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REDUCTIVE GROUP ACTIONS ON AFFINE SPACE WITH ONE-DIMENSIONAL QUOTIENT

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§ 0. Introduction

(0.0) Let G be a reductive complex algebraic group acting algebraically on X, where $X = \mathbb{A}^n$ is complex affine n-space. It has been conjectured by Kambayashi **[Ka]** that the action must then be linearizable, i.e. algebraically isomorphic to a representation of G. This conjecture seems a bit audacious when one considers the (somewhat) analogous situation of a smooth action of a compact Lie group K on $\mathbb{R}^n$. Here there are non-linearizable actions, in fact, often there are actions without any fixed points (e.g. if K is connected, non-abelian [**Br**, I.8.4]). However, if $\dim \mathbb{R}^n/K \leq 2$, then the action *is* linearizable [**Br**, IV.8.5], which leads us to focus on actions of G where the "orbit space" $X/\!\!/G$ (see §1) has low dimension.

We concentrate on the following

(0.1) PROBLEM (Luna). *Suppose that* $\dim X/\!\!/G = 1$. *Is the G-action on X linearizable?*

Linearizability in the case $\dim X/\!\!/G = 0$ is due to Luna **[Lu]**. Luna outlined an attack on the problem in 1981 (see (2.2) below), which has been our guide.

At the time of the Montreal conference (August, 1988) we only knew of positive results. Since then we have discovered non-linearizable actions (see **[S]**) which are intimately connected with G-vector bundles. This paper describes the techniques which are used to compute the relevant moduli spaces of G-actions on X (see §3). We describe when these moduli spaces are trivial, i. e. when G-actions are linearizable. We also present some examples of non-trivial moduli (see **[S]** for more details).

The easiest results to state at this point are the following.

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(0.2) THEOREM. *Suppose that* $\dim X/\!\!/G = 1$ *and that one of the following conditions holds:*

(1) *G acts semifreely on X.*
(2) *G^0 is simple.*
(3) *G is a torus.*
(4) $\dim X \leq 3$.

Then the action is linearizable.

We say that G acts *semifreely* if every closed orbit is either a fixed point or isomorphic to G (i.e. has trivial isotropy group). Linearization for torus actions with codimension one orbits is due to Bialynicki-Birula **[Bb]**, while the generalization to the case of a one-dimensional quotient is due to Kraft and Luna (unpublished).

(0.3) THEOREM. (1) *There is a non-linearizable action of* $\mathrm{O}_2 \times \mathbb{C}^*$ *on* $\mathbb{A}^4$ *with one-dimensional quotient which remains non-linearizable when restricted to* O_2.

(2) *There is a non-linearizable action of* $\mathrm{SL}_2 \times \mathbb{C}^*$ *on* $\mathbb{A}^7$ *with one-dimensional quotient which remains non-linearizable when restricted to* SL_2.

(0.4) We thank D. Luna for generously sharing his ideas and notes on the linearization problem. We thank J.-P. Serre for timely help concerning Galois cohomology.

§ 1. The Quotient Morphism

(1.0) For the moment, let X be an arbitrary affine G-variety. Then the algebra $\mathcal{O}(X)^G$ of invariant polynomial functions on X is finitely generated (see **[Kr**, II.3.2]). Let $X/\!\!/G$ denote the corresponding affine variety, and let $\pi_{X,G} : X \to X/\!\!/G$ be the morphism corresponding to the inclusion $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$.

(1.1) PROPOSITION (see **[Kr**, II.3.2]). (1) $\operatorname{Im} \pi_{X,G} = X/\!\!/G$.

