GEOMETRY OF REPRESENTATIONS OF QUIVERS

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INTRODUCTION

One of the first results about representations of quivers was Gabriel's characterization of the quivers of finite representation type and of their indecomposable representations [G1, G2] : The underlying graph of such a quiver is a union of Dynkin diagrams and the indecomposables are in one-to-one correspondence with the positive roots of the associated semi-simple Lie-algebra. Later Donovan-Freislich [DF] and independently Nazarova [N] discovered analogous relations between tame quivers and extended Dynkin diagrams. Since all remaining quivers are wild, there was little hope to get any further, except maybe in some special cases. Therefore Kac's spectacular paper [K1], where he describes the dimension types of all indecomposables of arbitrary quivers, came as a big surprise. In [K2] and [K3] Kac improved and completed his first results.

These notes are meant to be a guide to and through Kac's articles. In fact, most definitions and results are taken from his work. We reorganized them to give - we believe- a direct approach which is easy to follow. We refer to Kac's papers only for statements we do not prove completely.

Our point of view is of geometric nature -like in Kac's original work- and we use methods from algebraic geometry and transformation groups. The set of representations of a fixed dimension type is viewed as an algebraic variety on which the algebraic group of base change operators acts. In fact, it is a vectorspace with a linear group action. In this setting a number of interesting questions arise very naturally, for example the following :

What does the set of indecomposables look like ? How many components does it have, and what is the number of parameters ? What is the structure of the parameter space ? Is it always rational, and is there a (canonical) normal form ? How can one understand degenerations and deformations by means of representation theory ? What is the interpretation of the singularities in closures of isomorphism classes and of their tangent spaces ? What is the generic decomposition of the dimension type, and when is the generic representation indecomposable ? For which dimension types are there only finitely many isomorphism classes ?

Some of these questions were already answered by Kac and will be discussed in these notes too. But many of them are still open or have partial answers only in some very special cases. Furthermore it should be an important task to generalize Kac's results to quivers with relations. Again the set of representations of such a quiver of fixed dimension type forms an affine variety with the group of base change operators acting. But it will not be a vectorspace in general : it may have singularities and may even be reducible. Nevertheless the same questions as above can be asked here too, but -as far as we know- no real effort has been made yet to understand this more general situation from the geometric point of view. In particular, there is no handy description of the dimension types of the indecomposables even for finite or tame representation type. So once again there does not seem to be much hope...

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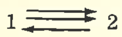
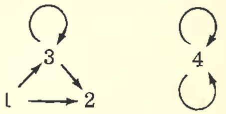
1. QUIVERS AND REPRESENTATIONS

1.1. A quiver Q consists of a set Q_0 of vertices, a set Q_1 of arrows and two maps $t, h : Q_1 \to Q_0$ assigning to an arrow φ its tail $t\varphi$ and its head $h\varphi$, respectively. We do not exclude loops nor multiple arrows ; i.e., $t\varphi$ and $h\varphi$ may coincide, and $t\varphi = t\psi$, $h\varphi = h\psi$ does not imply $\varphi = \psi$. We assume that Q_0 and Q_1 are finite, and we set $Q_0 = \{1, 2, \dots, n\}$.

Examples.

a) 

b) 

We fix an algebraically closed field k of arbitrary characteristic. A representation V of Q (over k) is a family $V(i)$, $i = 1, \dots, n$ of finite-dimensional k-vector spaces together with a k-linear map $V(\varphi) : V(th) \to V(h\varphi)$ for each arrow φ. The vector $\underline{\dim}\, V = (\dim V(1), \dots, \dim V(n)) \in \mathbb{N}^n$ is the dimension type of V. A morphism $f : V \to W$ is a family of k-linear maps $f(i) : V(i) \to W(i)$, $i = 1, \dots, n$, such that $W(\varphi) \circ f(t\varphi) = f(h\varphi) \circ V(\varphi)$ for all arrows φ.

The direct sum $V \oplus W$ of two representations V and W is defined by $(V \oplus W)(i) = V(i) \oplus W(i)$ and $(V \oplus W)(\varphi) = \begin{pmatrix} V(\varphi) & 0 \\ 0 & W(\varphi) \end{pmatrix}$. A representation V is called indecomposable if $V \neq 0$ and if $V = W_1 \oplus W_2$ implies $W_1 = 0$ or $W_2 = 0$.

1.2. Q is of finite representation type if Q has only finitely many indecomposable representations, up to isomorphism. For instance, the quiver

$$Q = 1 \to 2 \to \dots \to n-1 \to n$$

is of finite representation type ; the indecomposables are the $V_{i,j}$'s with $i \leq j$, which are defined by

$$V_{i,j}(\ell) = \begin{cases} k & \text{if } i \leq \ell \leq j \\ 0 & \text{otherwise} \end{cases}$$

$$V_{i,j}(\varphi) = \begin{cases} \mathbb{1} & \text{if } i \leq t\varphi < h\varphi \leq j \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}[column sep=small] V_{ij} = 0 \arrow[r, "0"] & 0 \ldots 0 \arrow[r, "0"] & k \arrow[r, "1"] & k \ldots k \arrow[r, "1"] & k \arrow[r, "0"] & 0 \ldots \arrow[r, "0"] & 0 . \\ & & i \arrow[u] & & j \arrow[u] & & \end{tikzcd}$$

If there exists a full embedding of the category of representations of $\circlearrowleft \cdot \circlearrowright$ into the category of representations of Q, Q is called **wild**. In this case, the problem of establishing a list of representatives of all indecomposables is considered hopeless. Finally, if Q is neither of finite representation type nor wild, it is said to be tame.

1.3. Tits form

The Tits form q_Q, a quadratic form on $\mathbb{Q}^n$ associated with Q, is defined as follows :

$$q(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i^2 - \sum_{\varphi\in Q_1} x_{t\varphi} x_{h\varphi} .$$

Obviously q_Q only depends on the non-oriented graph $\bar{Q}$ underlying Q.

The Cartan matrix C_Q describes the bilinear form $(\ ,\ )_Q$ associated with q_Q :

$$(x,y)_Q = q(x+y) - q(x) - q(y) = x C_Q y^T .$$

The components of $C_Q = (c_{ij})$ are :

$$c_{ij} = \begin{cases} 2 - 2\,\#\{\text{loops in } i\} & \text{if } i = j \\ -\,\#\{\text{edges linking } i \text{ and } j\} & \text{if } i \neq j . \end{cases}$$

Examples.

	Q	q_Q	C_Q
a)	$\circlearrowleft 1 \circlearrowright$	$-x_1^2$	(-2)
b)	$1 \to 2$	$x_1^2+x_2^2-x_1x_2$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
c)	$1 \to 2 \circlearrowright$	$x_1^2 - x_1x_2$	$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$

LEMMA [Ki, lemma 1.2]. Let Q be a connected quiver, q its Tits form and C its Cartan matrix.

a) q is positive definite if and only if $\bar{Q}$ is a Dynkin diagram.

b) q is positive semidefinite if and only if $\bar{Q}$ is an extended Dynkin diagram. In this case, rank $C = n-1$ and $\{\alpha \in \mathbb{N}^n : C\alpha \leq 0\} = \{\alpha \in \mathbb{N}^n : C\alpha = 0\} = \{\alpha\in\mathbb{N}^n : q(\alpha) = 0\} = \mathbb{N}\delta_Q$ for a unique $\delta_Q \in \mathbb{N}^n \setminus \{0\}$.

c) q is indefinite if and only if $C\alpha \geq 0$ for $\alpha \in \mathbb{N}^n$ implies $\alpha = 0$ and there exists an $\alpha \in \mathbb{N}^n$ such that $\alpha > 0$ and $C\alpha < 0$.

Here $\alpha > 0$ means that $\alpha(i) > 0$ and $\alpha \geq 0$ that $\alpha(i) \geq 0$ for all i. For a proof, see [B], [V].

Dynkin diagrams

A_m, $m \geq 1$ $\qquad 1 - 2 - \ldots - m{-}1 - m$

D_m, $m \geq 4$
$$\begin{tikzcd}[column sep=small, row sep=tiny] & & & & m{-}1 \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & \ldots \arrow[r, dash] & m{-}2 \arrow[ur, dash] \arrow[dr, dash] & \\ & & & & m \end{tikzcd}$$

E_m, $6 \leq m \leq 8$
$$\begin{tikzcd}[column sep=small] & & 4 \arrow[d, dash] & & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 5 \arrow[r, dash] & \ldots \arrow[r, dash] & m{-}1 \arrow[r, dash] & m \end{tikzcd}$$

Extended Dynkin diagrams $\qquad\qquad \delta_Q$

$\tilde{A}_m$, $m \geq 1$
$$\begin{tikzcd}[column sep=small] & & 0 \arrow[dll, dash] \arrow[drr, dash] & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & \ldots \arrow[r, dash] & m{-}1 \arrow[r, dash] & m \end{tikzcd} \qquad (11\ldots11)$$

$\tilde{D}_m$, $m \geq 4$
$$\begin{tikzcd}[column sep=small, row sep=tiny] 0 \arrow[dr, dash] & & & & & m{-}1 \\ & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \ldots \arrow[r, dash] & m{-}2 \arrow[ur, dash] \arrow[dr, dash] & \\ 1 \arrow[ur, dash] & & & & & m \end{tikzcd} \qquad (1122\ldots2211)$$

$\tilde{E}_6$
$$\begin{tikzcd}[column sep=small] & & 0 \arrow[d, dash] & & \\ & & 4 \arrow[d, dash] & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \end{tikzcd} \qquad (1123221)$$

$\tilde{E}_7$
$$\begin{tikzcd}[column sep=small] & & & 4 \arrow[d, dash] & & & \\ 0 \arrow[r, dash] & 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] & 7 \end{tikzcd} \qquad (12342321)$$

$\tilde{E}_8$
$$\begin{tikzcd}[column sep=small] & & 4 \arrow[d, dash] & & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 5 \arrow[r, dash] & \ldots \arrow[r, dash] & 8 \arrow[r, dash] & 0 \end{tikzcd} \qquad (124635432)$$

1.4. The following two fundamental results on representations of quivers are due to Gabriel (Theorem 1) and Donovan-Freislich and Nazarova (Theorem 2). Along with their generalizations to non algebraically closed fields, they can also be found in [DR].

Recall that for a Dynkin diagram $\bar{Q}$ the vectors $\alpha \in \mathbb{N}^n$ with

$q(\alpha) = 1$ are precisely the positive roots of the corresponding semisimple Lie algebra [B] . A similar statement holds for an extended Dynkin diagram $\bar{Q}$: the $\alpha \in \mathbb{N}^n$ with $q(\alpha) = 1$ are the positive real roots and the $\alpha \in \mathbb{N}\delta_Q \setminus \{0\}$ the positive imaginary roots of the corresponding infinite dimensional Kac-Moody algebra [K4] .

THEOREM 1 [G1, G2] . A connected quiver Q is of finite representation type if and only if $\bar{Q}$ is a Dynkin diagram. The map dim induces a bijection between isomorphism classes of indecomposable representations of Q and positive roots of Q .

