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The Geometry of Representations of A_m

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Introduction

0.1. Let $V_1, V_2, \ldots, V_m$ be finite dimensional vector spaces over some algebraically closed field k of characteristic zero. Consider the space

$$L := \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_3) \times \ldots \times \mathrm{Hom}(V_{m-1}, V_m)$$

whose elements

$$A : V_1 \xrightarrow{A_1} V_2 \xrightarrow{A_2} V_3 \longrightarrow \ldots \longrightarrow V_{m-1} \xrightarrow{A_{m-1}} V_m$$

can be viewed as representations of a certain fixed dimension of the Dynkin quiver A_m (oriented in the obvious way; cf. [2]). The group

$$G := \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \ldots \times \mathrm{GL}(V_m)$$

acts in a natural way on L and the orbits are just the equivalence classes of these representations: in particular there are only a finite number of orbits in L. It is an interesting task to describe the *degenerations* of these orbits and to determine the *singularities* in the closure $\bar{O}$ of a given orbit O. The first problem has been solved in [1] (cf. also 2.2 and 9.1). If $A \in L$ is a *complex* (i.e. $A_{i+1}A_i = 0$ for all i), Kempf has shown in [8] that the closure $\bar{O}_A$ of its orbit O_A has *rational singularities* (cf. 1.4 for definitions). We will prove this result in general.

0.2. Theorem. *For any $A \in L$ the closure $\bar{O}_A$ of its orbit is normal, Cohen-Macaulay with rational singularities.*

Our method is similar to the one developed in [13] and is based on the first fundamental theorem of invariant theory (1.2) (compare also with [10]). To the

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element $A=(A_1,\ldots,A_m)\in L$ we associate two other elements (5.4, basic construction):

$$\tilde{A}\in\tilde{L}:=\mathrm{Hom}(V_1,U_1)\times\mathrm{Hom}(U_1,V_2)\times\ldots\times\mathrm{Hom}(U_{m-1},V_m)$$

$$\tilde{A}:V_1\xrightarrow{\bar{A}_1}\!\!\!\rightarrow U_1\overset{I_1}{\hookrightarrow}V_2\xrightarrow{\bar{A}_2}\!\!\!\rightarrow U_2\overset{I_2}{\hookrightarrow}\ldots\ldots\xrightarrow{\bar{A}_{m-1}}\!\!\!\rightarrow U_{m-1}\xhookrightarrow{I_{m-1}}V_m$$

where $U_i:=\mathrm{Im}\,A_i\subset V_{i+1}$ and $A_i=I_i\bar{A}_i:V_i\twoheadrightarrow U_i\hookrightarrow V_{i+1}$ is the canonical decomposition, and

$$A'\in L':=\mathrm{Hom}(U_1,U_2)\times\mathrm{Hom}(U_2,U_3)\times\ldots\times\mathrm{Hom}(U_{m-2},U_{m-1})$$

$$A':U_1\xrightarrow{A'_1}U_2\xrightarrow{A'_2}U_3\longrightarrow\ldots\xrightarrow{A'_{m-2}}U_{m-1}$$

$A'_i=\bar{A}_{i+1}I_i$. Now we consider the two quotient maps (5.1, first fundamental theorem)

$$\begin{array}{ccc}\tilde{L} & \xrightarrow{\pi} & L' \\ \big\downarrow{\scriptstyle\varrho} & & \\ L & & \end{array}$$

defined by "composing":

$$\pi(B_1,C_1,B_2,C_2,\ldots)=(B_2C_1,B_3C_2,\ldots),$$
$$\varrho(B_1,C_1,B_2,C_2,\ldots)=(C_1B_1,C_2B_2,\ldots).$$

By construction we have $\pi(\tilde{A})=A'$, $\varrho(\tilde{A})=A$, and we will show (Proposition 5.5)

a) $\pi^{-1}(\bar{O}_{A'})=\bar{O}_{\tilde{A}}$ and $\mathrm{codim}_{\tilde{L}}\bar{O}_{\tilde{A}}=\mathrm{codim}_{L'}\bar{O}_{A'}$,
b) $\varrho(\bar{O}_{\tilde{A}})=\bar{O}_A$.

By induction (on the length m of the quiver) we may assume that $\bar{O}_{A'}$ is Cohen-Macaulay with rational singularities. This implies by a) that $\bar{O}_{\tilde{A}}$ is Cohen-Macaulay. Using a natural desingularization of $\bar{O}_{\tilde{A}}$ (Section 6) and a result of Kempf (Corollary 1.4) we are able to prove that $\bar{O}_{\tilde{A}}$ has rational singularities too (Proposition 7.1). Since ϱ is a quotient map and $\varrho(\bar{O}_{\tilde{A}})=\bar{O}_A$ by b), we get from Boutot's theorem (1.3b) that $\bar{O}_A$ is normal, Cohen-Macaulay with rational singularities.

0.3. Our second main theorem deals with the singularities arising in the closure $\bar{O}$ of an orbit $O\subsetneq L$. We describe the type (up to smooth equivalence) of the singularity of $\bar{O}$ in a *minimal degeneration* O', i.e. O' is an open orbit in $\bar{O}-O$. Using the description of the degenerations given in [1] we associate in a purely combinatorial way to each minimal degeneration $O'\subset\bar{O}$ a pair (p,q) of natural numbers, called the *index of the minimal degeneration* (9.1).

Theorem. *Let $O'\subset\bar{O}$ be a minimal degeneration of index (p,q). Then O' is of codimension $p+q-1$ in $\bar{O}$ and the singularity of $\bar{O}$ in O' is smoothly equivalent to the isolated singularity of the determinantal variety $D_{p,q}:=\{A\in M_{p,q}\mid\mathrm{rk}\,A\leqq 1\}$ in the origine.*

Our method here is similar to the one in [11]. We describe the orbits in L by a *diagram of boxes* (2.3), the rows representing the indecomposable factors of the corresponding representation. Given two orbits O_υ and O_λ, $O_\upsilon \subset \bar{O}_\lambda$, with associated diagram of boxes υ and λ, we give some reduction procedures which enable us to remove (under certain conditions) common rows (10.2) and common columns (10.1) to obtain a new pair of orbits $O_{\upsilon'}$ and $O_{\lambda'}$ in some smaller space L' with the same type of singularity of $\bar{O}_{\lambda'}$ in $O_{\upsilon'}$ as $\bar{O}_\lambda$ in O_υ. In case of a minimal degeneration of index (p,q) this procedure will end up in the pair $O_{\lambda'} = D_{p,q} - \{0\}$, $O_{\upsilon'} = \{0\}$, proving the claim (10.4).

1. Notations and Preliminaries

1.1. *Notations.* We always work over an algebraically closed field k of characteristic zero. For finite dimensional k-vector spaces $V_1, V_2, \ldots, V_m$ we define

$$L(V_1, V_2, \ldots, V_m):$$
$$= \operatorname{Hom}_k(V_1, V_2) \times \operatorname{Hom}_k(V_2, V_3) \times \ldots \times \operatorname{Hom}_k(V_{m-1}, V_m)$$

and

$$G(V_1, V_2, \ldots, V_m) := \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \ldots \times \mathrm{GL}(V_m).$$

This group acts in a natural way on $L(V_1, \ldots, V_m)$. For any $A \in L(V_1, \ldots, V_m)$ we denote by O_A its orbit in $L(V_1, \ldots, V_m)$.

1.2. *Quotients, First Fundamental Theorem.* If a reductive group G acts regularly on an affine variety Z a regular map $\pi: Z \to X$ from Z into some affine variety X will be called a *quotient with respect to* G if π induces a surjective map from the regular functions on X onto the G-invariant functions on Z.