(2) *$\pi_{X,G}$ separates disjoint closed G-invariant algebraic subsets of X.*
(3) *Every orbit contains a unique closed orbit in its closure, and $\pi_{X,G}$ sets up a bijection between the closed orbits in X and the points of $X/\!\!/G$.*

(1.2) PROPOSITION ([**Lu**]). *Let X be a smooth affine G-variety. Then there is a finite stratification $X/\!\!/G = \bigcup Z_i$ where the Z_i are locally closed smooth subvarieties with the following properties:*

(1) *If Gx is closed and $\pi_{X,G}(x) \in Z_i$, then the isotropy group G_x is conjugate to a fixed reductive subgroup H_i of G.*
(2) *The morphism $X_i := \pi^{-1}(Z_i) \to Z_i$ is an étale G-fibration.*
(3) *If $Z_i \cap \overline{Z_j} \neq \emptyset$, then H_j is conjugate to a subgroup of H_i.*

In (2), the fibration may not be locally trivial in the Zariski topology, but it is complex analytically locally trivial in the usual Hausdorff topology. If $X/\!\!/G$ is irreducible (e.g. X is irreducible), then one of the strata Z_i must be open and dense. We call this the *principal stratum* and the corresponding isotropy groups are called *principal*. We call the stratification by the Z_i the *Luna stratification* of $X/\!\!/G$.

(1.3) THEOREM (Luna). *Let (X,G) be as in (0.1). Then*

(1) *$X/\!\!/G$ is isomorphic to the affine line $\mathbb{A}$.*
(2) *Either $X^G \simeq X/\!\!/G \simeq \mathbb{A}$, or*
(3) *$X^G = \{x_0\}$ is a single point, and $\pi_{X,G}(x_0)$ and its complement are the Luna strata of $X/\!\!/G$.*

The main tool used to prove (1.3) is the Leray spectral sequence of $\pi_{X,G}$. In **[KPR]** part (1) is proved only assuming that $\dim X/\!\!/G = 1$ and that X is smooth and acyclic (i.e. has the $\mathbb{Z}$-homology of a point).

In case $X^G \simeq \mathbb{A}$ the action is *fix-pointed*, i.e. all the closed orbits are fixed points. It then follows from work of Kraft or Bass-Haboush **[BH]** that $X \simeq \mathbb{A} \times V$ where G acts trivially on $\mathbb{A}$ and V is a representation of G with $\mathcal{O}(V)^G = \mathbb{C}$.

(1.4) From now on we concentrate on the case where $X^G = \{x_0\}$. Let $V = T_{x_0}X$ with its canonical linear G-action. Clearly, if (X,G) is linearizable, then the corresponding representation is (V,G) ! It follows from Luna's slice theorem **[Lu]** that $\dim V/\!\!/G = \dim X/\!\!/G = 1$, hence $V/\!\!/G \simeq \mathbb{A}$. From now on we identify $X/\!\!/G$ and $V/\!\!/G$ with $\mathbb{A}$, and we arrange that $\pi_{V,G}(0) = \pi_{X,G}(x_0) = 0 \in \mathbb{A}$.

(1.5) REMARK: Let H be a principal isotropy group of (V,G), and let N denote $N_G(H)/H$. Then the representation (V^H, N) is semifree, and we can show that X^H is N-isomorphic to V^H. Now every closed orbit in X intersects X^H in a single closed N-orbit, thus, roughly, our N-isomorphism of X^H with V^H is a G-isomorphism of the closed orbits of X and V. Theorem (0.3) shows that, in general, one cannot extend the isomorphism over the non-closed orbits.

§ 2. Isomorphisms

(2.0) Let $\dot{\mathbb{A}}$ denote $\mathbb{A} \setminus \{0\}$, so that $\mathcal{O}(\dot{\mathbb{A}}) = \mathbb{C}[t,t^{-1}]$, where $\mathcal{O}(\mathbb{A}) = \mathbb{C}[t]$. Let $\hat{\mathbb{A}}$ and $\hat{\dot{\mathbb{A}}}$, respectively, correspond to the algebras $\mathbb{C}[[t]]$ (formal power series) and $\mathbb{C}((t))$ (quotient field of $\mathbb{C}[[t]]$). Set $\hat{X} := X \times_{\mathbb{A}} \hat{\mathbb{A}}$. Then $\mathcal{O}(\hat{X}) = \mathcal{O}(X) \otimes_{\mathcal{O}(\mathbb{A})} \mathcal{O}(\hat{\mathbb{A}})$, and $\hat{X}$ is the completion of X along the zero fiber $\pi_{X,G}^{-1}(0)$. Define $\dot{X}$, $\hat{\dot{X}}$, $\hat{V}$, $\dot{V}$ and $\hat{\dot{V}}$ analogously. Note that $\dot{X} = \pi_{X,G}^{-1}(\dot{\mathbb{A}}) = X \setminus \pi_{X,G}^{-1}(0)$, and similarly for $\dot{V}$.