THEOREM 2 [DF, N] . A connected quiver Q is tame if and only if $\bar{Q}$ is an extended Dynkin diagram. For each indecomposable V , $\dim V$ is a positive real or imaginary root of Q . For each positive real root α of Q , there exists a unique indecomposable V (up to isomorphism) with $\dim V = \alpha$. There exists a cofinite subset E_Q of $\mathbb{P}^1 k$ such that, for each positive imaginary root $\lambda\delta_Q$, the isomorphism classes of indecomposables V with $\dim V = \lambda\delta_Q$ are parametrized by E_Q .

2. THE REPRESENTATION SPACE OF A QUIVER

2.1. The representation space $R(Q,\alpha)$ of Q of dimension type $\alpha = (\alpha(1),\dots,\alpha(n)) \in \mathbb{N}^n$ is the set of representations

$$R(Q,\alpha) = \{V : V(i) = k^{\alpha(i)}, \ i = 1,\dots,n\} .$$

Since $V \in R(Q,\alpha)$ is determined by the maps $V(\varphi)$, we have

$$R(Q,\alpha) = \prod_{\varphi \in Q_1} \mathrm{Hom}_k\left(k^{\alpha(t\varphi)}, k^{\alpha(h\varphi)}\right) = \prod_{\varphi \in Q_1} M_\varphi ,$$

where M_φ is the set of matrices of size $\alpha(h\varphi) \times \alpha(t\varphi)$ with entries in k . We will consider $R(Q,\alpha)$ as an affine variety.

The algebraic group

$$GL(\alpha) = \prod_{i=1}^{n} GL(\alpha(i))$$

operates linearly (and regularly) on $R(Q,\alpha)$:

$$(g\cdot V)(\varphi) = g_{h\varphi} \circ V(\varphi) \circ g_{t\varphi}^{-1}$$

for $g = (g_1,\dots,g_n) \in GL(\alpha)$. The group $GL(\alpha)$ is the group of units of the finite dimensional k-algebra $M(\alpha) = \prod_{i=1}^{n} M(\alpha(i))$, where $M(s)$ is the algebra of $s\times s$-matrices. The group k^* diagonally embedded in $GL(\alpha)$ acts trivially, and we obtain an induced operation of

$$G(\alpha) = GL(\alpha)/k^*$$

on $R(Q,\alpha)$.

Using the notion of dimension for algebraic varieties, we can reinterpret the Tits form in the following way

$$q_Q(\alpha) = \dim GL(\alpha) - \dim R(Q,\alpha) .$$

2.2. By definition, the $GL(\alpha)$-orbits in $R(Q,\alpha)$ are just the isomorphism classes of representations. The stabilizer

$$C_{GL(\alpha)} V = \{g \in GL(\alpha) : g\cdot V = V\}$$

is the group Aut V of units in the endomorphism ring End $V \subseteq M(\alpha)$. Thus it is connected.

V is indecomposable if End V is local; i.e., the nilpotent endomorphisms form an ideal of codimension 1 . Equivalently, $k^* \subseteq C_{GL(\alpha)} V = \mathrm{Aut}\, V$ is a maximal torus, which means that every semisimple element of Aut V lies in k^* .

More generally, decomposing a representation V into indecomposables corresponds to choosing a maximal torus in Aut V . Indeed, if T is a maximal torus in Aut V , we can decompose

$$V(i) = \oplus_\chi V_\chi(i) \quad \text{with} \quad V_\chi(i) = \{v \in V(i) : t\cdot v = \chi(t)v \quad \text{for all} \quad t \in T\}$$

for all i , where $\chi : T \to k^*$ ranges over the characters of T . Then $V(\varphi)(V_\chi(t\varphi)) \subseteq V_\chi(h\varphi)$ for all arrows φ , and we thus obtain a decomposition $V = \oplus_\chi V_\chi$. Since T operates on V_χ by scalar multiplication, $k^* \subseteq \mathrm{Aut}\, V_\chi$ is a maximal torus, and therefore V_χ is indecomposable. Conversely, if $V = V_1 \oplus \dots \oplus V_r$ with V_i indecomposable, the product of the maximal tori $k^* \subseteq \mathrm{Aut}\, V_i$ is a maximal torus of Aut V .

The map $g \mapsto g \cdot V$ induces an isomorphism

$$GL(\alpha)/C_{GL(\alpha)}V \to \mathcal{O}_V ,$$

where $\mathcal{O}_V$ is the orbit of V. This implies

$$\dim \mathcal{O}_V + \dim \operatorname{End} V = \dim GL(\alpha) .$$

Since $k \subseteq \operatorname{End} V$ for any representation V, we get

$$\dim \mathcal{O}_V \leq \dim GL(\alpha) - 1 .$$

Using this inequality, Tits found a very nice argument, which proves part of theorem 1 in 1.4 [G2]. Assume that Q is a connected quiver of finite representative type and choose $\alpha \in \mathbb{N}^n \setminus \{0\}$. Since any representation of Q can be decomposed into a direct sum of indecomposables, $R(Q,\alpha)$ contains only finitely many $GL(\alpha)$-orbits. So one orbit must be dense and thus have the same dimension as $R(Q,\alpha)$. Therefore

$$\dim R(Q,\alpha) \leq \dim GL(\alpha) - 1$$

or equivalently

$$q_Q(\alpha) \geq 1 .$$

Since all off-diagonal entries of C_Q are non-positive, it follows that q_Q is positive definite on $\mathbb{Z}^n$ and hence on $\mathbb{Q}^n$. Thus $\bar{Q}$ is a Dynkin diagram (lemma 1.3).

2.3. In this paragraph we study $R(Q,\alpha)$, its decomposition into sheets (2.4) and the indecomposables in each sheet for a particular example, which should serve as motivation and illustration for the general definitions. The notations used here are adapted to those introduced later.

We consider the wild quiver

$$Q = \begin{tikzcd} 1 \arrow[r, shift left=2, "\varphi"] \arrow[r, "\psi"'] & 2 \arrow[l, shift left=4, "\chi"] \end{tikzcd}$$

and the dimension vector $\alpha = (2,1)$. We have

$$\dim R(Q,\alpha) = 6 ,$$
$$\dim GL(\alpha) = 5 ,$$
$$q_Q(x,y) = x^2 + y^2 - 3xy , \quad q_Q(\alpha) = -1 ,$$
$$(\alpha, (10))_Q = 1 , \quad (\alpha, (01))_Q = -4 .$$

The set $C = \left\{ V : \det\begin{pmatrix} V(\varphi) \\ V(\psi) \end{pmatrix} = 0 \right\}$ is closed and irreducible in $X = R(Q,\alpha)$. Every representation V in $X^{(1)} = X \setminus C$ has a unique representative of the form

$$V(\varphi) = (1\ 0) , \quad V(\psi) = (0\ 1) , \quad V(\chi) = \binom{\alpha}{\beta} \in k^2 .$$

Note that $X^{(1)}$ is irreducible. It consists of indecomposable representations with endomorphism ring k.

Every representation in C is isomorphic to precisely one of the following

i) $V(\varphi) = (\alpha\ 0)$, $V(\psi) = (\beta\ 0)$, $V(\chi) = \binom{0}{1}$ with $(\alpha\ \beta) \in \mathbb{P}^1 k$,

ii) $V(\varphi) = (\alpha\ 0)$, $V(\psi) = (\beta\ 0)$, $V(\chi) = \binom{1}{0}$ with $(\alpha\ \beta) \in k^2 \setminus \{0\}$,

iii) $V(\varphi) = (\alpha\ 0)$, $V(\psi) = (\beta\ 0)$, $V(\chi) = \binom{0}{0}$ with $(\alpha\ \beta) \in \mathbb{P}^1 k$,

iv) $V(\varphi) = V(\psi) = (0\ 0)$, $V(\chi) = \binom{1}{0}$,

v) $V(\varphi) = V(\psi) = (0\ 0)$, $V(\chi) = \binom{0}{0}$.

The representations of type i) are indecomposable with endomorphism ring $k[t]/(t^2)$, all others are decomposable with endomorphism rings of dimension 2, 3, 3 and 5 for the types ii), iii), iv) and v), respectively. Denote by $X^{(d)}$ the set of representations with d-dimensional endomorphism ring.

These sets can be described as follows :

$$X^{(2)} = \left\{ V \in X : \operatorname{rank}\begin{pmatrix} V(\varphi) \\ V(\psi) \end{pmatrix} = 1 , \ V(\chi) \neq 0 \right\} ,$$

$$X^{(3)} = \left\{ V \in X : \operatorname{rank}\begin{pmatrix} V(\varphi) \\ V(\psi) \end{pmatrix} = 1 , \ V(\chi) = 0 \right\} \cup \left\{ V \in X : V(\varphi) = V(\psi) = 0 , \ V(\chi) \neq 0 \right\} ,$$

$$X^{(5)} = \{0\} .$$

$X^{(3)}$ has two disjoint irreducible components of dimension 2 and 3, respectively, whereas $X^{(2)}$ and $X^{(5)}$ are irreducible. The set of indecomposables within each $X^{(d)}$ is closed. Indeed, $V \in X^{(2)}$ is indecomposable if and only if $V(\varphi) \circ V(\chi) = 0 = V(\psi) \circ V(\chi)$. But the set of all indecomposables is neither open nor closed nor locally closed (= open $\cap$ closed) in X. We will see in 2.5 that these are general facts.

Remark. This example shows that $R(Q,\alpha)$ may contain a dense open set of indecomposables without the set of all indecomposables being open. This contradicts the statements in [K1, 2.8], [K2, §4] which lead to the definition of the "canonical decomposition of α" [K1, (2.24)].

2.4. We introduce some notions and results used later for the general setting of an algebraic group G operating regularly on an irreducible variety Z (cf. [Kr 2, II.2)].

For any $z \in Z$, the orbit $G\cdot z$ is open in $\overline{G\cdot z}$. In particular, if $z' \in \overline{G\cdot z} \setminus G\cdot z$, then $\dim G\cdot z' < \dim G\cdot z$, or equivalently $\dim C_G z' > \dim C_G z$, where $C_G z = \{g \in G : g\cdot z = z\}$.

The fixed point set $Z^g = \{z \in Z : g\cdot z = z\}$ is closed in Z for any $g \in G$: identify Z^g with the inverse image of the diagonal under the regular map $Z \to Z \times Z$ given by $z \mapsto (z, g\cdot z)$. Thus

$$Z^G = \{z \in Z : g\cdot z = z, \ \forall g \in G\}$$

is closed as well.

For $s \in \mathbb{N}$ the set

$$Z_{(s)} = \{z \in Z : \dim G\cdot z = s\}$$

is locally closed in Z, since by Chevalley's theorem ([EGA IV, §13], cf. [Kr 2, II.2.6]) the function $z \mapsto \dim C_G z$ is upper semicontinuous. In particular, the union Z^{max} of all orbits of maximal dimension is open and dense in Z. An irreducible component $\mathcal{S}$ of a $Z_{(s)}$ is called a sheet of Z for the action of G. All orbits in $\mathcal{S}$ are closed in $\mathcal{S}$ and have the same dimension.