The *first fundamental theorem* of invariant theory tells us (cf. [15, II.6, Theorem 2.6A] or [14, Par. 3, Théorème 3]) that *the map*

$$\pi: \operatorname{Hom}(U, V) \times \operatorname{Hom}(V, W) \to \operatorname{Hom}(U, W)$$

given by $(A, B) \mapsto BA$ *is a quotient with respect to* $\mathrm{GL}(V)$.

1.3. *Some Properties of Quotients.* Let $\pi: Z \to X$ be a quotient map with respect to a reductive group G.

a) *If* $Y \subseteq Z$ *is G-stable and closed then* $\pi(Y) \subseteq X$ *is closed and* $\pi|_Y: Y \to X$ *is a quotient* [12, Chap. I, Sect. 2].

In particular we remark that if in addition Y is *normal* $\pi(Y)$ *is normal* too.

b) Boutot's Theorem: *If* Z *has rational singularities* (1.4) $\pi(Z)$ *has rational singularities too* (cf. [6])

1.4. *Rational Singularities.* Let X be an irreducible variety and $\varphi: Y \to X$ a *resolution of singularities*, i.e. Y is smooth and φ is birational and proper. One says that X has *rational singularities* if a) *X is normal and* b) *the direct images* $R^i\varphi_* \mathcal{O}_Y$ *of* *the structure sheaf $\mathcal{O}_Y$ vanish for $i>0$.* (One can show that this definition does not depend on the resolution φ.) One has the following result due to Kempf:

Proposition (cf. [7, p. 50, Proposition]). *Let $\varphi: Y \to X$ be a resolution of singularities, $n := \dim X$. The following statements are equivalent:*

(i) *X has rational singularities.*

(ii) *X is normal and Cohen-Macaulay and for every n-form ω defined on the smooth part of X the form $\varphi^*\omega$ extends to the whole variety Y.*

From this we immediately deduce the following criterion, which will be very useful for our task.

Corollary. *Let X be an irreducible Cohen-Macaulay variety and $\varphi: Y \to X$ a resolution of singularities. Assume that there is a closed subset $X' \subset X$ such that X has rational singularities in $X - X'$ and $\varphi^{-1}(X')$ has codimension at least 2 in Y. Then X has rational singularities.*

(In fact X is normal in $X - X'$, hence normal by Serre's criterion [4, IV, Théorème 5.8.6]. Given any n-form ω defined outside the singular locus of X, we can first extend $\varphi^*\omega$ to $\varphi^{-1}(X - X')$ since $X - X'$ has rational singularities, and then to the whole variety Y, since $\varphi^{-1}(X')$ has codimension at least 2.)

2. Degenerations of Orbits in L

2.1. We recall here some facts proved in [1] (cf. also [2]). To each element $A = (A_1, \ldots, A_{m-1}) \in L := L(V_1, \ldots, V_m)$ we associate an indexed set of integers $n^A = \{n^A_{(i,j)}\}_{1 \leqq i \leqq j \leqq m}$ defined by

$$n^A_{(i,j)} := \begin{cases} \operatorname{rk}(A_{j-1} \ldots A_{i+1} A_i) & \text{for } i<j \\ \dim V_i & \text{for } i=j. \end{cases}$$

It is clear that n^A depends only on the orbit O_A.

2.2. Proposition [1, Proposition 3.1]. *For $A, B \in L$ we have $O_B \subseteq \bar{O}_A$ if and only if $n^B_{(i,j)} \leqq n^A_{(i,j)}$ for all $1 \leqq i \leqq j \leqq n$.*

In particular the orbit O_A is determined by n^A.

2.3. We also introduce the following indexed set of integers $\lambda^A = (\lambda^A_{(i,j)})_{1 \leqq i \leqq j \leqq m}$:

$$\lambda^A_{(i,j)} := n^A_{(i,j)} - n^A_{(i-1,j)} - n^A_{(i,j+1)} + n^A_{(i-1,j+1)}.$$

(We put $n^A_{(r,s)} = 0$ if $r<1$ or $s>m$.)

These numbers have the following interpretation (cf. [1]): *Each representation $A \in L$ has a unique decomposition into a direct sum $A = \oplus E_\tau$ of indecomposable representations E_τ which are of the form*

$$E_{(i,j)}: \underset{1}{0} \longrightarrow \underset{2}{0} \longrightarrow \ldots \longrightarrow \underset{i}{k} \xrightarrow{\text{id}} k \xrightarrow{\text{id}} \ldots \xrightarrow{\text{id}} \underset{j}{k} \longrightarrow 0 \longrightarrow \ldots \longrightarrow \underset{m}{0}$$

for some $1 \leqq i \leqq j \leqq m$ (cf. [2, 2.2]). Now *$\lambda^A_{(i,j)}$ is the number of indecomposable factors E_τ of A of type $E_{(i,j)}$.*

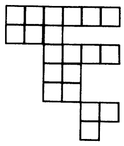
Clearly n^A is determined by λ^A:

$$n^A_{(i,j)} = \sum_{r \leqq i \leqq j \leqq s} \lambda^A_{(r,s)}.$$

It is convenient to represent λ^A as a "*diagram of boxes*", each row starting at i and ending at j stands for one indecomposable factor of type $E_{(i,j)}$.

E.g. the following diagram represents λ^A for A isomorphic to

$$E_{(1,6)} \oplus E_{(1,3)} \oplus E_{(3,6)} \oplus E_{(3,4)} \oplus E_{(3,4)} \oplus E_{(5,6)} \oplus E_{(5,5)}:$$



2.4. Conversely any indexed set $\lambda = (\lambda_{(i,j)})_{1 \leqq i \leqq j \leqq m}$ of natural numbers determines an orbit in $L(V_1, V_2, \ldots, V_m)$ provided $\dim V_i = \hat{\lambda}_i := \sum_{r \leqq i \leqq s} \lambda_{(r,s)}$ ($= \#$ boxes in the i^{th} column of λ). We will shortly call such an indexed set a *diagram*, define $\dim \lambda := (\hat{\lambda}_1, \ldots \hat{\lambda}_m) \in \mathbb{N}^m$ to be its *dimension* and denote by O_λ the *corresponding orbit*. For any $A \in O_\lambda$ we have by definition

$$n^A_{(i,j)} = n^\lambda_{(i,j)} := \sum_{r \leqq i \leqq j \leqq s} \lambda_{(r,s)}.$$

In this way we get a bijection between the orbits in $L(V_1, \ldots, V_m)$ and the diagrams of dimension $(\dim V_1, \ldots, \dim V_m)$.

If O_ν, O_λ are orbits in $L(V_1, \ldots, V_m)$, *we say* $\nu \leqq \lambda$ *if* $O_\nu \subseteq \bar{O}_\lambda$ i.e. if $n^\nu_{(i,j)} \leqq n^\lambda_{(i,j)}$ for all (i,j) (cf. Proposition 2.2). This define an *ordering* on the set of all diagrams of a fixed dimension.

3. Dimension Formula for Stabilizers

3.1. For any $A \in L := L(V_1, \ldots, V_m)$ we denote by $\operatorname{Stab} A$ the *stabilizer* of A in the group $G := G(V_1, \ldots, V_m)$:

$$\operatorname{Stab} A = \{g \in G \mid gA = A\}.$$

It is easy to see that its Lie algebra $\operatorname{Lie}\operatorname{Stab} A \subset \operatorname{End}(V_1) \times \ldots \times \operatorname{End}(V_m)$ is the *endomorphism ring* of A:

$$\operatorname{Lie}\operatorname{Stab} A = \operatorname{End}_G A.$$

Furthermore we have

$$\operatorname{Hom}_G(E_{(r,s)}, E_{(i,j)}) \cong \begin{cases} k & i \leqq r \leqq j \leqq s \\ 0 & \text{otherwise} \end{cases}$$

for the indecomposable representations $E_{(i,j)}$ (2.3). Using the decomposition of A into indecomposable factors this proves the first part of the following lemma; the second follows by direct computation from the definition of $\lambda^A_{(i,j)}$ in 2.3.