(2.1) PROPOSITION. *Suppose:*

(1) *There are G-isomorphisms $\dot{\varphi}$ and $\hat{\varphi}$ such that the following diagrams commute*

$$\begin{tikzcd} \dot{X} \arrow[r, "\dot{\varphi}", "\sim"'] \arrow[d] & \dot{V} \arrow[d] \\ \dot{\mathbb{A}} \arrow[r, equal] & \dot{\mathbb{A}} \end{tikzcd} \qquad \begin{tikzcd} \hat{X} \arrow[r, "\hat{\varphi}", "\sim"'] \arrow[d] & \hat{V} \arrow[d] \\ \hat{\mathbb{A}} \arrow[r, equal] & \hat{\mathbb{A}} \end{tikzcd}$$

where the vertical maps are induced by $\pi_{X,G}$ and $\pi_{V,G}$.

(2) *The morphisms $\dot{\varphi}$ and $\hat{\varphi}$ induce the same isomorphism of $\hat{\dot{X}}$ and $\hat{\dot{V}}$.*

Then X is G-isomorphic to V.

PROOF: Note that $\mathcal{O}(\dot{\mathbb{A}}) \cap \mathcal{O}(\hat{\mathbb{A}}) = \mathcal{O}(\mathbb{A})$ (intersecting inside $\mathcal{O}(\hat{\dot{\mathbb{A}}})$). It is automatic that $\pi_{X,G} : X \longrightarrow \mathbb{A}$ is flat, and it follows that $\mathcal{O}(\dot{X}) \cap \mathcal{O}(\hat{X}) = \mathcal{O}(X)$, and similarly for $\mathcal{O}(V)$. An easy argument then shows that $\dot{\varphi}^*$ (or $\hat{\varphi}^*$) induces a G-isomorphism of $\mathcal{O}(V)$ and $\mathcal{O}(X)$. ∎

(2.2) The proposition above gives the broad outline of Luna's plan to establish linearizability. The existence of the isomorphism $\hat{\varphi}$ is an immediate consequence of Luna's slice theorem. In the rest of this section we outline how to show the existence of $\dot{\varphi}$. In general, one cannot arrange for (2.1.2) to hold.

(2.3) Recall that by (1.2) and (1.3), $\dot{X} \longrightarrow \dot{\mathbb{A}}$ and $\dot{V} \longrightarrow \dot{\mathbb{A}}$ are étale G-fiber bundles. Their fibers are G-isomorphic (slice theorem). Let $F = \pi_{V,G}^{-1}(1) \simeq \pi_{X,G}^{-1}(1)$ denote the fiber, and let $L = \mathrm{Aut}_G(F)$ denote the group of G-automorphisms of F. Then one can show that L is a linear algebraic group, and $\dot{X} \longrightarrow \dot{\mathbb{A}}$ and $\dot{V} \longrightarrow \dot{\mathbb{A}}$ correspond to principal L-bundles P and Q, respectively, over $\dot{\mathbb{A}}$. As usual, $P \simeq Q$ if and only if $\dot{X} \simeq \dot{V}$ over $\dot{\mathbb{A}}$.

We are able to show the following:

(2.4) THEOREM. *Let L be a linear algebraic group, and let P and Q be principal L-bundles over $\dot{\mathbb{A}}$ (in an étale sense). Then*

(1) *P and Q become trivial over a finite cyclic cover of $\dot{\mathbb{A}}$.*
(2) *$P \simeq Q$ if and only if $P/L^0 \simeq Q/L^0$, where L^0 denotes the identity component of L.*

Note that P/L^0 and Q/L^0 are principal L/L^0-bundles, where L/L^0 is a finite group Λ. Now two principal Λ-bundles over $\dot{\mathbb{A}}$ are isomorphic if and only if they are topologically isomorphic. Thus (2) above implies that the algebraic and topological classifications of principal L-bundles over $\dot{\mathbb{A}}$ are the same.