As an example, we consider the operation of $G = GL(n)$ on $Z = M(n)$ by conjugation. With a matrix A having eigenvalues $\lambda_1, \dots, \lambda_r$ and Jordan blocks with eigenvalue λ_i of size $\rho_{i1} \geq \rho_{i2} \geq \dots \geq \rho_{in} \geq 0$ for $i = 1, \dots, r$, we associate the partition $p_A = (p_1, \dots, p_n)$ of n, where $p_j = \sum_{i=1}^{r} \rho_{ij}$. This is the partition corresponding to the dimensions of the invariant factor modules for the n-dimensional $k[T]$-module given by A. It is easy to see that all matrices A yielding the same partition belong to the same sheet of Z. In fact, we have the following result, which is due to Dixmier, Peterson, Kraft (cf. [Kr 1], [P]).

PROPOSITION. The map $A \mapsto p_A$ induces a bijection between sheets of $M(n)$ and partitions of n. The sheets are disjoint. They are smooth, and each one contains exactly one nilpotent conjugacy class and a dense open set of semisimple matrices. The orbit space $\mathcal{S}_p/GL(n)$ of the sheet corresponding to $p = (p_1, \dots, p_n)$ is isomorphic to k^{p_1}.

2.5. Fix Q and set $R(\alpha) = R(Q,\alpha)$. Put

$$R(\alpha)^{(d)} = \{V \in R(\alpha) : \dim \operatorname{End} V = d\}$$

for $d \in \mathbb{N}$.

PROPOSITION. a) $R(\alpha)^{(d)}$ is locally closed in $R(\alpha)$; $R(\alpha)^{max}$ is open and dense in $R(\alpha)$.

b) $R(\alpha)^{(d)}_{ind} = \{V \in R(\alpha)^{(d)} : V \text{ indecomposable}\}$ is closed in $R(\alpha)^{(d)}$.

As a consequence, $R(\alpha)_{ind} = \bigcup_{d \in \mathbb{N}} R(\alpha)^{(d)}_{ind}$ is a constructible set; i.e., a finite union of locally closed sets.

Proof. a) follows from 2.4, since

$$R(\alpha)^{(d)} = R(\alpha)_{\bar{d}}$$

with $\bar{d} = \dim GL(\alpha) - d$.

For b) consider the closed subvariety

$$N = \{(V, \rho) \in R(\alpha) \times M(\alpha) : \rho \in \operatorname{End} V, \ \rho \text{ nilpotent}\}$$

and the projection

$$p : N \to R(\alpha).$$

The fiber $p^{-1}(V)$ of a representation $V \in R(\alpha)$ consists of the nilpotent endomorphisms of V. Since the zero section $R(\alpha) \to N$ meets every irreducible component of every fiber, the function $V \mapsto \dim p^{-1}(V)$ is upper semicontinuous (theorem of Chevalley). But $V \in R(\alpha)$ is indecomposable if and only if $\dim p^{-1}(V) \geq \dim \operatorname{End} V - 1$, and so

$$R(\alpha)^{(d)}_{ind} = \{V \in R(\alpha)^{(d)} : \dim p^{-1}(V) \geq d-1\}$$

is closed in $R(\alpha)^{(d)}$.

Remark (Happel). If $V = V' \oplus V'' \in R(\alpha)^{max}$, then $\mathrm{Ext}^1(V', V'') = 0$. Indeed, if there exists a non-split extension

$$0 \longrightarrow V'' \longrightarrow W \longrightarrow V' \longrightarrow 0,$$

Then $\mathcal{O}_V \subseteq \bar{\mathcal{O}}_W \setminus \mathcal{O}_W$, which contradicts $V \in R(\alpha)^{max}$.

We embed $GL(\alpha)$ as blocks along the diagonal into $M(N)$, $N = \sum_{i=1}^{n} \alpha(i)$. A minisheet of $GL(\alpha)$ is an irreducible component of the intersection $\mathcal{S} \cap GL(\alpha)$, where $\mathcal{S}$ is a sheet of $M(N)$ with respect to the operation of $GL(N)$ by conjugation (2.4). Part a) of the following lemma implies that each minisheet is contained in a sheet of $GL(\alpha)$, where we consider the action of $GL(\alpha)$ on itself by conjugation.

LEMMA. a) The functions $g \mapsto \dim C_{GL(\alpha)}g$ and $g \mapsto \dim R(\alpha)^g$ are constant on minisheets.

b) Each minisheet contains a dense set of semisimple elements.

Proof. a) $GL(\alpha)$ operates on $\mathrm{Hom}_k(k^{\alpha(i)}, k^{\alpha(j)})$ by $g \circ f = g_j \circ f \circ g_i^{-1}$ for $g = (g_1, \dots, g_n)$, and the function $g \mapsto \dim \mathrm{Hom}(k^{\alpha(i)}, k^{\alpha(j)})^g$ is upper semicontinuous on $GL(\alpha)$. On the other hand, we have

$$M(N) = \prod_{i,j} \mathrm{Hom}(k^{\alpha(i)}, k^{\alpha(j)})$$

and

$$M(N)^g = \prod_{i,j} \mathrm{Hom}(k^{\alpha(i)}, k^{\alpha(j)})^g$$

for $g \in GL(\alpha)$. The function $g \mapsto \dim M(N)^g$ is constant on each sheet $\mathcal{S}$ of $M(N)$ and hence constant on minisheets. Therefore the functions $g \mapsto \dim \mathrm{Hom}(k^{\alpha(i)}, k^{\alpha(j)})^g$ are also constant on minisheets. But we have

$$R(\alpha)^g = \prod_{\varphi \in Q_1} \mathrm{Hom}(k^{\alpha(t\varphi)}, k^{\alpha(h\varphi)})^g$$

and

$$C_{GL(\alpha)}g = \{\text{units of } \prod_{i=1}^{n} (\mathrm{End}\ k^{\alpha(i)})^g\}$$

for $g \in GL(\alpha)$.

b) Let $\mathcal{S}$ be a sheet of $M(N)$ and $\mathcal{S}'$ an irreducible component of $\mathcal{S} \cap GL(\alpha)$. Choose an element $x \in \mathcal{S}'$ which does not lie in any other irreducible component of $\mathcal{S} \cap GL(\alpha)$. Considering the component of x in each $GL(\alpha(i))$ separately, we may suppose that x is in Jordan normal form. As an easy consequence of the description of sheets in $M(N)$ given in 2.4, we find an invertible diagonal matrix $d \in \mathcal{S}$ such that the line

$$L = \{\lambda s + (1-\lambda)d : \lambda \in k\}$$

is contained in $\mathcal{S}$. Hence $L' = L \cap GL(\alpha)$ is an irreducible curve in $\mathcal{S} \cap GL(\alpha)$ containing x and d. By the choice of x, L' -and thus d- is contained in $\mathcal{S}'$.

So we found one semisimple element in $\mathcal{S}'$. But the set of semisimple elements in $\mathcal{S}$ is open and dense (2.4), and therefore $\mathcal{S}'$ contains a dense set of semisimples.

2.6. DEFINITION. $V \in R(\alpha)$ is stably indecomposable if there exists an open neighborhood of V consisting of indecomposable representations.

THEOREM. V is stably indecomposable if and only if $\mathrm{End}\ V = k$.

Proof. If $\mathrm{End}\ V = k$, all representations in the dense open set $R(\alpha)^{max}$ have endomorphism ring k.

Conversely, suppose that V is indecomposable and has an automorphism $g_0 \notin k^*$. Choose an open neighborhood U of V. We want to show that U contains a representation admitting a semisimple automorphism outside of k^*. Then V cannot be stably indecomposable. Set

$$S = \{g \in GL(\alpha) : \dim R(\alpha)^g = \dim R(\alpha)^{g_0}\}$$

and

$$E = \bigcup_{W \in U} \mathrm{End}\ W \subseteq M(\alpha).$$

Since $g_0 \notin k^*$, S does not intersect k^*. Moreover, S contains a dense set of semisimple elements, since it is a union of minisheets (2.5). The following lemma implies that $E \cap S$ is open in S. Hence E contains semisimple elements, and the theorem follows.

LEMMA. Let G be an algebraic group operating linearly on a vectorspace V, $U \subseteq V$ an open subset, $g_0 \in G$. Set $S = \{g \in G : \dim V^g = \dim V^{g_0}\}$.

Then

$S' = \{ g \in S \ : \ \exists u \in U \ \text{ with } \ g \cdot u = u \}$

is open in S .

Proof. Consider the vector bundle

$p_1 \ : S \times V \to S$.

By the definition of S ,

$L = \{ (s, v) : s \cdot v = v \}$

is a subbundle. Since the restriction $\varphi = p_1/L \ : \ L \to S$ is flat, the image

$S' = \varphi(L \cap S \times U)$

is open.

2.7. Generic decomposition.

PROPOSITION. For $\alpha \in \mathbb{N}^n$ there exists a unique decomposition $\alpha = \alpha_1 + \ldots + \alpha_s$ such that the set

$R(\alpha)_{gen} = \{ V \in R(\alpha) : V = V_1 \oplus \ldots \oplus V_s \ , \ \dim V_i = \alpha_i \ , \ V_i \text{ indecomposable} \}$

contains an open and dense set of $R(\alpha)$.

$\alpha = \alpha_1 + \ldots + \alpha_s$ is called the generic decomposition of α , representations in $R(\alpha)^{max}_{gen} = R(\alpha)_{gen} \cap R(\alpha)^{max}$ are called generic representations of type α .

Remarks.

a) As example 2.3 shows, $R(\alpha)_{gen}$ is not necessarily open in $R(\alpha)$.

b) The generic decomposition depends on the orientation of Q : choose $\bar{Q} =$  and $\alpha = \begin{smallmatrix} & 2 & \\ 1 & & 1 \end{smallmatrix}$. For the orientation  , the generic decomposition is $\begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix} + \begin{smallmatrix} & 1 & \\ 0 & & 0 \end{smallmatrix}$, whereas α is generically indecomposable for the orientation  .

Proof. For each decomposition $\alpha = \beta_1 + \ldots + \beta_t$, $\beta_i \in \mathbb{N}^n$, the set of representations V such that $V = V_1 \oplus \ldots \oplus V_t$ with $\dim V_i = \beta_i$ is constructible, since it can be viewed as the image of $GL(\alpha) \times R(\beta_1) \times \ldots \times R(\beta_t)$ under the map $(g, V_1, \ldots, V_t) \mapsto g \cdot (V_1 \oplus \ldots \oplus V_t)$. Thus the set

$R(\alpha ; \beta_1, \ldots, \beta_t) = \{ V : V = V_1 \oplus \ldots \oplus V_t \ , \ \dim V_i = \beta_i \ , \ V_i \text{ indecomposable} \}$

is constructible as well, and $R(\alpha)$ is the disjoint union of the $R(\alpha \ ; \beta_1, \ldots, \beta_t)$, taken over the finite set of all distinct decompositions of α . So precisely one of these sets, say $R(\alpha \ ; \alpha_1, \ldots, \alpha_s)$, contains an open dense set of $R(\alpha)$.