3.2. Lemma.

$$\begin{aligned}\dim \operatorname{Stab} A &= \sum_{1 \leqq i \leqq r \leqq j \leqq s \leqq m} \lambda^A_{(i,j)} \cdot \lambda^A_{(r,s)} \\ &= \sum_{1 \leqq i \leqq j \leqq m} (n^A_{(i,j)} - n^A_{(i,j+1)})(n^A_{(i,j)} - n^A_{(i-1,j)}).\end{aligned}$$

4. An Induction Lemma

4.1. Let $W_1, W_2, \ldots, W_n$ be finite dimensional vector spaces, $n > 2$, and fix an integer $t \in \{2, 3, \ldots, n-1\}$. Consider the map

$$\pi := \pi_t : L := L(W_1, \ldots, W_n) \to L' := L(W_1, \ldots, \hat{W}_t, \ldots, W_n)$$

given by $A = (A_1, \ldots, A_{n-1}) \mapsto (A_1, \ldots, A_t A_{t-1}, \ldots, A_{n-1})$ i.e. by composing at the t^{th} point. This is a *quotient map* with respect to $\mathrm{GL}(W_t)$ (first fundamental theorem, cf. 1.2).

4.2. Proposition. *Assume* $\dim W_{t-1}, \dim W_{t+1} \leqq \dim W_t$.

a) *The map* π *is smooth in* $L^0 := \{A = (A_1, \ldots, A_{n-1}) \in L \mid A_{t-1}$ or A_t has maximal rank$\}$.

For any $A = (A_1, \ldots, A_{n-1}) \in L$ *such that* A_{t-1} *and* A_t *have maximal rank we have*

b) $\dim \operatorname{Stab} A = \dim \operatorname{Stab} \pi(A) + (\dim W_t - \operatorname{rk} A_t) \cdot (\dim W_t - \operatorname{rk} A_{t-1})$ *and*

c) $\pi^{-1}(\pi(\bar{O}_A)) = \bar{O}_A$ *and* $\operatorname{codim}_L \bar{O}_A = \operatorname{codim}_{L'} \bar{O}_{\pi(A)}$.

Proof. a) The tangent map of π in $A \in L$ is given by

$$(d\pi)_A(X_1, \ldots, X_{n-1}) = (X_1, \ldots, X_t A_{t-1} + A_t X_{t-1}, \ldots, X_{n-1}).$$

Hence $(d\pi)_A$ is surjective if either A_{t-1} is injective or A_t is surjective, i.e. if $A \in L^0$.

b) It is easy to describe the decomposition of $\pi(A)$ into indecomposable factors from the decomposition of A. For this let us label the quiver corresponding to L' by

$$\underset{1}{\bullet} \to \underset{2}{\bullet} \to \ldots \to \underset{t-1}{\bullet} \to \underset{t+1}{\bullet} \to \ldots \to \underset{n}{\bullet}$$

and define for $i \in \{1, 2, \ldots, n\}$ the integers i' and i'' by $i' = i'' = i$ for $i \neq t$ and $t' = t+1$, $t'' = t-1$. Then the factors of A of type $E_{(t,t)}$ disappear and a factor of type $E_{(i,j)}$ of A with $(i,j) \neq (t,t)$ becomes a factor of type $E_{(i',j'')}$ of $\pi(A)$. Since by assumption A_{t-1} is injective and A_t is surjective, we don't have factors of type $E_{r,t-1}$ or $E_{t+1,j}$ in A, and therefore $\lambda^A_{(r,t-1)} = \lambda^A_{(t+1,j)} = 0$. It follows that

$$\sum_{\substack{i \leqq r \leqq j \leqq s \\ (r,j) \neq (t,t)}} \lambda^A_{(i,j)} \lambda^A_{(r,s)} = \sum_{i' \leqq r' \leqq j'' \leqq s''} \lambda^{\pi(A)}_{(i',j'')} \lambda^{\pi(A)}_{(r',s'')}.$$

Hence from the dimension formula 3.2 we get

$$\dim \operatorname{Stab} A = \dim \operatorname{Stab} \pi(A) + \sum_{\substack{1 \leqq i \leqq t \\ t \leqq s \leqq n}} \lambda^A_{(i,t)} \cdot \lambda^A_{(t,s)}.$$

The last sum is equal to $\left(\sum_{i \leqq t} \lambda^A_{(i,t)}\right) \cdot \left(\sum_{t \leqq s} \lambda^A_{(t,s)}\right)$ and it is easy to see that

$$\sum_{i \leqq t} \lambda^A_{(i,t)} = \dim \operatorname{Ker} A_t = \dim W_t - \operatorname{rk} A_t$$

and

$$\sum_{t \leqq s} \lambda^A_{(t,s)} = \operatorname{codim} \operatorname{Im} A_{t-1} = \dim W_t - \operatorname{rk} A_{t-1}.$$

c) We have $\pi(O_A) = O_{\pi(A)}$, hence $\pi(\bar{O}_A) = \bar{O}_{\pi(A)}$ by 1.3a). Furthermore any $C \in L$ satisfies $n^C_{(i,j)} = n^{\pi(C)}_{(i,j)}$ for $i \neq t$, $j \neq t$, $i \leqq j$, and by assumption $n^A_{(i,t)} = n^A_{(i,t-1)}$, $n^A_{(t,j)} = n^A_{(t+1,j)}$ for $i < t$ and $j > t$. Now for $B \in L$ with $\pi(B) \in \bar{O}_{\pi(A)}$ we get for all (i,j) $n^{\pi(B)}_{(i,j)} \leqq n^{\pi(A)}_{(i,j)}$ by 2.2. Hence $n^B_{(i,j)} \leqq n^A_{(i,j)}$ for $i \neq t$ and $j \neq t$ and therefore

$$n^B_{(i,t)} \leqq n^B_{(i,t-1)} = n^{\pi(B)}_{(i,t-1)} \leqq n^{\pi(A)}_{(i,t-1)} = n^A_{(i,t-1)} = n^A_{(i,t)}$$

for $i < t$ and

$$n^B_{(t,j)} \leqq n^B_{(t+1,j)} = n^{\pi(B)}_{(t+1,j)} \leqq n^{\pi(A)}_{(t+1,j)} = n^A_{(t+1,j)} = n^A_{(t,j)}$$

for $j > t$. This proves $B \in \bar{O}_A$ by 2.2, i.e. $\pi^{-1}(\pi(\bar{O}_A)) = \bar{O}_A$. From b) we get (setting $d_i := \dim W_i$)

$$\begin{aligned}
\operatorname{codim}_L \bar{O}_A &= \dim L - \dim G + \dim \operatorname{Stab} A \\
&= \dim L - \dim G + \dim \operatorname{Stab} \pi(A) \\
&\quad + (d_t - d_{t+1})(d_t - d_{t-1}) \\
&= (\dim L - d_t d_{t-1} - d_{t+1} d_t + d_{t+1} d_{t-1}) \\
&\quad - (\dim G - d_t^2) + \dim \operatorname{Stab} \pi(A) \\
&= \dim L' - \dim G' + \dim \operatorname{Stab} \pi(A) \\
&= \operatorname{codim}_{L'} \bar{O}_{\pi(A)}. \quad \text{qed}
\end{aligned}$$

4.3. *Remark.* It is immediate from the proof above that the proposition holds also for $t = 1$ and $t = n$, i.e. for the two projections

$$\pi_1 : L(W_1, \ldots, W_n) \longrightarrow L(W_2, \ldots, W_n),$$
$$\pi_n : L(W_1, \ldots, W_n) \longrightarrow L(W_1, \ldots, W_{n-1}).$$

(We set $W_i = 0$ for $i < 1$ or $i > n$ and $A_i = 0$ if $i < 1$ or $i > n - 1$.)