(2.5) COROLLARY. *$\dot{X}$ and $\dot{V}$ are isomorphic G-fiber bundles.*

PROOF: Luna's slice theorem implies that X and V are complex analytically isomorphic over a small neighborhood of $0 \in \mathbb{A}$. It follows easily that $\dot{X}$ and $\dot{V}$ are topologically (hence algebraically) isomorphic. ∎

The proof of the main result above, theorem (2.4), requires techniques from the theories of group schemes and Galois cohomology. Using results of Steinberg **[St]** on group schemes over curves we are able to reduce to the case where L is a torus. We then establish the existence of a finite cover as in (2.4.1). Let Γ denote the corresponding Galois group. Then (2.4.2) can be translated into statements about the vanishing of Γ-cohomology groups with values in Γ-modules which, as abelian groups, are several copies of $\mathbb{C}[t, t^{-1}]$. We compute that the requisite cohomology groups vanish.

§ 3. The Moduli Space

(3.0) As remarked following theorem (1.3), we can establish everything so far if we replace "$X = \mathbb{A}^n$" by "X is smooth and acyclic," which we do from now on. We fix V (where, of course, $\dim V/\!\!/G = 1$), and consider the possible corresponding X's.

(3.1) Let $\hat{\dot{\mathfrak{A}}}$ denote the group of G-isomorphisms of $\hat{\dot{V}}$ which induce the identity on $\hat{\dot{\mathbb{A}}}$, and define $\dot{\mathfrak{A}}$ and $\hat{\mathfrak{A}}$ similarly. Note that $\hat{\dot{\mathfrak{A}}}$ is (the opposite group to) the group of G- and $\mathcal{O}(\hat{\dot{\mathbb{A}}})$-automorphisms of $\mathcal{O}(\hat{\dot{V}}) = \mathcal{O}(V) \otimes_{\mathcal{O}(\mathbb{A})} \mathcal{O}(\hat{\dot{\mathbb{A}}})$, and similarly for $\dot{\mathfrak{A}}$ and $\hat{\mathfrak{A}}$.

Let X, $\dot{\varphi}$ and $\hat{\varphi}$ be as in (2.1.1). Then the composition $\hat{\dot{\varphi}} := \dot{\varphi} \circ \hat{\varphi}^{-1}$ lies in $\hat{\dot{\mathfrak{A}}}$. Now $\dot{\varphi}$ is only well-defined up to composition with an element $\dot{\alpha} \in \dot{\mathfrak{A}}$, and similarly for $\hat{\varphi}$. Thus the double coset of $\hat{\dot{\varphi}} = \dot{\varphi}\hat{\varphi}^{-1}$ in $\mathfrak{A}(\dot{\mathbb{A}})\backslash\mathfrak{A}(\hat{\dot{\mathbb{A}}})/\mathfrak{A}(\hat{\mathbb{A}})$ is well-defined, and we denote it by $[\hat{\dot{\varphi}}(X)]$. We say that an element $\hat{\dot{\alpha}} \in \mathfrak{A}(\hat{\dot{\mathbb{A}}})$ is *realized by* X if $\hat{\dot{\alpha}} \in [\hat{\dot{\varphi}}(X)]$.

(3.2) Let $\mathcal{M}_{V,\mathbb{A}}$ denote the set of isomorphism classes of contractible affine G-varieties X with fixed quotient mapping $\pi : X \longrightarrow \mathbb{A} \simeq X/\!\!/G$ such that

(1) $X^G = \{x_0\}$ is a single fixed point,
(2) $T_{x_0}X$ is G-isomorphic with V,
(3) $\pi(x_0) = 0 \in \mathbb{A}$.

(If φ is an allowable isomorphism of G-varieties X, X' satisfying our conditions, then φ induces the identity on $\mathbb{A}$). Let $\mathcal{M}_V$ be defined in the same way as $\mathcal{M}_{V,\mathbb{A}}$, except that we do not fix an isomorphism of $X/\!\!/G$ with $\mathbb{A}$. Clearly, $\mathcal{M}_V$ is the quotient of $\mathcal{M}_{V,\mathbb{A}}$ by an action of $\mathbb{C}^*$, and $\mathcal{M}_V$ is trivial (i.e. a point) if and only if $\mathcal{M}_{V,\mathbb{A}}$ is trivial.