As a consequence of this and theorem 2.6 we obtain

COROLLARY. Let $V = V_1 \oplus \ldots \oplus V_s$ be generic, V_i indecomposable. Then $\operatorname{End} V_i = k$, for all i, and $\operatorname{Ext}^1(V_i, V_j) = 0$ for $i \neq j$. In particular, if the generic representation V is indecomposable, we have $R(\alpha)^{max} \subseteq R(\alpha)_{ind}$ and $\operatorname{End} V = k$.

3. THE FUNDAMENTAL SET

3.1. Denote by $\varepsilon_1, \ldots, \varepsilon_n$ the standard basis of $\mathbb{Q}^n$.

The fundamental set F_Q is defined by

$F_Q = \{ \alpha \in \mathbb{N}^n \setminus \{0\} \ : \ (\alpha, \varepsilon_i) \leq 0 \ , \ \operatorname{supp} \alpha \text{ connected} \}$.

Here $(\ , \ )$ is the bilinear form $(\ , \ )_Q$ defined in 1.3, and $\operatorname{supp} \alpha$ denotes the full subquiver of Q whose vertices are $\{ i : \alpha(i) \neq 0 \}$.

The following result is an easy consequence of lemma 1.3.

LEMMA 1. Let Q be connected.

a) $F_Q = \emptyset$ if and only if $\bar{Q}$ is a Dynkin diagram.

b) $F_Q = \mathbb{N}\delta \setminus \{0\}$ for some $\delta \neq 0$ if and only if $\bar{Q}$ is an extended Dynkin diagram ; in this case $\delta = \delta_Q$.

c) If $q_Q(\alpha) = 0$ for some $\alpha \in F_Q$, then $\operatorname{supp} \alpha$ is tame (i.e. $\operatorname{supp} \alpha$ is an extended Dynkin diagram).

LEMMA 2. Let $\alpha = \beta_1 + \ldots + \beta_r \in F_Q$ with $r \geq 2$ and $\beta_1, \ldots, \beta_r \in \mathbb{N}^n \setminus \{0\}$ such that $q(\alpha) \geq q(\beta_1) + \ldots + q(\beta_r)$. Then $\operatorname{supp} \alpha$ is tame, and α is a multiple of $\delta_{\operatorname{supp} \alpha}$.

Proof. We first consider the case $r = 2$ and set $\beta_1 = \gamma = \Sigma c_i \varepsilon_i$, $\beta_2 = \delta = \Sigma d_i \varepsilon_i$, $\alpha = \Sigma a_i \varepsilon_i$. We may suppose $Q = \operatorname{supp} \alpha$. By assumption we have $(\gamma,\delta) = q(\alpha) - q(\gamma) - q(\delta) \geq 0$. An easy computation based on $a_i = c_i + d_i$ and $c_{ij} = c_{ji}$ for the coefficients of the Cartan matrix $C = (c_{ij})$ of Q (1.3) yields

$$0 \leq (\gamma,\delta) = \sum_{i,j} c_{ij} c_i d_j$$

$$= \sum_j \frac{c_j d_j}{a_j} \sum_i c_{ij} a_i + \tfrac{1}{2} \sum_{i \neq j} c_{ij} \left(\frac{c_i}{a_i} - \frac{c_j}{a_j} \right)^2 a_i a_j \; .$$

Since

$$(\alpha, \varepsilon_j) = \sum_i c_{ij} a_i \leq 0 \quad \text{for all } j \text{ and } c_{ij} \leq 0 \quad \text{for all } i \neq j \; ,$$

this inequality implies

$$\frac{c_i}{a_i} = \frac{c_j}{a_j} \quad \text{if } c_{ij} \neq 0 \; .$$

But Q is connected, and therefore α and γ are proportional. As a consequence, we have $(\alpha, \varepsilon_j) = 0$ for all j , hence $C\alpha = 0$. But then Q is tame and $\alpha \in \mathbb{N}\delta_Q$ (lemma 1.3c)).

In case $r > 2$, we have

$$(\alpha,\alpha) = \sum_i (\alpha, \beta_i) \geq \sum_i (\beta_i, \beta_i) \; .$$

This implies

$$(\alpha - \beta_i, \beta_i) \geq 0$$

for some i , and we apply what we already proved to $\gamma = \beta_i$, $\delta = \alpha - \beta_i$.

3.3. THEOREM. If α lies in F_Q and $\operatorname{supp}\alpha$ is not tame, then the generic representation in $R(\alpha)$ is indecomposable.

Proof. Let $\alpha = \alpha_1 + \ldots + \alpha_s$ be the generic decomposition, and suppose $s \geq 2$. Set

$$R' = R(\alpha_1) \times \ldots \times R(\alpha_s)$$

and

$$G' = GL(\alpha_1) \times \ldots \times GL(\alpha_s) \; .$$

The image of

$$\varphi : GL(\alpha) \times R' \longrightarrow R(\alpha)$$
$$(g, V) \longmapsto g \cdot V$$

is dense in $R(\alpha)$ by construction, and φ is constant on the orbits of the free action of G' on $GL(\alpha) \times R'$ given by $h \cdot (g, V) = (gh^{-1}, h \cdot V)$. As a consequence,

$$\dim GL(\alpha) + \dim R' - \dim G' \geq \dim R(\alpha) \; ,$$

which implies

$$q(\alpha) \geq q(\alpha_1) + \ldots + q(\alpha_s) \; ,$$

in contradiction with lemma 2 of 3.2.

3.4. Number of parameters.

Let G be an algebraic group acting on a variety Z . If $X \subseteq Z$ is a G-stable subset, we write

$$X = \cup X_{(s)}$$

with

$$X_{(s)} = \{ x \in X : \dim \mathcal{O}_x = s \} \; .$$

DEFINITION. The number of parameters of X , is

$$\mu(X) = \max_s (\dim X_{(s)} - s) \; .$$

Here $\dim X_{(s)}$ denotes the dimension of the closure of $X_{(s)}$ in Z .

Example. If the generic representation in $R(\alpha)$ is indecomposable, we have

$$\mu(R(\alpha)^{\max}) = \dim R(\alpha) - (\dim GL(\alpha) - 1) = 1 - q(\alpha)$$

(corollary 2.7).

THEOREM. If α lies in F_Q and $\operatorname{supp}\alpha$ is not tame, then

$$\mu(R(\alpha)_{\mathrm{ind}}) = \mu(R(\alpha)^{\max}) = 1 - q(\alpha) > \mu(R(\alpha)^{(d)}_{\mathrm{ind}})$$

for all $d > 1$.

For the proof we need :

PROPOSITION. Let G act on Z , and suppose that G contains a finite number of unipotent conjugacy classes. Then the number of parameters of

$$X = \{ z \in Z : G_z \text{ unipotent} \}$$

satisfies

$$\mu(X) \leq \max_{\substack{u \in G \\ u \text{ unipotent}}} (\dim Z^u - \dim G_u) \; .$$

Remark. The Jordan normal form shows that $GL(\alpha)$ and also $G(\alpha)$ contain only finitely many unipotent conjugacy classes. In fact, this holds for any reductive group [L].

Proof. Consider the closed subvariety

$$L = \{(g,z) \in G\times Z : g\cdot z = z\}$$

of $G\times Z$ and the projections

$$\varphi = pr_2 : L \to Z \quad , \quad \psi = pr_1 : L \to G .$$

For $z \in Z$, we have

$$\varphi^{-1}(z) = C_G z \times \{z\} .$$

If z lies in $X_{(s)}$, $\dim C_G z = \dim G - s$, and therefore

$$\dim \varphi^{-1}(X_{(s)}) = \dim X_{(s)} + \dim G - s .$$

Consequently,

$$\mu(X) = \max_s (\dim X_{(s)} - s) = -\dim G + \max_s \dim \varphi^{-1}(X_{(s)})$$

$$= -\dim G + \dim \varphi^{-1}(X) .$$

The definition of X implies $\varphi^{-1}(X) \subseteq \psi^{-1}(U)$, where

$$U = \{u \in G : u \text{ unipotent}\} .$$

Since U consists of a finite number of conjugacy classes

$$C_u = \{gug^{-1} : g \in G\},$$

we obtain

$$\mu(X) \le -\dim G + \max_u \dim \psi^{-1}(C_u) .$$

But

$$\psi^{-1}(g) = \{g\} \times Z^g \quad \text{for } g \in G ,$$

and thus

$$\dim \psi^{-1}(C_u) = \dim Z^u + \dim C_u = \dim Z^u + \dim G - \dim G_u .$$

This proves the proposition.

Proof of the theorem. Recall that $V \in R(\alpha)$ is indecomposable if and only if $C_{G(\alpha)}V$ is unipotent, where $G(\alpha) = GL(\alpha)/k^*$ (2.1). By 3.3, $R(\alpha)^{max}$ is contained in $R(\alpha)_{ind}$, and we already saw that $\mu(R(\alpha)^{max}) = 1 - q(\alpha)$. Set $\bar{R} = R(\alpha)\setminus R(\alpha)^{max}$. The lemma below implies that

$$\dim \bar{R}^u - \dim G(\alpha)_u = \dim \bar{R}^u - \dim GL(\alpha)_u + 1 < 1 - q(\alpha)$$

for any unipotent element $u \neq 1\!\!1$. For $u = 1\!\!1$,

$$\dim \bar{R} - \dim G(\alpha) < \dim R - \dim GL(\alpha) + 1 < 1 - q(\alpha) .$$

Applying the proposition to $\bar{R}$ and $G(\alpha)$, we find

$$\mu(\bar{R}_{ind}) = \max_{d>1} \mu(R(\alpha)^{(d)}_{ind}) < 1 - q(\alpha) .$$

LEMMA. If α belongs to F_Q and $\operatorname{supp}\alpha$ is not tame, then $\dim GL(\alpha)_g - \dim R(\alpha)^g > q(\alpha)$ for $g \in GL(\alpha)\setminus k^*$.

Proof. The left hand side being constant on minisheets (2.5), we may suppose that g is semisimple. Let $\alpha = \alpha_1 + \ldots + \alpha_s$ be the decomposition obtained from the eigen space decomposition of g, and note that $s \ge 2$ since $g \notin k^*$. Then we have

$$GL(\alpha)_g = \prod GL(\alpha_i) \quad \text{and} \quad R(\alpha)^g = \prod R(\alpha_i)$$

and consequently (3.2)

$$\dim GL(\alpha)_g - \dim R(\alpha)^g = \sum_{i=1}^{s} q(\alpha_i) > q(\alpha) .$$

Remark. The theorem shows that for $\alpha \in F_Q$ with $\operatorname{supp}\alpha$ not tame, the number of parameters of indecomposables in the maximal sheet is strictly bigger than in all other sheets. In fact, the proof given in chapter 5 that this is true whenever the generic representation is indecomposable. It is an open question whether the maximal number of parameters of indecomposables always occurs in $R(\alpha)^{(d)}_{ind}$ for the smallest number d with $R(\alpha)^{(d)}_{ind} \neq \phi$.

3.5. Remark. From the classification of indecomposables for extended Dynkin diagrams (cf. [DR]) one obtains: if Q is tame, $R(\alpha)_{ind}$ is contained in $R(\alpha)^{max}$. For $\alpha = \lambda\delta_Q$, the number of parameters is λ for $R(\alpha)^{max}$ and $1 = 1 - q(\alpha)$ for $R(\alpha)_{ind}$; the generic decomposition is $\alpha = \delta_Q + \ldots + \delta_Q$.