5. The Basic Construction

5.1. Let $V_1, V_2, \ldots, V_m$ and $U_1, U_2, \ldots, U_{m-1}$ be finite dimensional vector spaces. Consider the following two maps π and ϱ

$$\begin{array}{l}
\tilde{L} := L(V_1, U_1, V_2, U_2, \ldots, U_{m-1}, V_m) \xrightarrow{\pi} L' := L(U_1, \ldots, U_{m-1}) \\
\quad \Big\downarrow \varrho \\
L := L(V_1, \ldots, V_m)
\end{array}$$

given by

$$\pi(B_1, C_1, B_2, C_2, \ldots, B_{m-1}, C_{m-1}) = (B_2C_1, B_3C_2, \ldots, B_{m-1}C_{m-2}),$$
$$\varrho(B_1, C_1, B_2, C_2, \ldots, B_{m-1}, C_{m-1}) = (C_1B_1, C_2B_2, \ldots, C_{m-1}B_{m-1}).$$

Define

$$G := G(V_1, \ldots, V_m),$$
$$G' := G(U_1, \ldots, U_{m-1})$$

and

$$\tilde{G} := G(V_1, U_1, V_2, U_2, \ldots, U_{m-1}, V_m).$$

The group $\tilde{G}$ is canonically isomorphic to $G \times G'$, the maps π and ϱ are equivariant with respect to the projections $\tilde{G} \to G'$ and $\tilde{G} \to G$ and are *quotient maps* with respect to G and G' respectively (cf. first fundamental theorem 1.2).

5.2. We remark that the map π is a composition $\pi = \pi_{m-1} \cdot \pi_{m-2} \cdot \ldots \cdot \pi_2 \cdot p$ of the following form:

(*)
$$\begin{tikzcd}
\tilde{L} \arrow[r, "p"'] \arrow[ddddr, "\pi"'] & L_1 := L(U_1, V_2, \ldots, V_{m-1}, U_{m-1}) \arrow[d, "\pi_2"] \\
& L_2 := L(U_1, U_2, V_3, \ldots, V_{m-1}, U_{m-1}) \arrow[d, "\pi_3"] \\
& L_3 := L(U_1, U_2, U_3, V_4, \ldots, U_{m-1}) \arrow[d] \\
& \vdots \arrow[d, "\pi_{m-1}"] \\
& L' = L_{m-1} := L(U_1, U_2, \ldots, U_{m-1})
\end{tikzcd}$$

where p is the canonical projection and π_i is the quotient with respect to $\mathrm{GL}(V_i)$, i.e. it is of the form given in 4.1.

5.3. Consider the following two subsets of $\tilde{L}$:

$$\tilde{L}^0 := \{(B_1, C_1, \ldots, B_{m-1}, C_{m-1})|$$

for each $i = 1, \ldots, m-2$ either C_i is injective or B_{i+1} is surjective$\}$

$$\tilde{L}^{\max} := \{(B_1, C_1, \ldots, B_{m-1}, C_{m-1})|$$

all B_i are surjective and all C_i are injective$\}$.

Proposition. *Assume* $\dim U_i \leqq \dim V_i, \dim V_{i+1}$ *for* $i = 1, \ldots, m-1$. *Then* $\tilde{L}^0 \supset \tilde{L}^{\max}$ *are not empty and*

a) π *is smooth in* $\tilde{L}^0$,

b) $\varrho|_{\tilde{L}^{\max}} : \tilde{L}^{\max} \longrightarrow \varrho(\tilde{L}^{\max})$ *is a fibration with typical fibre* $G(U_1, \ldots, U_{m-1})$.

(Here fibration means locally trivial in the étale topology.)

'roof. The first assertion is clear. For a) consider the decomposition of π given in ı.2. The assumption implies that each L_i in (*) satisfies the condition of 'roposition 4.2 and that we can apply 4.2a) to the image of $\tilde{L}^0$ in L_i. Since p is learly smooth this implies that π is smooth in $\tilde{L}^0$. For b) consider the group $I := \prod_{i=1}^{m-1} (\mathrm{GL}(V_i) \times \mathrm{GL}(U_i) \times \mathrm{GL}(V_{i+1}))$ which operates in a natural way on $\tilde{L}$ nd on L:

$$\begin{aligned}&(g_1, h_1, g_1', g_2, h_2, g_2', \ldots)(B_1, C_1, B_2, C_2, \ldots)\\ &\quad = (h_1 B_1 g_1^{-1}, g_1' C_1 h_1^{-1}, h_2 B_2 g_2^{-1}, g_2' C_2 h_2^{-1}, \ldots),\\ &(g_1, h_1, g_1', g_2, h_2, g_2', \ldots)(D_1, D_2, \ldots) = (g_1' D_1 g_1^{-1}, g_2' D_2 g_2^{-1}, \ldots).\end{aligned}$$

Now clearly $\tilde{L}^{\max}$ is an orbit under H and ϱ is H-equivariant. This implies that the nap $\varrho|_{\tilde{L}^{\max}}$ is of the form $H/H_1 \to H/H_2$ with two subgroups $H_1 \subset H_2 \subset H$, hence ocally trivial in the étale topology. Furthermore it is obvious that for each $C \in \tilde{L}^{\max}$ he fibre $\varrho^{-1}(\varrho(C))$ is the $G(U_1, \ldots, U_{m-1})$ orbit of C and the stabilizer of C in $\mathfrak{Z}(U_1, \ldots, U_{m-1})$ is trivial. qed

ı.4. Now let us start with a fixed element $A \in L$,

$$A : V_1 \xrightarrow{A_1} V_2 \xrightarrow{A_2} \ldots \longrightarrow V_{m-1} \xrightarrow{A_{m-1}} V_m .$$

Defining $U_i := \operatorname{Im} A_i \subset V_i$, $i = 1, \ldots, m-1$, we get the following basic construction:

$$\begin{tikzcd}
U_1 \arrow[r, "A_1'"] \arrow[dr, "I_1"] & U_2 \arrow[r, "A_2'"] \arrow[dr, "I_2"] & \ldots \arrow[r] & U_{m-2} \arrow[r, "A_{m-2}'"] \arrow[dr, "I_{m-2}"] & U_{m-1} \arrow[dr, "I_{m-1}"] & \\
V_1 \arrow[u, "\bar{A}_1"] \arrow[r, "A_1"'] & V_2 \arrow[u, "\bar{A}_2"] \arrow[r, "A_2"'] & V_3 \arrow[r] & \ldots \arrow[r] & V_{m-1} \arrow[u, "\bar{A}_{m-1}"] \arrow[r, "A_{m-1}"'] & V_m
\end{tikzcd}$$

Here $A_i = I_i \cdot \bar{A}_i$ is the canonical factorization and $A_i' := \bar{A}_{i+1} I_i$. In this way we have associated to $A \in L$ two elements $\tilde{A} \in \tilde{L}$ and $A' \in L'$:

$$\tilde{A} : V_1 \xrightarrow{\bar{A}_1} U_1 \xrightarrow{I_1} V_2 \xrightarrow{\bar{A}_2} \ldots \longrightarrow V_{m-1} \xrightarrow{\bar{A}_{m-1}} U_{m-1} \xrightarrow{I_{m-1}} V_m ,$$

$$A' : U_1 \xrightarrow{A_1'} U_2 \xrightarrow{A_2'} U_3 \longrightarrow \ldots \longrightarrow U_{m-2} \xrightarrow{A_{m-2}'} U_{m-1} ,$$

such that $\pi(\tilde{A}) = A'$ and $\varrho(\tilde{A}) = A$. Let us denote by $O_{\tilde{A}}$, $O_{A'}$ and O_A the corresponding orbits in $\tilde{L}$, L' and L respectively.