(3.3) LEMMA. *Let $\hat{\dot{\alpha}} \in \hat{\dot{\mathfrak{A}}}$. Then there is a Zariski open subset $U \subseteq \mathbb{A}$ containing 0 and a G-automorphism $\tilde{\dot{\alpha}}$ of $\pi^{-1}(U \cap \dot{\mathbb{A}})$ inducing the identity on $U \cap \dot{\mathbb{A}}$ which is in the same double coset as $\hat{\dot{\alpha}}$.*

(3.4) THEOREM. *The correspondence*

$$\mathcal{M}_{V,\mathbb{A}} \xrightarrow{[\hat{\dot{\varphi}}]} \mathfrak{A}(\dot{\mathbb{A}})\backslash\mathfrak{A}(\hat{\dot{\mathbb{A}}})/\mathfrak{A}(\hat{\mathbb{A}}), \quad X \longmapsto [\hat{\dot{\varphi}}(X)]$$

is a bijection.

PROOF OF (3.4): One easily generalizes (2.1) to show that $[\hat{\dot{\varphi}}]$ is injective. Given $\tilde{\dot{\alpha}}$ as in the lemma, one uses it to identify $\dot{V}$ and $\tilde{V}$ over $\tilde{\dot{V}}$. The resulting space X is an affine G-variety, smooth and contractible, which realizes $\hat{\dot{\alpha}}$. ∎

(3.5) REMARK: We know of no examples where we obtain an X not isomorphic to $\mathbb{A}^n$.

(3.6) We now consider the structure of the double coset space of (3.4). We begin by making the groups $\dot{\mathfrak{A}}$ and $\hat{\dot{\mathfrak{A}}}$ more explicit: Since $\mathcal{O}(V)^G$ is graded of dimension 1, we may assume that $\pi = \pi_{V,G} : V \longrightarrow \mathbb{A}$ is homogeneous, say of degree d. As before, set $F = \pi^{-1}(1)$ and $L = \mathrm{Aut}_G(F)$. Let Γ denote the dth roots of unity, and let $\mathbb{B} = \mathrm{Spec}\,\mathbb{C}[s]$ denote another copy of the affine line. We let Γ act on $\mathbb{B}$ by: $(\gamma, b) \mapsto b\gamma^{-1}$, and we identify the quotient $\mathbb{B}/\Gamma$ with $\mathbb{A}$ by the embedding $\mathcal{O}(\mathbb{A}) = \mathbb{C}[t] \hookrightarrow \mathbb{C}[s] = \mathcal{O}(\mathbb{B})$, sending t into s^d. Define $\dot{\mathbb{B}}$, etc. as usual. Then the canonical map

$$\rho : \dot{\mathbb{B}} \times^{\Gamma} F \longrightarrow \dot{V}, \quad [b, v] \mapsto bv$$

is an isomorphism, where $\dot{\mathbb{B}} \times^{\Gamma} F$ is the quotient of $\dot{\mathbb{B}} \times F$ by the action: $\gamma(b, v) = (b\gamma^{-1}, \gamma v)$; $\gamma \in \Gamma$, $b \in \dot{\mathbb{B}}$, $v \in F$. Moreover, ρ is an isomorphism over $\dot{\mathbb{A}}$ via $\pi : \dot{V} \longrightarrow \dot{\mathbb{A}}$ and the map $[b, v] \mapsto b^d$ of $\dot{\mathbb{B}} \times^{\Gamma} F$ to $\dot{\mathbb{A}}$. The principal bundle P associated to $\dot{V}$ is easily seen to be $\dot{\mathbb{B}} \times^{\Gamma} L$ where $\Gamma \subseteq L$ acts on L by left multiplication. We then get

(3.7) PROPOSITION (Luna). *The map ρ induces isomorphisms*

(1) $\dot{\mathfrak{A}} \simeq L(\dot{\mathbb{B}})^{\Gamma}$,
(2) $\hat{\dot{\mathfrak{A}}} \simeq L(\hat{\dot{\mathbb{B}}})^{\Gamma}$.