Examples.

a) $Q = 1 \rightrightarrows 2$, $\alpha = (2,2)$.

$$R(\alpha)^{max} = GL(\alpha)\cdot \Big\{ V : V(\varphi) = 1\!\!1 ,\ V(\psi) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} ,\ \lambda_1 \neq \lambda_2 \in k$$

$$\text{or } V(\psi) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} ,\ \lambda \in k \Big\} \cup$$

$$GL(\alpha)\cdot\left\{V : V(\psi) = \mathbb{1}\ ,\ V(\varphi) = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},\quad \lambda_1 \neq \lambda_2 \in k\right.$$

$$\left.\text{or } V(\varphi) = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\ ,\ \lambda \in k\right\}.$$

$$R(\alpha)^{\mathrm{ind}} = GL(\alpha)\cdot\left\{V : V(\varphi) = \mathbb{1}\ ,\ V(\psi) = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\ ,\ \lambda \in k\right\} \cup$$
$$GL(\alpha)\cdot\left\{V : V(\varphi) = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\ ,\ \lambda \in k\ ,\quad V(\psi) = \mathbb{1}\right\}.$$

b) $Q = $
$$\begin{tikzcd} & 2 \arrow[dr, "\psi"] & \\ 1 \arrow[ur, "\varphi"] \arrow[rr, "\chi"'] & & 3 \end{tikzcd}$$
$,\quad \alpha = (1,2,1)$.

$R(\alpha)_{\mathrm{ind}} = GL(\alpha)\cdot S$ with $S(\varphi) = \begin{pmatrix}1\\0\end{pmatrix}$, $S(\psi) = (0\ 1)$, $S(\chi) = 1$,

$$R(\alpha)^{\max} = GL(\alpha)\cdot\left\{V : V(\varphi) = \begin{pmatrix}1\\0\end{pmatrix}\ ,\ V(\psi) = (1\ 0)\ ,\ V(\chi) = \lambda \in k\right\}$$
$$\dot{\cup}\ R(\alpha)_{\mathrm{ind}}\ .$$

4. INDECOMPOSABLES AND ROOT SYSTEMS

4.1. Reflection functors

Let Q and α be as before. Fix a source i of Q, and suppose

$$\sum_{t\varphi=i} \alpha(h\varphi) \geq \alpha(i)\ .$$

Consider the set

$$R'(Q,\alpha) = \left\{V \in R(Q,\alpha)\ :\ [V(\varphi)] : V(i) \longrightarrow \bigoplus_{t\varphi=i} V(h\varphi)\right\}\ \text{injective}\ .$$

Obviously $R(Q,\alpha)_{\mathrm{ind}}$ is contained in $R'(Q,\alpha)$, and $R(\alpha)_{\mathrm{ind}} = \emptyset$ if α does not satisfy the required inequality, unless $\alpha = \epsilon_i$.

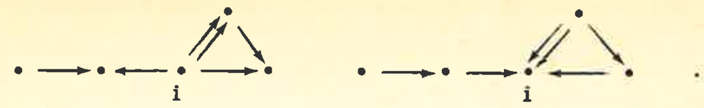
The quiver Q^* is obtained from Q by reversing all arrows with tail i, and α^* is given by

$$\alpha^*(k) = \begin{cases}\alpha(k) & \text{for } k \neq i\\ \sum_{t\varphi=i} \alpha(h\varphi) - \alpha(i) & \text{for } k = i\ .\end{cases}$$

We have

$$\sum_{h\varphi=i} \alpha^*(t\varphi) \geq \alpha^*(i)\ .$$

Example.



$Q \qquad\qquad Q^*$

We set

$$R'(Q^*,\alpha^*) = \left\{V \in R(Q^*,\alpha^*)\ :\ [V(\varphi)] : \bigoplus_{h\varphi=i} V(t\varphi) \longrightarrow V(i)\ \text{surjective}\right\}.$$

PROPOSITION. There exists a homeomorphism

$$R'(Q,\alpha)/GL(\alpha) \xrightarrow{\sim} R'(Q^*,\alpha^*)/GL(\alpha^*)$$

such that corresponding representations have isomorphic endomorphism rings.

(We use the quotient topology of the Zariski topology).

Proof. Set $m = \sum_{\substack{\varphi\in Q_1\\ t\varphi=i}} \alpha(h\varphi)$, $W = k^m$, and

$$\bar{R} = \prod_{\substack{\varphi\in Q_1\\ t\varphi\neq i}} \mathrm{Hom}(k^{\alpha(t\varphi)}, k^{\alpha(h\varphi)})\ ,\qquad \bar{G} = \prod_{j\neq i} GL(\alpha(j))\ .$$

The required homeomorphism is obtained from the following diagram, in which we use the isomorphisms

$$R'(Q,\alpha)/GL(\alpha(i)) \xrightarrow{\sim} \bar{R} \times Gr_{\alpha(i)}W$$
$$V \longmapsto \left((V(\varphi))_{t\varphi\neq i},\ \mathrm{im}[V(\varphi)]_{t\varphi=i}\right)$$

$$R'(Q^*,\alpha^*)/GL(\alpha^*(i)) \xrightarrow{\sim} \bar{R} \times Gr_{\alpha(i)}W$$
$$V \longmapsto \left((V(\varphi))_{h\varphi\neq i},\ \ker[V(\varphi)]_{h\varphi=i}\right).$$

Here $Gr_{\alpha(i)}W$ denotes the Grassmann variety of $\alpha(i)$-dimensional subspaces of W.

$$\begin{tikzcd}
R'(Q,\alpha) \arrow[d, "/GL(\alpha(i))"] & & R'(Q^*,\alpha^*) \arrow[d, "/GL(\alpha^*(i))"] \\
R'(Q,\alpha)/GL(\alpha(i)) \arrow[dr, "\sim"] \arrow[dd, "/\bar{G}"] & & R'(Q^*,\alpha^*)/GL(\alpha^*(i)) \arrow[dl, "\sim"'] \arrow[dd, "/\bar{G}"] \\
& \bar{R} \times Gr_{\alpha(i)}W \arrow[dl] \arrow[dr] & \\
R'(Q,\alpha)/GL(\alpha) \arrow[rr, dashed, "\sim"] & & R'(Q^*,\alpha^*)/GL(\alpha^*)
\end{tikzcd}$$

The claim about endomorphism rings follows from

$$C_{\overline{G} \times GL(\alpha(i))} V \xrightarrow{\sim} C_{\overline{G}} \overline{V} ,$$

$$C_{\overline{G} \times GL(\alpha^*(i))} V^* \xrightarrow{\sim} C_{\overline{G}} \overline{V}^*$$

for $V \in R(Q,\alpha)$, $V^* \in R(Q^*, \alpha^*)$, where $\overline{V}$ and $\overline{V}^*$ are the images in $\overline{R} \times Gr_{\alpha(i)} W$.

COROLLARY. <u>The number of parameters as well as the number of irreducible components of maximal dimension coincide for</u> $R(Q,\alpha)^{(d)}_{ind}$ <u>and</u> $R(Q^*, \alpha^*)^{(d)}_{ind}$, <u>for all</u> $d \in \mathbb{N}$.

<u>Remarks</u>. The isomorphism above is induced from the "reflection functor" of Bernstein-Gel'fand-Ponomarev [BGP] , which plays a crucial role in the proof of the two theorems of chapter 1. Independently it was introduced by Sato and Kimura under the name of "Castling transform" [SK] .

We could have started from a sink instead of a source, considering $i \in Q_0^*$ first. An <u>admissible vertex</u> is a source or a sink. In particular, no loop is attached at an admissible vertex. We will say that (Q^*, α^*) is obtained from (Q, α) by applying the "reflection" R_i at the admissible vertex i of Q .

It follows from the preceding proposition that all the results we proved in chapter 3 for $\alpha \in F_Q$ still hold for representations of a quiver $\widetilde{Q}$ of dimension type $\widetilde{\alpha}$, provided that $(\widetilde{Q}, \widetilde{\alpha})$ is obtained from (Q, α) by applying a series of reflections $R_{i_1}, R_{i_2}, \ldots, R_{i_s}$ to (Q,α) , where i_1 is admissible in Q , i_2 is admissible after reversing the arrows with extremity i_1 and so on.

4.2. <u>Real and imaginary roots</u>.

With each vertex i of Q to which no loop is attached we associate a <u>reflection</u> $r_i : \mathbb{Z}^n \to \mathbb{Z}^n$ given by $r_i(\alpha) = \alpha - (\alpha, \epsilon_i)\epsilon_i$. The <u>Weyl group</u> $W = W_Q$ is the subgroup of $GL(\mathbb{Z}^n)$ generated by the r_i . It is contained in the orthogonal group $O(\mathbb{Z}^n, q_Q)$.

A <u>root</u> of Q is a vector $\alpha \in \mathbb{N}^n$ such that $R(Q,\alpha)$ contains an indecomposable representation. Roots have connected support. If for a root α we have $\mu(R(\alpha)_{ind}) \geq 1$, α is called <u>imaginary</u>, and <u>real</u> otherwise. So a root α is real if and only if $R(\alpha)_{ind}$ contains a finite number of orbits. We will see as a consequence of the main theorem that in this case $R(\alpha)_{ind}$ is one single orbit. We denote by $\Delta = \Delta(Q)$, $\Delta_{re} = \Delta(Q)_{re}$ and $\Delta_{im} = \Delta(Q)_{im}$ the sets of all roots, the real roots, and imaginary roots, respectively.

A <u>simple root</u> is a vector ϵ_i , where i is a vertex in which no loop is attached. Equivalently, ϵ_i does not lie in $F = F_Q$ (3.1). The set of simple roots is denoted by $\Pi = \Pi_Q$. Clearly $\Pi \subseteq \Delta_{re}$.

If Q is a Dynkin quiver, then $W\Pi = \Delta \cup -\Delta$ is the corresponding root system and W its Weyl group [B] .

4.3. <u>Kac's theorem</u>.

THEOREM. a) $\Delta(Q)_{re} = W\Pi \cap \mathbb{N}^n$; <u>if</u> $\alpha \in \Delta(Q)_{re}$, <u>then</u> $R(\alpha)_{ind}$ <u>is one orbit</u>.

b) $\Delta(Q)_{im} = WF_Q$; <u>if</u> $\alpha \in \Delta(Q)_{im}$, <u>then</u> $\mu(R(Q,\alpha)_{ind}) = 1 - q(\alpha)$.

The proof rests on the following crucial lemma, which we will prove in chapter 5.

FUNDAMENTAL LEMMA. <u>For</u> $\alpha \in \mathbb{N}^n$ <u>the number of isomorphism classes of indecomposables</u> V <u>with</u> <u>dim</u> $V = \alpha$ <u>as well as</u> $\mu(R(Q,\alpha)_{ind})$ <u>only depend on the underlying graph</u> $\overline{Q}$ <u>of</u> Q (and not on the orientation).