Remark. It follows from the construction that the diagram of A' is obtained from the diagram of A by *removing the last box of each row*:

$$\lambda^{A'}_{(i,j)} = \lambda^{A}_{(i,j+1)} .$$

5.5. The main induction step for the proof of Theorem 0.2 is the following result.

Proposition. *Set* $N_A := \pi^{-1}(\bar{O}_{A'})$. *Then*

(i) $O_{\tilde{A}} = \varrho^{-1}(O_A) \subset \tilde{L}^{\max}$ *and* $\pi(O_{\tilde{A}}) = O_{A'}$,

(ii) $\bar{O}_{\tilde{A}} = N_A$ *and* $\varrho(N_A) = \bar{O}_A$,

(iii) $\operatorname{codim}_{\tilde{L}} N_A = \operatorname{codim}_{L'} \bar{O}_{A'}$.

Proof. Since $\pi(\tilde{A}) = A'$ and $\varrho(\tilde{A}) = A$ we get $\pi(O_{\tilde{A}}) = O_{A'}$ and $\varrho(O_{\tilde{A}}) = O_A$. Furthermore $\tilde{A} \in \tilde{L}^{\max}$ and $\varrho^{-1}(A)$ is an orbit under $G(U_1, \ldots, U_{m-1})$ by construction (cf. proof of Proposition 5.3), hence $\varrho^{-1}(O_A) = O_{\tilde{A}}$. This proves i). For ii) and iii) we use the decomposition (*) of π (5.2) and remark that each L_i in (*) satisfies the assumption of Proposition 4.2. Since p is smooth and $\bar{A}_1$ and I_{m-1} are of maximal rank we first get $\operatorname{codim}_{\tilde{L}} \bar{O}_{\tilde{A}} = \operatorname{codim}_{L_1} p(\bar{O}_{\tilde{A}})$ and $p^{-1}(p(\bar{O}_{\tilde{A}})) = \bar{O}_{\tilde{A}}$. Now the image of $\tilde{A}$ in each L_i satisfies the assumption of 4.2c). Hence we get by induction $\bar{O}_{\tilde{A}} = \pi^{-1}(\pi(\bar{O}_{\tilde{A}})) = N_A$ and $\operatorname{codim}_{\tilde{L}} N_A = \operatorname{codim}_{L'} \bar{O}_{A'}$. qed

5.6. *Remark.* The construction of the orbit $O_{A'}$ from the orbit O_A does not depend on the special choice of the vector spaces U_i, but only on their dimensions. More precisely we can formulate the Proposition 5.5 in the following way:

If O_λ *is an orbit in* $L(V_1, \ldots, V_m)$ *with associated diagram* λ *and* $U_1, \ldots, U_{m-1}$ *are vector spaces of dimension* $\dim U_i = n^\lambda_{(i,i+1)}$ *and if we denote by* λ' *the diagram obtained from* λ *by removing the last box in each row* (i.e. $\lambda'_{(i,j)} = \lambda_{(i,j+1)}$), *then we have*

(i) $\varrho^{-1}(O_\lambda)$ *is an orbit in* $\tilde{L}$ *contained in* $\tilde{L}^{\max}$, $O_{\lambda'} = \pi(\varrho^{-1}(O_\lambda))$,

(ii) $\overline{\varrho^{-1}(O_\lambda)} = \pi^{-1}(\bar{O}_{\lambda'})$,

(iii) $\operatorname{codim}_{\tilde{L}} \pi^{-1}(\bar{O}_{\lambda'}) = \operatorname{codim}_{L'} \bar{O}_{\lambda'}$.

6. A Resolution of Singularities

6.1. We fix an element $C \in L := L(W_1, W_2, \ldots, W_n)$, $C = (C_1, C_2, \ldots, C_{n-1})$.

For each $i = 1, 2, \ldots, n$ consider the *flag variety* $\mathscr{F}_i$ of flags $F_i = (F_i^0 = W_i \supseteqq F_i^1 \supseteqq F_i^2 \supseteqq \ldots \supseteqq F_i^{i-1})$ of nationality $(n^C_{(i,i)}, n^C_{(i-1,i)}, n^C_{(i-2,i)}, \ldots, n^C_{(1,i)})$ (i.e. $\dim F_i^t = n^C_{(i-t,i)}$).

E.g. $\mathscr{F}_1$ is a point and $\mathscr{F}_2$ is a grassmanian. For $B_j : W_j \to W_{j+1}$, $F_j \in \mathscr{F}_j$ and $F_{j+1} \in \mathscr{F}_{j+1}$ we shortly write $B_j F_j \subseteqq F_{j+1}$ if $B_j F_j^t \subset F_{j+1}^{t+1}$ for all $t = 0, 1, \ldots, j-1$. Of course $G := G(W_1, \ldots, W_n)$ acts in the obvious way on $\mathscr{F}_C := \mathscr{F}_1 \times \mathscr{F}_2 \times \ldots \times \mathscr{F}_n$. *Define* $Y := Y_C \subset L \times \mathscr{F}_C$ *to be the subvariety of pairs* (B, F), $B = (B_1, \ldots, B_{n-1})$, $F = (F_1, \ldots, F_n)$ *satisfying*

$$B_j F_j \subseteqq F_{j+1}, \quad j = 1, \ldots, n-1.$$

It is clear that Y_C is a *closed and G-stable subvariety*. Let us consider the two maps

$$\begin{tikzcd} Y_C \arrow[r, "\varphi"] \arrow[d, "p"'] & L \\ \mathscr{F}_C & \end{tikzcd}$$

induced by the two projections of $L \times \mathscr{F}_C$. It is easy to see that $p: Y_C \to \mathscr{F}_C$ is a *subvectorbundle* of the trivial bundle $pr: L \times \mathscr{F}_C \to \mathscr{F}_C$. In particular Y_C is a *smooth* variety. Furthermore Φ is *proper*, since $Y_C \subset L \times \mathscr{F}_C$ is closed and $\mathscr{F}_C$ is projective.

6.2. Lemma. *$\Phi(Y_C) = \bar{O}_C$ and the induced map $\varphi: Y_C \to \bar{O}_C$ is proper and birational*, i.e. φ is a resolution of singularities (1.4).

Proof. The definition of Y_C implies that for each $(B, F) \in Y_C$, $B := (B_1, \ldots, B_{n-1})$, $F := (F_1, \ldots, F_n)$ we have $B_{j-1} \ldots B_i(W_i) \subset F_j^{j-i}$ for $j > i$, hence $n^B_{(i,j)} \leqq \dim F_i^{j-i} = n^C_{(i,j)}$. Therefore $B \in \bar{O}_C$ by Proposition 2.2 which implies $\Phi(Y_C) \subseteq \bar{O}_C$. Furthermore $\varphi^{-1}(C) = \{(C, F_C)\}$, $F_C = (F_1, \ldots, F_n)$ defined by $F_i^t := C_{i-1} \ldots C_{i-t}(W_{i-t})$ for $t < i$ $(F_1 := W_1)$. In particular $C \in \Phi(Y_C)$ and hence $\Phi(Y_C) = \bar{O}_C$ (since Y_C is G-stable and closed and Φ is proper), and φ induces an isomorphism $\varphi^{-1}(O_C) \xrightarrow{\sim} O_C$ and so φ is birational. qed

7. Proof of the First Main Theorem

7.1. Now we go back to the basic construction 5.4. The proof of the theorem 0.2 follows immediately by induction from the following result.