Here $L(\dot{\mathbb{B}})$, etc. denotes the group of morphisms from $\dot{\mathbb{B}}$ to L, etc. Γ acts on $L(\dot{\mathbb{B}})$ by: $(\gamma\dot{\alpha})(b) = \gamma\dot{\alpha}(b\gamma)\gamma^{-1}$, where $\dot{\alpha} \in L(\dot{\mathbb{B}})$, $b \in \dot{\mathbb{B}}$, $\gamma \in \Gamma$; and similarly for $L(\hat{\dot{\mathbb{B}}})$. Note that the action of Γ on $L(\dot{\mathbb{B}})$, etc. arises from the action of Γ on L as automorphisms by conjugation.

Of course, we can also consider $L(\hat{\mathbb{B}})$ and $L(\hat{\mathbb{B}})^{\Gamma}$. Let $L(\hat{\mathbb{B}})_r$ denote the morphisms $\hat{\alpha} : \hat{\mathbb{B}} \longrightarrow L$ such that $\hat{\alpha} - I$ vanishes to order r at $0 \in \mathbb{B}$, where I is the identity element of L and $r \in \mathbb{Z}^{+}$.

We can prove:

(3.8) PROPOSITION. *Let L be any linear algebraic group with an action by Γ as automorphisms. Then*

$$L(\hat{\dot{\mathbb{B}}})^{\Gamma} = L(\dot{\mathbb{B}})^{\Gamma} L(\hat{\mathbb{B}})_1^{\Gamma}.$$

The proof of the proposition again involves showing the vanishing of a Γ-cohomology group. The case $\Gamma = \{1\}$ was communicated to us by Luna. Note that $L(\hat{\mathbb{B}})_1^{\Gamma} = L^0(\hat{\mathbb{B}})_1^{\Gamma}$.

Unfortunately, $\hat{\mathfrak{A}} \neq L(\hat{\mathbb{B}})^{\Gamma}$, in general. In fact, since L is the automorphism group of the general fiber of π, while $\hat{\mathfrak{A}}$ consists of automorphisms which preserve the zero fiber, a strong connection is somewhat surprising. We have shown, however:

(3.9) PROPOSITION. *Via ρ, consider $\hat{\mathfrak{A}}$ as a subgroup of $L(\hat{\dot{\mathbb{B}}})^{\Gamma} \simeq \hat{\dot{\mathfrak{A}}}$. Then*

(1) *There is an $r > 0$ such that $L(\hat{\mathbb{B}})_r^{\Gamma} \subseteq \hat{\mathfrak{A}}$.*
(2) $\hat{\mathfrak{A}}_1 \subseteq L(\hat{\mathbb{B}})_1^{\Gamma}$.

Here $\hat{\mathfrak{A}}_1 \subseteq \hat{\mathfrak{A}} \subseteq \mathrm{Aut}(\hat{V})$ denotes those automorphisms whose differential at $0 \in \hat{V}$ is the identity.

(3.10) DEFINITION: We say that L has the *approximation property* if $L(\hat{\mathbb{B}})_1^{\Gamma} \subseteq L(\mathbb{B})_1^{\Gamma} L(\hat{\mathbb{B}})_r^{\Gamma}$ for all $r > 0$.

Obviously, we have:

(3.11) PROPOSITION. *If L has the approximation property, then $\mathcal{M}_V$ is trivial.*

It is easy to see that a torus does not have the approximation property, but this is more or less the only example.

We can show:

(3.12) THEOREM. *Suppose that the radical and unipotent radical of L coincide (e.g. L^0 is semisimple). Then L has the approximation property.*

(3.13) Let L' denote the subgroup of L^0 generated by a maximal connected semisimple subgroup and the unipotent radical. Then L' is normal in L, hence Γ-stable, and L^0/L' is our problematic torus. Let $\mathfrak{l}$ (resp. $\mathfrak{l}'$) denote the Lie algebra of L (resp. L').

(3.14) Let D be a derivation of $\mathcal{O}(V)$, i.e. a polynomial vector field. We say that D has *degree p* if D maps polynomials of degree q to polynomials of degree $p + q$ for all q.