<u>Proof of the theorem</u>. For $r_i \in W$ and $\alpha \in \Delta \backslash \{\epsilon_i\}$, $r_i\alpha$ lies again in Δ . Indeed, by the fundamental lemma we may assume that i is admissible, apply 4.1 and the fundamental lemma again. So $\Delta \backslash W\Pi$ is stable under W ; we want to show that it is contained in WF , where $F = F_Q$. For $\alpha \in \Delta \backslash W\Pi$ we choose $\beta \in W\alpha$ with minimal height $ht(\beta) = \sum_{i=1}^{n} \beta(i)$. Then

$$ht(r_i\beta) = ht(\beta) - (\beta, \epsilon_i) \geq ht(\beta)$$

for all $r_i \in W$, which implies that $\beta \in F$. Note that $(\beta, \epsilon_j) \leq 0$ for $\epsilon_j \in F$. We conclude that $\Delta \subseteq (W\Pi \cap \mathbb{N}^n) \cup WF$. Conversely, F and thus WF lies in $\Delta \backslash W\Pi$ by theorem 3.3 and remark 3.5.

In order to prove

$$\mu(R(Q,\alpha)_{ind}) = 1 - q(\alpha)$$

for $\alpha \in \Delta(Q)_{im}$, we write $\alpha = r_{i_s} \dots r_{i_1}\beta$, $\beta \in F$, and proceed by induction on s. For $s = 0$, the result follows from theorem 3.4 and remark 3.5. For $s \geq 1$, we set $\gamma = r_{i_{s-1}} \dots r_{i_1}\beta$ and choose an orientation Q' for which i_s is admissible. By Q'' we denote the quiver obtained from Q' by reversing the arrows with extremity i_s. Using the fundamental lemma and 4.1, we find

$$\mu(R(Q,\alpha)_{ind}) = \mu(R(Q',\alpha)_{ind}) = \mu(R(Q'',\gamma)_{ind}) = \mu(R(Q,\gamma)_{ind}) ,$$

which by the induction hypothesis is equal to

$$1 - q(\gamma) = 1 - q(\alpha) .$$

An analogous argument for $\alpha \in W\Pi \cap \mathbb{N}^n$ finishes the proof.

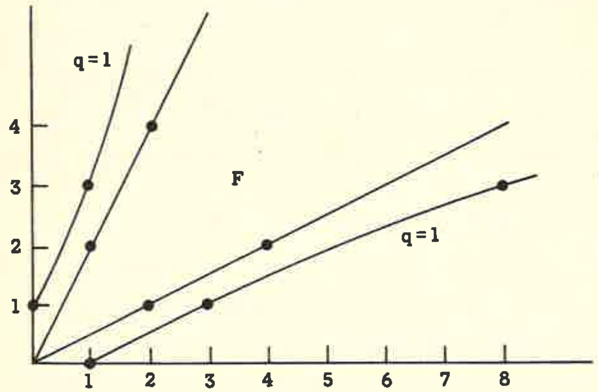
The following proposition from [K1, prop.1.6] generalizes theorems 1 and 2 of chapter 1.

PROPOSITION. Let Q be a connected quiver whose proper subquivers are all either of finite or of tame type. Then

$$\Delta(Q)_{re} = \{\alpha \in \mathbb{N}^n : q(\alpha) = 1\} ,$$
$$\Delta(Q)_{im} = \{\alpha \in \mathbb{N}^n \setminus \{0\} : q(\alpha) \leq 0\} .$$

Example.

$$Q = 1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow}\; 2$$



5. PROOF OF THE FUNDAMENTAL LEMMA.

5.1. Starting from a quiver Q, any orientation on $\bar{Q}$ can be obtained in several steps by reversing one arrow at the time. For any arrow $\psi : i \to j$ of Q, we can write

$$R(Q,\alpha) = \bar{R} \times H ,$$

where

$$H = \mathrm{Hom}(k^{\alpha(i)}, k^{\alpha(j)})$$

and

$$\bar{R} = \prod_{\varphi \neq \psi} \mathrm{Hom}(k^{\alpha(t\varphi)}, k^{\alpha(h\varphi)}) .$$

Thus we have to compare the numbers of non-isomorphic indecomposables in $\bar{R} \times H$ and $\bar{R} \times H^*$, where

$$H^* \cong \mathrm{Hom}(k^{\alpha(j)}, k^{\alpha(i)})$$

is the dual vector space of H. Unfortunately, there seems to be no way of doing this directly over an arbitrary algebraically closed field k.

However, a result of Brauer shows that this comparison is possible over finite fields (5.5). Counting points of varieties in finite fields yields the fundamental lemma for the algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$, for any prime p (5.6). Finally, interpreting the representations in $R(Q,\alpha)$ over an algebraically closed field k as the k-valued points of a scheme over $\mathbb{Z}$, one obtains the lemma in characteristic zero (5.7). So this is one of the examples where the only proof known for a result about fields of characteristic zero passes via fields of positive characteristic.

For any field k -not necessarily algebraically closed- we denote by $R(Q,\alpha)(k)$ the set of k-representations of Q of dimension type α :

$$R(Q,\alpha)(k) = \prod_{\varphi \in Q_1} \mathrm{Hom}_k(k^{\alpha(t\varphi)}, k^{\alpha(h\varphi)}) .$$

Isomorphism classes of k-representations correspond to orbits of $GL(\alpha, k)$ in $R(Q,\alpha)(k)$.

5.2. Fix a prime number p and an algebraic closure k of $\mathbb{F}_p$. Recall that k contains precisely one field $\mathbb{F}_{p^r}$ with p^r elements for $r \in \mathbb{N}$, which is the fixed field in k of the r-th power of the Frobenius automorphism

$x \mapsto x^p$ of k. All fields K, L, E, F occuring up to section 5.6 are finite subfields of k.

Let $K \subset L$ be an extension of degree r, and consider the functors

$$L \otimes_K \ : R(Q,\alpha)(K) \longrightarrow R(Q,\alpha)(L)$$

and, for a fixed choice of a k-basis $L = K^r$,

$$|K : R(Q,\alpha)(L) \longrightarrow R(Q, r\alpha)(K) \ .$$

Obviously $(L \otimes_K V)|K$ is isomorphic to V^r for any representation V in $R(Q,\alpha)(K)$. In combination with the theorem of Krull-Schmidt, this implies the following.

LEMMA 1. If for two representations V_1, V_2 in $R(Q,\alpha)(K)$ the representations $L \otimes_K V_1$ and $L \otimes_K V_2$ are isomorphic in $R(Q,\alpha)(L)$, then V_1 and V_2 are isomorphic in $R(Q,\alpha)(K)$.

Let Γ be the Galois group of L over K, which is cyclic of order r, and denote by $L\Gamma$ the skew group algebra of Γ over L: As a L-(left) vector space, $L\Gamma$ has the elements of Γ as a basis, and

$$x\sigma y\tau = x\sigma(y)\sigma\tau$$

for x, y in L and σ, τ in Γ. We let Γ operate on $L\Gamma$ by left multiplication. The fixed point set $(L\Gamma)^\Gamma$ for this operation is obviously a K-left and L-right vector space.

LEMMA 2. The map

$$\mu : L \otimes_K (L\Gamma)^\Gamma \longrightarrow L\Gamma$$

given by the multiplication is an isomorphism of (left and right) L-vector spaces.

Proof. Choosing a normal basis $\{\sigma(x), \sigma \in \Gamma\}$ for L over K, one sees immediately that

$$\dim_K (L\Gamma)^\Gamma = r \ .$$

Thus the two L-vector spaces $L \otimes_K (L\Gamma)^\Gamma$ and $L\Gamma$ have the same dimension r. The theorem about the linear independence of distinct σ in Γ within $\mathrm{End}_K L$ shows that any L-linear form on $L\Gamma$ which annihilates $\mathrm{im}\mu$ is zero. Therefore μ is surjective.

Remark. For any representation W in $R(Q,\alpha)(L)$ we have the decomposition

$$L\Gamma \otimes_L W = \bigoplus_{\sigma \in \Gamma} {}^\sigma W$$

of $L\Gamma \otimes_L W$ as a direct sum of L-representations

$${}^\sigma W = \{\sigma \otimes w : w \in W\}$$

of dimension type α.

The representation ${}^\sigma W$ can be described as follows: for any vertex i of Q, the L-vector space ${}^\sigma W(i)$ has the same underlying abelian group as $W(i)$, but the product $a \cdot w$ is given by $\sigma^{-1}(a)w$ for $a \in L$ and $w \in {}^\sigma W(i)$; for any arrow φ, ${}^\sigma W(\varphi)$ equals $W(\varphi)$. Obviously ${}^\sigma W|K$ is isomorphic to $W|K$.

5.3. An indecomposable representation V in $R(Q,\alpha)(K)$ is called absolutely indecomposable if $k \otimes_K V$ is indecomposable. Equivalently, $L \otimes_K V$ is indecomposable for any extension L of K. An extension L of K is called a splitting field for a representation V in $R(Q,\alpha)(K)$ if $L \otimes_K V$ is a direct sum of absolutely indecomposable representations in $R(Q,\alpha)(L)$.

LEMMA 1. Let V be indecomposable in $R(Q,\alpha)(K)$. Then $L = \mathrm{End}\, V/\mathrm{rad}\,\mathrm{End}\, V$ is a splitting field for V. Moreover $r = [L:K]$ divides α, and we have

$$L \otimes_K V \simeq L\Gamma \otimes_L W = \bigoplus_{\sigma \in \Gamma} {}^\sigma W \ ,$$

where $\Gamma = \mathrm{Gal}(L:K)$, for some absolutely indecomposable representation W in $R(Q, \frac{\alpha}{r})(L)$, and the representations ${}^\sigma W$ are pairwise non-isomorphic.

Proof. Since V is indecomposable, $L = \mathrm{End}\, V/\mathrm{rad}\,\mathrm{End}\, V$ is a (finite) skew field and thus a field. We have

$$\mathrm{End}(L \otimes_K V) \simeq L \otimes_K \mathrm{End}\, V \ ,$$

and also

$$\mathrm{rad}\ \mathrm{End}\,(L \otimes_K V) \simeq L \otimes_K \mathrm{rad}\,\mathrm{End}\, V \ ,$$

since K is perfect. So we find an isomorphism

$$\mathrm{End}(L \otimes_K V)/\mathrm{rad}\ \mathrm{End}(L \otimes_K V) \simeq L \otimes_K L \simeq L^r \ .$$

Therefore $L \otimes_K V$ can be decomposed in $R(Q,\alpha)(L)$ as a direct sum

$$L \otimes_K V = W_1 \oplus \ldots \oplus W_r ,$$

and the W_i's are pairwise non-isomorphic and have L as endomorphism algebra. As

$$\mathrm{End}(E \otimes W_i)/\mathrm{rad}\,\mathrm{End}(E \otimes W_i) \xrightarrow{\sim} E$$

for any extension field E of L, W_i is absolutely indecomposable for $i = 1,\ldots,r$. For any σ in Γ, the representations $L \otimes_K V$ and

$${}^{\sigma}(L \otimes_K W) = {}^{\sigma}W_1 \oplus \ldots \oplus {}^{\sigma}W_r$$

are isomorphic. Using Krull-Schmidt, we find that $\{W_1,\ldots,W_r\}$ is one Γ-orbit, up to isomorphism. This proves the lemma.

