Let G be a commutative reductive group (i.e., a product of a torus and a finite commutative group) and let V be a representation of G. Then every G-vector bundle on V is trivial.*

An essential ingredient in the proof is the theorem of GUBELADZE [Gu88] saying that every vector bundle on a normal affine toric variety is trivial. Here, it can be applied to the "algebraic quotient" $V/\!\!/G$ (= the maximal spectrum of the invariant ring) since G is commutative.

In the last few years a lot of work has been done in the first open case, namely for $\mathbb{C}^*$-actions on $\mathbb{A}^3$. We have already seen in §2 and §4 that this problem is strongly related to the *Characterization Problem* and the *Embedding Problem*. A number of special cases have been settled earlier (see [KoR86/89ab], [Kr90]), some others led to a list of acyclic hypersurfaces in $\mathbb{A}^4$ as possible counterexamples, one of them being Example 1 in §2. As already mentioned earlier, KALIMAN and MAKAR-LIMANOV have been able to prove that all these hypersurfaces are non-isomorphic to $\mathbb{A}^3$. It is now reasonable to assume that the following conjecture will be proved in the near future. We refer to the forthcoming report [KoR95] of KORAS and RUSSELL for a thorough investigation of the problem.

Conjecture. *Every $\mathbb{C}^*$-action on $\mathbb{A}^3$ is linearizable.*

On the other hand, there is absolutely no progress concerning automorphisms of finite order of $\mathbb{A}^3$. For example,

- *Is every involution of $\mathbb{A}^3$ with an isolated fixed point equivalent to* -id*?*
- *Is every involution of $\mathbb{A}^3$ whose fixed point set is of dimension 2 equivalent to a reflection on a linear plane?*

In their approach to the *Linearization Problem* KRAFT and SCHWARZ [KrS92] realized that all non-trivial G-vector bundles became trivial (and the corresponding action linearizable) if one allows *holomorphic* changes of coordinates. The explanation of this phenomena was given by a general equivariant Oka-principle proved by HEINZNER and KUTZSCHEBAUCH [HeK94]:

Theorem 9. *Let V be a representation of a complex reductive group. Then every holomorphic G-vector bundle on V is trivial.*

As a consequence, all examples of non-linearizable actions obtained so far become linearizable if we allow holomorphic changes of coordinates. Thus the linearization problem in the holomorphic setting is completely open.

Concerning the *Fixed Point Problem* there are a number of results from topological transformation groups which can be applied to the algebraic situation. E.g., for every action of a *torus* $T = \mathbb{C}^{*m}$ on $\mathbb{A}^n$ the fixed point set is an acyclic smooth subvariety and in particular non-empty. One also knows that every *finite cyclic group* acting on $\mathbb{A}^n$ has fixed points (see [PeR86]), a result which does not hold in the topological setting. It is based on the existence of an equivariant completion which allows to apply Lefschetz type arguments. For more details we refer again to the survey [Kr89a, §3]. So far, we have not been able to construct reductive group actions on affine n-space without fixed points although we believe that such actions exist for all semisimple groups. But there is an interesting example over $\mathbb{R}$ given by Dovermann, Masuda and Petrie [DMP89]:

> *For every $n \geq 24$ there exists an effective fixed point free real algebraic action of the icosahedral group on a real algebraic variety diffeomorphic to $\mathbb{R}^n$.*

Substantial progress was made recently by Fankhauser [Fa95]. He was able to extend several results of Hsiang and Straume [HsS86] about compact Lie group actions on acyclic manifolds to the algebraic setting. Among other things he shows:

> *There exist always fixed points provided the algebraic quotient $\mathbb{A}^n /\!\!/ G$ has at most dimension 3 or is small compared with the rank of G.*

We should point out that all these results about fixed points hold more generally for actions on *acyclic smooth affine varieties* and do not use the fact that the underlying variety is affine n-space.

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