Proposition (Notations of Sect. 5). *If $\bar{O}_{A'}$ has rational singularities* (1.4), *then $N_A = \pi^{-1}(\bar{O}_{A'})$ and $\bar{O}_A$ have rational singularities too.*

Proof. Since $\bar{O}_A = \varrho(N_A)$ [Proposition 5.5ii and since ϱ is a quotient map, the first assertion implies the second by Boutot's theorem (1.3b)]. From Lemma 6.2 we get a desingularization of $N_A = \bar{O}_{\tilde{A}}$ (5.5ii):

$$\varphi: Y_{\tilde{A}} \longrightarrow N_A = \bar{O}_{\tilde{A}},$$

$$Y_{\tilde{A}} \subset \tilde{L} \times \mathscr{F}_{\tilde{A}}, \tilde{L} := L(V_1, U_1, V_2, U_2, \ldots, U_{m-1}, V_m),$$

$\mathscr{F}_{\tilde{A}} := \mathscr{F}_1 \times \mathscr{F}'_1 \times \mathscr{F}_2 \times \mathscr{F}'_2 \times \ldots \times \mathscr{F}'_{m-1} \times \mathscr{F}_m$ (in obvious notations). The codimension result 5.4 iii) implies that the schematic inverse image $\pi^{-1}(\bar{O}_{A'})$ is Cohen-Macaulay [4, IV, Proposition 15.4.2c')$\Rightarrow$a)]. Since the map π is smooth in $N_A^0 := N_A \cap \tilde{L}^0$ (5.3a) and since $\bar{O}_{A'}$ has rational singularities, $\pi^{-1}(\bar{O}_{A'})$ is a reduced Cohen-Macaulay variety, being smooth in the dense open set $O_{\tilde{A}} \subset N_A^0$ (5.5i, ii), [4, IV, 5.8.5], with rational singularities in N_A^0. The following lemma enables us to apply Corollary 1.4 which implies the claim. qed

7.2. Lemma. $\operatorname{codim}_{Y_{\tilde{A}}} \varphi^{-1}(N_A - N_A^0) \geqq 2$.

Proof. Let $C := N_A - N_A^0$. It is enough to prove that for each $F \in \mathscr{F}_{\tilde{A}}$ the set $\varphi^{-1}(C) \cap p^{-1}(F)$ has codimension at least 2 in $p^{-1}(F)$ $(p: Y_{\tilde{A}} \to \mathscr{F}_{\tilde{A}}$ is the projection 7.1). The fibre $p^{-1}(F)$ is a product $H_1 \times K_1 \times H_2 \times \ldots \times H_{m-1} \times K_{m-1}$ of subspaces $H_i \subset \operatorname{Hom}(V_i, U_i)$, $K_i \subset \operatorname{Hom}(U_i, V_{i+1})$. By construction $\varphi^{-1}(C) \cap p^{-1}(F)$ is a union of subvarieties $D_j \subset H_1 \times K_1 \times \ldots \times H_{m-1} \times K_{m-1}$, $j = 1, \ldots, m-1$, D_j defined by the condition that the element in H_j is not surjective and the element in K_j is not injective. Since the generic element of H_j is surjective and the generic element of K_j is injective the subvariety D_j is of codimension greater or equal to 2. qed

8. Determinantal Singularities

8.1. *Definition* (cf. [5] 1.7). Consider two varieties X, Y and two points $x \in X$, $y \in Y$. Then the singularity of X in x is called *smoothly equivalent* to the singularity of Y in y if there is a variety Z, a point $z \in Z$ and two maps

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & X \\ {\scriptstyle\psi}\downarrow & & \\ Y & & \end{array}$$

such that $\varphi(z)=x$, $\psi(z)=y$, and φ and ψ are smooth in z.

This clearly defines an *equivalence relation* between pointed varieties (X, x). We denote the equivalence class of (X, x) by $\operatorname{Sing}(X, x)$.

Assume that an algebraic group G acts regularly on the variety X. Then $\operatorname{Sing}(X, x)=\operatorname{Sing}(X, x')$ if x and x' belong to the same orbit O. In this case we denote the equivalence class also by $\operatorname{Sing}(X, O)$.

8.2. For two vector spaces U, V of dimension p, q respectively consider the orbit $O_{\min} \subset L(U, V)$ of maps of rank 1. Then $\bar{O}_{\min}=O_{\min} \cup \{0\}$ and $0 \in \bar{O}_{\min}$ is an isolated rational singularity (Theorem 0.2, cf. [9, Sect. 2]). We denote its equivalence class by $d_{p,q}$:

$$\operatorname{Sing}(\bar{O}_{\min}, 0)=d_{p,q}.$$

We have $\dim \bar{O}_{\min}=p+q-1$ *and* $\bar{O}_{\min}$ *is smooth if and only if* $p=1$ *or* $q=1$ [i.e. $\bar{O}_{\min}=L(U, V)$].

Assuming $p, q>1$, one gets for any $(r, s): d_{r,s}=d_{p,q}$ if and only if $(r, s)=(p, q)$ or $(r, s)=(q, p)$. [Since both are represented by isolated singularities one has $p+q-1=r+s-1$. Since $L(U, V)$ is the Zariskitangentspace of $\bar{O}_{\min}$ in 0 we must also have $p \cdot q=r \cdot s$.]

8.3. It is easy to describe a resolution of singularities of $\bar{O}_{\min}$.

Choose a basis $e_1, \ldots, e_p$ of U and a basis $f_1, \ldots, f_q$ of V and consider the one dimensional subspace

$$M:=\{A \in L(U, V) \mid Ae_1 \subset kf_1, Ae_i=0 \text{ for } i=2, \ldots, p\}.$$

Then the stabilizer P of M in $G:=G(U, V)$ is the parabolic $P=P_{e_1} \times P_{f_1} \subset \mathrm{GL}(U) \times \mathrm{GL}(V)=G$, where P_{e_1}, P_{f_1} are the stabilizers of the lines ke_1, kf_1.

Now $G/P=\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, the associated bundle

$$G \times^P M \longrightarrow G/P$$

is the line bundle $\mathcal{O}_{\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}}(-1)$ (given by the Segre embedding) and the canonical map

$$\varphi: G \times^P M \longrightarrow \bar{O}_{\min}$$

is a resolution of singularities [with $\varphi^{-1}(0)=$ zero section of the line bundle]. Hence we obtain the singularity $d_{p,q}$ *by "collapsing" the line bundle* $\mathcal{O}_{\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}}(-1)$ (cf. [9]).

8.4. Let us recall that an orbit O' is called a *minimal degeneration* of the orbit $O \subsetneq L$ if O' is open in $\bar{O} - O$, or equivalently if for each orbit O'' such that $O' \subsetneq \bar{O}'' \subsetneqq \bar{O}$ we have either $O'' = O$ or $O'' = O'$. In the following Sect. 9 we will associate to any minimal degeneration a pair (p, q) of natural numbers called the *index of the minimal degeneration*; it can easily be read off the diagrams of the orbits.

Now we can reformulate our second main theorem (cf. 0.3).

Theorem. *Let $O' \subset \bar{O}$ be a minimal degeneration of orbits in $L(V_1, \ldots, V_m)$ with index (p, q). Then*

$$\operatorname{codim}_{\bar{O}} O' = p + q - 1 \quad \textit{and} \quad \operatorname{Sing}(\bar{O}, O') = d_{p,q}.$$

The proof will be given in section 10; it uses two general reduction results for singularities in closures of orbits $O \subset L(V_1, \ldots, V_m)$ (10.1, 10.2). From 8.2 we obtain the following corollary:

Corollary. *If $O' \subset \bar{O}$ is a minimal degeneration with* $\operatorname{codim}_{\bar{O}} O' \leqq 2$, *then $\bar{O}$ is smooth in O'.*

In particular the singular locus of $\bar{O}$ can be strictly smaller than the boundary $\partial O = \bar{O} - O$, even if ∂O has codimension $\geqq 2$.