Let W be in $R(Q,\alpha)(L)$. A subfield $K \subseteq L$ is a <u>field of definition</u> for W if there is a representations V in $R(Q,\alpha)(K)$ such that W is induced from V; i.e., W is isomorphic to $L \otimes_K V$.

LEMMA 2. <u>A subfield K of L such that ${}^{\sigma}W$ is isomorphic to W for all $\sigma \in \Gamma = \mathrm{Gal}(L:K)$ is a field of definition for W.</u>

<u>Proof.</u> Clearly we may suppose that the indecomposables in a decomposition of W form one Γ-orbit up to isomorphism. By our hypothesis, $L\Gamma \otimes_L W$ is isomorphic to W^r, and lemma 2 of 5.2 tells us that $L\Gamma \otimes_L W$ is induced from $(L\Gamma)^{\Gamma} \otimes_L W$ in $R(Q, r\alpha)(K)$. Thus we have

$$W^r \xrightarrow{\sim} L \otimes_K V_1 \oplus \ldots \oplus L \otimes_K V_t$$

for some indecomposable K-representations $V_1,\ldots,V_t$. Our assumption on the indecomposables occuring in W implies that $L \otimes_K V_i$ is isomorphic to some W^{s_i} for all i. So we have found an integer s and an indecomposable K-representation V such that W^s is induced from V.

Set $F = \mathrm{End}\,V/\mathrm{rad}\,\mathrm{End}\,V$, and let E be a common extension field of F and L in k. By lemma 1,

$$E \otimes_K V \xrightarrow{\sim} E \otimes_L W^s$$

is a direct sum of pairwise non-isomorphic indecomposables, which implies $s = 1$.

COROLLARY. a) <u>Any splitting field E for an indecomposable representation V in $R(Q,\alpha)(K)$ contains $L = \mathrm{End}\,V/\mathrm{rad}\,\mathrm{End}\,V$.</u>

b) <u>Any field of definition K for a representation W in $R(Q,\alpha)(L)$ contains the fixed field L^G, where</u> $G = \{\sigma \in \mathrm{Gal}(L:\mathbb{F}_p) : {}^{\sigma}W \xrightarrow{\sim} W\}$.

We call the field L of a) the <u>minimal splitting field</u> of V and the field L^G of b) <u>the minimal field of definition</u> of W.

<u>Proof.</u> a) For any splitting field E for V, we have a decomposition

$$E \otimes_K V = U_1 \oplus \ldots \oplus U_r ,$$

where the U_i's are absolutely indecomposable and where $r = [L:K]$, independently of E by lemma 1. As a consequence of lemma 2, the U_i's form one orbit up to isomorphism under $\mathrm{Gal}(E:K)$, since V is indecomposable. Thus r divides $[E:K]$, which implies $L \subseteq E$.

b) If K is a field of definition for W, obviously the group $\mathrm{Gal}(L:K)$ is contained in $G = \mathrm{Gal}(L:L^G)$. Then $[L:K]$ divides $[L:L^G]$, which implies that $[L^G:\mathbb{F}_p]$ divides $[K:\mathbb{F}_p]$, and thus $L^G \subseteq K$.

5.4. We sum up our results in form of the following lemma. For any power q of p we set :

$\nu(Q,\alpha;q)$ = number of isomorphism classes of indecomposable representations in $R(Q,\alpha)(\mathbb{F}_q)$,

$\nu^a(Q,\alpha;q)$ = number of isomorphism classes of absolutely indecomposable representations in $R(Q,\alpha)(\mathbb{F}_q)$,

$\rho(Q,\alpha;q)$ = number of isomorphism classes of absolutely indecomposable representations in $R(Q,\alpha)(\mathbb{F}_q)$ with minimal field of definition $\mathbb{F}_q$.

LEMMA.

a) $\nu^a(Q,\alpha;q) = \sum_{\mathbb{F}_{q'} \subseteq \mathbb{F}_q} \rho(Q,\alpha;q')$

b) $\nu(Q,\alpha;q) = \nu^a(Q,\alpha;q) + \sum_r \frac{1}{r}\,\rho(Q,\frac{\alpha}{r};q^r)$, <u>where</u> r <u>ranges</u>

over all integers greater than 1 for which $\frac{\alpha}{r}$ belongs to $\mathbb{N}^n$.

Proof. a) is clear, and b) says that the isomorphism classes of indecomposables in $R(Q,\alpha)(K)$ with minimal splitting field $\mathbb{F}_{q^r}$ are in bijection with the isomorphism classes of $\mathrm{Gal}(\mathbb{F}_{q^r} : \mathbb{F}_q)$ -orbits of indecomposables in $R(Q,\frac{\alpha}{r})(\mathbb{F}_{q^r})$ with minimal field of definition $\mathbb{F}_{q^r}$.

5.5. The "fundamental lemma for finite fields".

Fix a finite field $F \subset k$.

LEMMA. The number of isomorphism classes of representations in $R(Q,\alpha)(F)$ is independent of the orientation of Q .

Proof. Using the notations of 5.1 , we have to show that $GL(\alpha, F)$ has the same number of orbits in $\bar{R} \times H$ as in $\bar{R} \times H^*$. Choose representatives $V_1, \ldots, V_r$ from the (finitely many !) $GL(\alpha, F)$-orbits in $\bar{R}$. A $GL(\alpha, F)$-orbit in $\bar{R} \times H$ or $\bar{R} \times H^*$ corresponds to the choice of a V_i and of a $C_{GL(\alpha,F)} V_i$-orbit in H or H^* , respectively.

Choose i in $\{1,\ldots,r\}$, and set $G = C_{GL(\alpha,F)} V_i$. Consider the $\mathbb{C}$-vector space $\mathbb{C}^H$ of maps from H to $\mathbb{C}$, and let G operate on $\mathbb{C}^H$ by

$$(g \cdot f)(h) = f(g^{-1} \cdot h) .$$

A fixed point of G in $\mathbb{C}^H$ is a map which is constant on G-orbits in H . Thus the number of G-orbits in H equals the dimension of the vector space $(\mathbb{C}^H)^G$ of fixed points of G in $\mathbb{C}^H$.

To prove our assertion, it suffices to exhibit a G-equivariant isomorphism

$$F : \mathbb{C}^H \to \mathbb{C}^{H^*} .$$

The Fourier transform, which is defined as follows, does the trick :

$$(Ff)(\varphi) = \frac{1}{|H|} \sum_{h \in H} f(h) \exp\left(\frac{2\pi i}{p} \operatorname{tr}(\varphi(h))\right)$$

for $f \in \mathbb{C}^H$, $\varphi \in \mathbb{C}^{H^*}$. Here $|H|$ is the cardinality of H , and $\operatorname{tr} : F \to \mathbb{F}_p$ the usual trace map. An easy computation shows that

$$(F(Ff))(h) = \frac{1}{|H|} f(-h) .$$

So F is an isomorphism, which is G-equivariant by definition.

Remark. The result that a finite group operating linearly on a finite vector space V has the same number of orbits in V and V^* seems to be due to Brauer, but we have not been able to find a precise reference.

COROLLARY 1. The number of isomorphism classes of indecomposable representations in $R(Q,\alpha)(F)$ is independent of the orientation of Q .

Proof. Use induction on $ht(\alpha) = \sum_{i=1}^{n} \alpha(i)$ and Krull-Schmidt.

COROLLARY 2. The numbers $\nu^a(Q,\alpha\,;q)$ and $\rho(Q,\alpha\,;q)$ defined in 5.4 are independent of the orientation of Q for any $q = p^s$ and any α .

Proof. This is clearly true for any q if $ht(\alpha) = 1$. Suppose it is true for any q and any β with $ht(\beta) < ht(\alpha)$. Then formula b) of Lemma 5.4 implies that $\nu^a(Q,\alpha\,;q)$ is independent of the orientation, as $\nu(Q,\alpha\,;q)$ is by corollary 1.

To prove the independence of $\rho(Q,\alpha\,;p^s)$ of the orientation, we keep α fixed and apply induction on s . For $s = 1$, we have

$$\rho(Q,\alpha;\,p) = \nu^a(Q,\alpha\,;p) .$$

The general case follows from formula a) of lemma 5.4, which says that

$$\nu^a(Q,\alpha;p^s) = \rho(Q,\alpha\,;\,p^s) + \sum_{\substack{t/s \\ t \neq s}} \rho(Q,\alpha;p^t) .$$

5.6. The fundamental lemma for $k = \overline{\mathbb{F}}_p$.

Let $\mathcal{U} \subseteq k^N$ be a locally closed subset which is stable under the Frobenius automorphism σ of k^N . For any finite subfield $\mathbb{F}_q$ of k , set

$$\mathcal{U}(\mathbb{F}_q) = \mathcal{U} \cap \mathbb{F}_q^N .$$

This set may well be empty for small q's . However, the famous result of Lang and Weil about counting points of varieties in finite fields ([LW] , see also [Sch] for an elementary approach) implies the cardinality of $\mathcal{U}(\mathbb{F}_q)$

behaves as follows :

PROPOSITION. If $\mathcal{U}$ is stable under σ, then $\# \mathcal{U}(\mathbb{F}_q) \approx c\, q^{\dim \mathcal{U}}$ where c is the number of irreducible components of $\mathcal{U}$ of maximal dimension.

The symbol $f(q) \approx g(q)$ means that either $f(q) = g(q) = 0$ for q large or that $f(q), g(q) \neq 0$ for q large and $\frac{f(q)-g(q)}{f(q)} \underset{q\to\infty}{\longrightarrow} 0$.

The set $R(Q,\alpha)^{(d)}_{\mathrm{ind}}$ of indecomposable representations of Q over k of dimension type α and with d-dimensional endomorphism algebra is a locally closed subset of k^N, where $N = \sum_{\varphi \in Q_1} \alpha(t\varphi)\alpha(h\varphi)$ (2.5). Let V be in $R(Q,\alpha)^{(d)}_{\mathrm{ind}}$. All coefficients of all matrices $V(\varphi)$, $\varphi \in Q_1$, lie in some finite subfield $K \subset k$, and we may view V as an absolutely indecomposable representation in $R(Q,\alpha)(K)$ with d-dimensional endomorphism algebra over K. An easy computation shows that applying the Frobenius σ to all coefficients of all matrices $V(\varphi)$ while keeping the bases of all $V(i)$ fixed yields a representation isomorphic to ${}^{\sigma}V$ (5.2, remark), where σ is viewed in $\mathrm{Gal}(K : \mathbb{F}_p)$. Therefore $R(Q,\alpha)^{(d)}_{\mathrm{ind}}$ is stable under σ, and we may apply the preceding proposition. We find :

$$\# R(Q,\alpha)^{(d)}_{\mathrm{ind}}(\mathbb{F}_q) \approx c(d)\, q^{\mu(d) + \dim GL(\alpha) - d},$$

where $\mu(d)$ denotes the number of parameters (3.4) and $c(d)$ the number of irreducible components of maximal dimension of $R(Q,\alpha)^{(d)}_{\mathrm{ind}}$.