9. The Index of a Minimal Degeneration

In order to define the index of a minimal degeneration $O' \subset \bar{O}$ we have to recall the main result on degenerations of orbits in $L = L(V_1, \ldots, V_m)$ proved in [1].

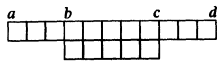
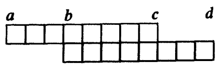
9.1. Proposition. *Let $O' \subset \bar{O}$ be a minimal degeneration, $\nu < \lambda$ the associated diagrams. Then there are natural numbers $a < b$, $b - 1 \leqq c < d$ such that*

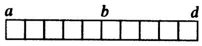
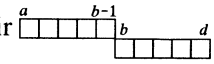
$$\nu_{(i,j)} = \begin{cases} \lambda_{(i,j)} & \textit{for} \quad (i,j) \neq (a,d), (a,c), (b,d), (b,c) \\ \lambda_{(a,c)} + 1 & \textit{for} \quad (i,j) = (a,c) \\ \lambda_{(b,d)} + 1 & \textit{for} \quad (i,j) = (b,d) \\ \lambda_{(a,d)} - 1 & \textit{for} \quad (i,j) = (a,d) \\ \lambda_{(b,c)} - 1 & \textit{for} \quad (i,j) = (b,c) \quad \textit{if} \quad b \leqq c. \end{cases}$$

Clearly the numbers a, b, c, d are uniquely determined by the two diagrams $\nu < \lambda$.

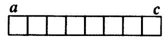
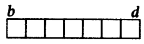
Definition. With the notations of the proposition above the pair $(\nu_{(a,c)}, \nu_{(b,d)})$ is called *the index of the minimal degeneration.*

9.2. In more geometric terms Proposition 9.1 means that for a minimal degeneration $O_\nu \subset \bar{O}_\lambda$ the diagram ν is obtained from the diagram λ performing one of the following two operations:

a) *a shift*, i.e. a pair of rows of λ of the form  is replaced by the pair ,

b) *a cut*, i.e. a row  of λ is replaced by the pair 

(here $c=b-1$),
and all other rows remain unchanged.

In both cases the index (p,q) is given by the numbers of the rows  and  in the diagram ν.

Clearly a cut or a shift always define a degeneration. But in general such a degeneration is not minimal, as one easily sees from examples. The following lemma gives additional conditions for a minimal degeneration showing that certain rows cannot appear in λ and ν. (The proof is purely combinatorial and left to the reader.)

Lemma. *Let $O_\nu \subset \bar{O}_\lambda$ be a minimal degeneration and consider the numbers $a<b$, $b-1 \leqq c<d$ given in Proposition 9.1. Then*

$$\lambda_{(i,j)} = \nu_{(i,j)} = 0 \quad \textit{for} \quad a \leqq i \leqq b, c \leqq j \leqq d \quad \textit{and}$$
$$(i,j) \neq (a,c), (a,d), (b,c), (b,d).$$

9.3. We now can prove the codimension formula given in Theorems 0.3 and 8.4.

Proposition. *Let $O' \subset \bar{O}$ be a minimal degeneration with index (p,q). Then*

$$\operatorname{codim}_{\bar{O}} O' = p+q-1.$$

Proof. In order to simplify the notations let us denote by $[a,b]$ the diagram consisting in one row starting at a and ending at b. Furthermore define for two diagrams α and β:

$$h(\alpha,\beta) = \sum_{i \leqq r \leqq j \leqq s} \alpha_{(i,j)} \cdot \beta_{(r,s)}$$

From 3.2 we have for $A \in O_\lambda$ that $\dim \operatorname{Stab} A = h(\lambda,\lambda)$, hence

$$\operatorname{codim}_{\bar{O}_\lambda} O_\nu = h(\nu,\nu) - h(\lambda,\lambda).$$

Now one easily checks the following equations:

$$h([a,c]+[b,d],[i,j]) - h([a,d]+[b,c],[i,j]) = \begin{cases} 1 & \text{for} \quad a \leqq i < b, c \leqq j < d \\ 0 & \text{otherwise} \end{cases}$$

$$h([i,j],[a,c]+[b,d]) - h([i,j],[a,d]+[b,c]) = \begin{cases} 1 & \text{for} \quad a < i \leqq b, c < j \leqq d \\ 0 & \text{otherwise.} \end{cases}$$

($\alpha+\beta$ denotes the diagram consisting of the union of the rows of α and β.)

Since $\nu<\lambda$ is a minimal degeneration, there are numbers $a<b$, $b-1 \leqq c<d$ satisfying the equations of Proposition 9.1, i.e. $\lambda=[a,d]+[b,c]+\delta$, $\nu=[a,c]+[b,d]+\delta$ for a suitable diagram δ (delet $[b,c]$ if $b>c$). By definition of the index (p,q) we have $\delta=(p-1)[a,c]+(q-1)[b,d]+\delta'$, δ' a diagram with $\delta'_{(a,c)}=\delta'_{(b,d)}=0$. Furthermore it follows from Lemma 9.2 that

$$\delta'_{(i,j)}=0 \quad \text{for} \quad a \leqq i \leqq b, \quad c \leqq j \leqq d, \quad (i,j) \neq (a,d), (b,c).$$

Hence we find

$$\begin{aligned}
&h(v, v) - h(\lambda, \lambda)\\
&\quad = h([a,c]+[b,d],[a,c]+[b,d]) - h([a,d]+[b,c],[a,d]+[b,c])\\
&\qquad + h([a,c]+[b,d],\delta) - h([a,d]+[b,c],\delta)\\
&\qquad + h(\delta,[a,c]+[b,d]) - h(\delta,[a,d]+[b,c])\\
&\quad = 1+(p-1)+(q-1)\,. \quad \text{qed}
\end{aligned}$$

10. Some Reduction Results, Proof of the Second Main Theorem

Let O_v, O_λ be orbits in $L(V_1, \ldots, V_m)$, $O_v \subset \bar{O}_\lambda$, $v \leqq \lambda$ the associated diagrams (2.4). In this section we will describe some procedures in order to obtain from the pair $v \leqq \lambda$ a new pair $v' \leqq \lambda'$ of diagrams with less boxes such that

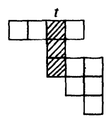
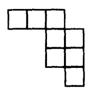
$$\operatorname{Sing}(\bar{O}_\lambda, O_v) = \operatorname{Sing}(\bar{O}_{\lambda'}, O_{v'})\,.$$

(Compare [11] where the same is done for conjugacy classes of matrices.)

10.1. For the first reduction result let us consider the quotient map

$$\pi := \pi_t : L(V_1, \ldots, V_m) \longrightarrow L(V_1, \ldots, \hat{V}_t, \ldots, V_m)$$

for some fixed $t \in \{1, 2, \ldots, m\}$, given by $(A_1, \ldots, A_{m-1}) \mapsto (A_1, \ldots, A_t A_{t-1}, \ldots, A_m)$ (4.2, 4.3; we always set $V_i = 0$ if $i < 1$ or $i > m$ and $A_i = 0$ if $i < 1$ or $i > m-1$). If $O_\lambda \subset L(V_1, \ldots, V_m)$ is an orbit with associated diagram λ the diagram λ' of the orbit $\pi(O_\lambda)$ is obtained from λ by removing the t^{th} column (and pushing the two parts together).

E.g. $\lambda =$  gives $\lambda' =$  .