For an absolutely indecomposable representation V in $R(Q,\alpha)(\mathbb{F}_q)$, we have

$$GL(\alpha)\cdot V \cap \mathbb{F}_q^N = GL(\alpha, \mathbb{F}_q)\cdot V$$

by lemma 1 of 5.2. If V lies in $R(Q,\alpha)^{(d)}_{\mathrm{ind}}$, its $GL(\alpha)$-orbit is a locally closed irreducible σ-stable subset of dimension $GL(\alpha) - d$, and we find :

$$\# (GL(\alpha)V)(\mathbb{F}_q) \approx q^{\dim GL(\alpha) - d}.$$

As a consequence, the number of $GL(\alpha, \mathbb{F}_q)$-orbits in $R(Q,\alpha)^{(d)}_{\mathrm{ind}}(\mathbb{F}_q)$

behaves like $c(d)\, q^{\mu(d)}$, and thus the number $\nu^a(Q,\alpha;q)$ of $GL(\alpha,\mathbb{F}_q)$-orbits in $R(Q,\alpha)_{\mathrm{ind}}(\mathbb{F}_q)$ like $c q^{\mu}$, where $\mu = \max_d \mu(d)$ is the number of parameters of $R(Q,\alpha)_{\mathrm{ind}}$ (3.4) and $c = \sum_{\mu(d)=\mu} c(d)$.

But $\nu^a(Q,\alpha;q)$ is independent of the orientation of Q (5.5). Therefore the number of parameters μ and the number c of irreducible components of $R(Q,\alpha)_{\mathrm{ind}}$ with number of parameters μ are independent as well. Since for $\mu = 0$, c is the number of orbits, this ends the proof.

Remark. As we may now apply Kac's theorem to $\overline{\mathbb{F}}_p$, we see that $\mu = 1 - q(\alpha)$, independently of p. Moreover, there exists precisely one irreducible component of $R(Q,\alpha)_{\mathrm{ind}}$ for which the number of parameters is maximal, which means that $c = 1$. Indeed, this is true for $\alpha \in F_Q$, it remains true if we replace (Q,α) by (Q^*,α^*) for some reflection at an admissible vertex (4.1), and it still holds when we change orientation. In particular, if the generic representation of dimension type α is indecomposable, the number $1-q(\alpha)$ of parameters of $R(\alpha)^{\max}$ (corollary 2.7) is strictly greater than the one of any other irreducible component of $R(\alpha)_{\mathrm{ind}}$.

5.7. End of the proof.

In order to transfer our results in characteristic p to characteristic zero, we have to define our varieties over $\mathbb{Z}$. Let $\mathbb{Z}[X_{\varphi;st}]$ be the polynomial ring over $\mathbb{Z}$ in the variables $X_{\varphi;st}$, where φ ranges over the arrows of Q and $1 \le s \le \alpha(t\varphi)$, $1 \le t \le \alpha(h\varphi)$, and set

$$\mathcal{R} = \mathcal{R}(Q,\alpha) = \operatorname{Spec} \mathbb{Z}[X_{\varphi;st}].$$

Obviously $\mathcal{R}$ is affine N-space over $\mathbb{Z}$ with $N = \sum_{\varphi\in Q_1} \alpha(h\varphi)\,\alpha(t\varphi)$. For any (commutative !) ring A, the A-valued points $\mathcal{R}(A)$ of $\mathcal{R}$ can be identified with

$$\prod_{\varphi \in Q_1} \mathrm{Hom}_A(A^{\alpha(t\varphi)}, A^{\alpha(h\varphi)}).$$

In particular, if k is an algebraically closed field, we have

$$\mathcal{R}(k) = R(Q,\alpha)(k).$$

Next we want to define a scheme whose k-valued points are pairs (V, f) consisting of a representation V and an endomorphism f of V. To this end, we set

$$\mathcal{M} = \operatorname{Spec} \mathbb{Z}[Y_{i;jk}] ,$$

where i ranges over the points of Q and $1 \le j, k \le \alpha(i)$. We denote by Y_i the $\alpha(i) \times \alpha(i)$-matrix whose entries are the $Y_{i;jk}$ and by X_φ the $\alpha(h\varphi) \times \alpha(t\varphi)$ - matrix with entries $X_{\varphi;st}$. For each $\varphi \in Q_1$, the equation

$$X_\varphi \cdot Y_{t\varphi} - Y_{h\varphi} \cdot X_\varphi = 0$$

yields $\alpha(t\varphi)\alpha(h\varphi)$ linear equations in

$$\mathbb{Z}[X_{\varphi;st}] \otimes_{\mathbb{Z}} \mathbb{Z}[Y_{i;jk}] = \mathbb{Z}[X_{\varphi;st}, Y_{i;jk}] .$$

Denote by $\mathcal{I}$ the ideal generated by these linear equations, and let

$$\mathcal{Z} = \operatorname{Spec} k[X_{\varphi;st}, Y_{i;jk}] / \mathcal{I}$$

be the corresponding closed subscheme of $\mathcal{R} \times \mathcal{M}$. Obviously we have for any algebraically closed field

$$\mathcal{Z}(k) = \{(V, f) : V \in R(Q, \alpha)(k) , f \in \operatorname{End} V\} .$$

Consider the first projection

$$\pi : \mathcal{Z} \to \mathcal{R}$$

and define the subset

$$\mathcal{R}^{(d)} = \{x \in \mathcal{R} : \dim \pi^{-1}(x) = d\} .$$

Since sending x to $(x, 0)$ is a section for π and since all fibers of π are irreducible, being affine spaces, $\mathcal{R}^{(d)}$ is locally closed by Chevalley's theorem [EGA IV, 13.1]. We endow $\mathcal{R}^{(d)}$ with the structure of a reduced scheme. Clearly we have

$$\mathcal{R}^{(d)}(k) = R^{(d)}(Q, \alpha)(k)$$

for any algebraically closed field k.

In order to define the "subscheme of indecomposables in $\mathcal{R}^{(d)}$", we use the characterization of 2.5. Let

$$\mathcal{J} \subseteq \mathbb{Z}[Y_{i;jk}]$$

be the ideal generated by the equations which express that Y_i is nilpotent, or equivalently that $Y_i^{\alpha(i)} = 0$, for all vertices i. Then

$$\mathcal{N} = \operatorname{Spec} \mathbb{Z}[X_{\varphi;st}, Y_{i;jk}] / \mathcal{J} + \mathcal{I}$$

is a closed subscheme of $\mathcal{Z}$. Applying, Chevalley's theorem to the first projection

$$\pi' : \mathcal{N} \to \mathcal{R} ,$$

we see that the subset

$$\mathcal{S}^{(t)} = \{x \in \mathcal{R} : \dim \pi'^{-1}(x) \ge t\}$$

of $\mathcal{R}$ is closed for all t. Indeed, the zero section for π' meets all irreducible components of each fiber. We give the intersection

$$\mathcal{R}^{(d)}_{ind} = \mathcal{R}^{(d)} \cap \mathcal{S}^{(d-1)} ,$$

which is closed in $\mathcal{R}^{(d)}$, the structure of a reduced scheme. For any algebraically closed field k, $\mathcal{R}^{(d)}_{ind}(k)$ consists of those representations with d-dimensional endomorphism algebra for which the nilpotent endomorphism form a subvariety of dimension $d-1$; this is just $R(Q, \alpha)^{(d)}_{ind}(k)$.

Summing up, we have defined for each d a locally closed reduced subscheme $\mathcal{R}^{(d)}$ of $\mathcal{R}$ and a closed reduced subscheme $\mathcal{R}^{(d)}_{ind}$ of $\mathcal{R}^{(d)}$ such that

$$\mathcal{R}^{(d)}(k) = R(Q, \alpha)^{(d)}(k) \quad \text{and} \quad \mathcal{R}^{(d)}_{ind}(k) = R(Q, \alpha)^{(d)}_{ind}(k)$$

for any algebraically closed field k.

Let k be algebraically closed and choose an algebraic closure k_0 within k of the prime subfield of k. For any d, the varieties $\mathcal{R}^{(d)}_{ind}(k)$ and $\mathcal{R}^{(d)}_{ind}(k_0)$ have the same dimension and the same number of irreducible components of maximal dimension [EGA IV, 4.4]. Therefore the fundamental lemma is true for k if it holds for k_0. It remains to prove it for an algebraic closure of $\mathbb{Q}$.

Consider the canonical morphism

$$f_d : \mathcal{R}^{(d)}_{ind} \to \operatorname{Spec} \mathbb{Z} .$$

There is a non-empty open subset U_d of $\mathbb{Z}$ such that

$$\dim f_d^{-1}(0) = \dim f_d^{-1}(p)$$

for all p in U_d ([EGA IV, 9.2]. If $\bar{\mathbb{Q}}$ and $\bar{\mathbb{F}}_p$ denote algebraic closures of $\mathbb{Q}$ and $\mathbb{F}_p$, then $\mathcal{R}^{(d)}_{ind}(\bar{\mathbb{Q}})$ and $\mathcal{R}^{(d)}_{ind}(\bar{\mathbb{F}}_p)$ are the varieties of $\bar{\mathbb{Q}}$-valued points of $f_d^{-1}(0)$ and $\bar{\mathbb{F}}_p$-valued points of $f_d^{-1}(p)$, respectively. We find

$$\dim \mathcal{R}^{(d)}_{ind}(\bar{\mathbb{Q}}) = \dim \mathcal{R}^{(d)}_{ind}(\bar{\mathbb{F}}_p)$$

for all p in the open set $U = \bigcap_d U_d$ and all d.

For any algebraically closed field k, the number of parameters of $R(Q,\alpha)_{ind}(k)$ is defined as

$$\mu(k) = \max_d (\dim R(Q,\alpha)^{(d)}_{ind}(k) - \dim GL(\alpha) + d) \qquad \text{(see 3.4).}$$

As $\mu(\bar{\mathbb{F}}_p) = 1 - q(\alpha)$ for all p, we conclude that $\mu(\bar{\mathbb{Q}}) = 1 - q(\alpha)$, independently of the orientation of Q.

Suppose $1 - q(\alpha) = 0$. For any prime p there is a unique d_p such that $R(Q,\alpha)^{(d_p)}_{ind}(\bar{\mathbb{F}}_p)$ is non-empty, and then it is a single $GL(\alpha, \bar{\mathbb{F}}_p)$-orbit. As $d_p = d$ is constant on U, we see that $R(Q,\alpha)^{(d')}_{ind}(\bar{\mathbb{Q}})$ is empty for $d' \neq d$. Moreover, $\mathcal{R}^{(d)}_{ind}(\bar{\mathbb{Q}})$ is connected by [EGA IV, 9.7], and thus consists of a single orbit. This ends the proof of the fundamental lemma for $\bar{\mathbb{Q}}$.

Remark. Refining the last argument one can show that for any algebraically closed field k the number $1 - q(\alpha)$ of parameters is reached for precisely one irreducible component of precisely one $R^{(d)}_{ind}(Q,\alpha)(k)$.

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