Proposition. *Assume* $\dim V_{t+1}$, $\dim V_{t-1} \leqq \dim V_t$. *Let* $A, B \in L(V_1, \ldots, V_m)$, $B \in \bar{O}_A$, *such that* A_{t-1} *and* A_t *have maximal rank and either* B_{t-1} *is injective or* B_t *is surjective. Then*

$$\operatorname{Sing}(\bar{O}_A, O_B) = \operatorname{Sing}(\bar{O}_{\pi(A)}, O_{\pi(B)})\,.$$

Proof. The claim follows immediately from Proposition 4.2 (and Remark 4.3). In fact we have by assumption O_A, $O_B \subset L^0 := \{C \in L(V_1, \ldots, V_m) \mid C_{t-1}$ or C_t has maximal rank$\}$ and $\bar{O}_A = \pi^{-1}(\pi(\bar{O}_A)) = \pi^{-1}(\bar{O}_{\pi(A)})$ (4.2c). Hence π induces a smooth map

$$\bar{\pi} : \bar{O}_A \cap L^0 \longrightarrow \bar{O}_{\pi(A)}$$

[4.2a)] with $\bar{\pi}(O_B) = O_{\pi(B)}$ and so by Definition 8.1

$$\operatorname{Sing}(\bar{O}_A, O_B) = \operatorname{Sing}(\bar{O}_{\pi(A)}, O_{\pi(B)})\,. \quad \text{qed}$$

10.2. Our second reduction result is obtained in the following way. For $A \in L(V_1, \ldots, V_m)$, $C \in L(W_1, \ldots, W_m)$ the sum $A \oplus B$ is defined using the canonical embedding

$$L(V_1, \ldots, V_m) \oplus L(W_1, \ldots, W_m) \hookrightarrow L(V_1 \oplus W_1, \ldots, V_m \oplus W_m).$$

If λ and δ are the associated diagrams to O_A and O_C the diagram of $O_{A \oplus C}$ is $\lambda + \delta$, given by

$$(\lambda + \delta)_{(i,j)} := \lambda_{(i,j)} + \delta_{(i,j)}$$

i.e. $\lambda + \delta$ consists in the union of the rows of λ and δ.

Proposition. *Let* $A, B \in L(V_1, \ldots, V_m)$, $C \in L(W_1, \ldots, W_m)$.

a) *We have* $O_B \subset \bar{O}_A$ *if and only if* $O_{B \oplus C} \subset \bar{O}_{A \oplus C}$. *Furthermore if* $O_{B \oplus C} \subset \bar{O}_{A \oplus C}$ *is a minimal degeneration then* $O_B \subset \bar{O}_A$ *is a minimal degeneration.*

b) *If* $O_B \subset \bar{O}_A$ *and if* $\operatorname{codim}_{\bar{O}_A} O_B = \operatorname{codim}_{\bar{O}_{A \oplus C}} O_{B \oplus C}$, *then*

$$\operatorname{Sing}(\bar{O}_A, O_B) = \operatorname{Sing}(\bar{O}_{A \oplus C}, O_{B \oplus C}).$$

Proof. a) follows from 2.2 since $n^{A \oplus C}_{(i,j)} = n^A_{(i,j)} + n^C_{(i,j)}$ for all (i,j).

For b) consider the subgroup $H = G(V_1, \ldots, V_m) \times G(W_1, \ldots, W_m) \subset G := G(V_1 \oplus W_1, \ldots, V_m \oplus W_m)$, operating on $E := L(V_1, \ldots, V_m) \oplus L(W_1, \ldots, W_m)$.

Then $O := O_A \times O_C$ and $O' := O_B \times O_C$ are orbits under H in E and $\bar{O} = \bar{O}_A \times \bar{O}_C$. Hence $\operatorname{Sing}(\bar{O}, O') = \operatorname{Sing}(\bar{O}_A, O_B)$ and $\operatorname{codim}_{\bar{O}} O' = \operatorname{codim}_{\bar{O}_A} O_B$. Furthermore we have the decomposition

$$\operatorname{Lie} G = \bigoplus_{i=1}^{m} \operatorname{End}(V_i \oplus W_i) = \operatorname{Lie} H \oplus N$$

with $N := \bigoplus_{i=1}^{m} (\operatorname{Hom}(V_i, W_i) \oplus \operatorname{Hom}(W_i, V_i))$. Now under the operation of $\operatorname{Lie} G$ on $L(V_1 \oplus W_1, \ldots, V_m \oplus W_m)$ we clearly have $NE \cap E = 0$. Since $\bar{O}_{A \oplus C}$ is normal (0.2) the claim follows from Lemma 10.3 below. qed

Remark. In terms of diagrams $\nu \leq \lambda$ the proposition says that we can remove some common rows of λ and ν without changing the type of singularity, provided the codimension does not change. More precisely, *if* $\lambda = \lambda' + \delta$, $\nu = \nu' + \delta$ *with some diagram* δ *such that* $\operatorname{codim}_{\bar{O}_\lambda} O_\nu = \operatorname{codim}_{\bar{O}_{\lambda'}} O_{\nu'}$, *then* $\operatorname{Sing}(\bar{O}_\lambda, O_\nu) = \operatorname{Sing}(\bar{O}_{\lambda'}, O_{\nu'})$.

10.3. Lemma [11, Proposition 4.2]. *Let G be an algebraic group,* $H \subset G$ *a subgroup, L a G-module and* $E \subset L$ *an H-stable subspace. Consider elements* $x \in E$ *and* $y \in \overline{Hx}$ *and assume that*

(i) *there is a decomposition* $\operatorname{Lie} G = \operatorname{Lie} H \oplus N$ *such that* $Ny \cap E = 0$,

(ii) $\operatorname{codim}_{\overline{Hx}} Hy = \operatorname{codim}_{\overline{Gx}} Gy$,

(iii) $\overline{Gx}$ *is normal in* y.

Then $\operatorname{Sing}(\overline{Gx}, y) = \operatorname{Sing}(\overline{Hx}, y)$.

10.4. Now we can prove our main result on singularities in closures of orbits (Theorems 0.3 and 8.4). Let $O_\nu \subset \bar{O}_\lambda$ be a minimal degeneration of index (p, q). By Proposition 9.1 there are numbers a, b, c, d with $a < b$ and $b - 1 \leq c < d$ and a diagram δ such that

$$\lambda = [a,d] + [b,c] + (p-1)[a,c] + (q-1)[b,d] + \delta$$
$$\upsilon = p[a,c] + q[b,d] + \delta$$

(cf. proof of Proposition 9.3; delet $[b,c]$ if $c=b-1$). Using Proposition 9.3 our second reduction result 10.2 implies that we may assume $\delta = \emptyset$ (cf. Remark 10.2). If $c \geqq b$ it follows from the first reduction result 10.1 that

$$\operatorname{Sing}(\bar{O}_\lambda, O_\upsilon) = \operatorname{Sing}(\bar{O}_{\lambda'}, O_{\upsilon'}),$$

where λ', υ' are obtained from λ, υ by removing the c^{th} column, i.e

$$\lambda' = [a,d-1] + [b,c-1] + (p-1)[a,c-1] + (q-1)[b,d-1]$$
$$\upsilon' = p[a,c-1] + q[b,d-1].$$

Hence by induction we may suppose that $c=b-1$, i.e

$$\lambda = [a,d] + (p-1)[a,b-1] + (q-1)[b,d]$$

$$\upsilon = p[a,b-1] + q[b,d].$$

Finally if $a<b-1$ (or $b<d$) we can again apply Proposition 10.1, this time to the a^{th} column (or the d^{th} column), ending up at the pair

$$\lambda = [1,2] + (p-1)[1,1] + (q-1)[2,2]$$
$$\upsilon = p[1,1] + q[2,2],$$

i.e. $O_\lambda = O_{\min} \subset M_{q,p}$, $O_\upsilon = \{0\}$. qed

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