Suppose that D is G-invariant and annihilates π. Then D induces a G-derivation of $\mathcal{O}(F)$, denoted $D|_F$, and $D|_F \in \mathfrak{l}$.

(3.15) DEFINITION: We say that (V, G) is *rigid* if there are G-invariant derivations $D_1, \ldots, D_r$ of degree $\leq d = \deg \pi$, which annihilate π, such that $\{D_i|_F\}$ spans $\mathfrak{l}/\mathfrak{l}'$.

Using the exponential map we can show:

(3.16) THEOREM. (1) *(V, G) is rigid if and only if the canonical morphism from $\hat{\mathfrak{A}}_1$ to $(L^0/L')(\hat{\mathbb{B}})_1^{\Gamma}$ is surjective.*

(2) *(V, G) is rigid if and only if $\mathcal{M}_V$ is trivial.*
(3) *In general, there is a bijection between $\mathcal{M}_{V,\mathbb{A}}$ and the quotient of an affine space $\mathbb{A}^r$ by a linear action of Γ.*

(3.17) Consider the case where (V, G) is semifree and G is connected. Then $F \simeq G \simeq L$, and L/L' is the image of the center Z of G. The action of Z on F is the restriction of its linear action on V, so clearly (V, G) is rigid, by (3.16.1).

Theorem (0.2) is a consequence of the following more general result:

(3.18) THEOREM. *(V, G) is rigid in the following cases:*

(1) *(V, G^0) is rigid and G/G^0 acts faithfully on $V/\!\!/G^0 \simeq \mathbb{A}$.*
(2) *G^0 is simple.*
(3) *The principal isotropy group H of (V, G) is central in G.*
(4) *(V, G^0) is self dual.*
(5) $\dim V \leq 3$.

(3.19) There are many examples where (V, G) is not rigid: Let $(V, G) = (V_1 \oplus V_2, G' \times \mathbb{C}^*)$ where $\dim V_1/\!\!/G' = 1$ and $\mathbb{C}^*$ acts trivially on V_1 and by scalar multiplication on V_2. Then $V/\!\!/G \simeq V_1/\!\!/G' \simeq \mathbb{A}$. We suppose that $\mathcal{M}_{V_1}$ is trivial. Let $\mathrm{VEC}_{G'}(V_1, V_2)$ denote the isomorphism classes of G'-vector bundles with base V_1 and fiber V_2 at $0 \in V_1$. (The total spaces of these bundles are isomorphic to affine space by the work of Quillen and Suslin (solution to the Serre conjecture)). Then $\mathrm{VEC}_{G'}(V_1, V_2)$ is isomorphic to an affine space $\mathbb{A}^r$.

The group Γ' corresponding to (V_1, G') (as in (3.6)) lies in $\mathrm{GL}(V_1)^{G'}$, hence operates on $\mathrm{VEC}_{G'}(V_1, V_2) \simeq \mathbb{A}^r$ by pull-back. The Γ'-action on $\mathbb{A}^r$ is linear, and $\mathcal{M}_{V,\mathbb{A}} \simeq \mathbb{A}^r/\Gamma'$.

(3.20) THEOREM. *Let $(V,G) = (V_1 \oplus V_2, G' \times \mathbb{C}^*)$ and r be as above.*

(1) *Let $G' = \mathrm{SL}_2$, $V_1 = R_2$, and $V_2 = R_m$, $m \geq 1$, where R_m denotes the SL_2-module of binary forms of degree m. Then $r = [(m-1)^2/4]$. In particular, there is a non-linearizable action of $\mathrm{SL}_2 \times \mathbb{C}^*$ on $\mathbb{A}^7$ (One can even show that the SL_2-action is not linearizable).*

(2) *Let $G' = \mathrm{O}_2 = \mathbb{C}^* \rtimes \mathbb{Z}/2$, $V_1 = W_1$ and $V_2 = W_m$, $m \geq 1$, where W_m is the irreducible representation of dimension 2 of O_2 with $\mathbb{C}^*$-weights m and $-m$. Then $r = m-1$. In particular, there is a non-linearizable action of $\mathrm{O}_2 \times \mathbb{C}^*$ on $\mathbb{A}^4$.*

(3) *Let $G' = \mathrm{O}_2$, $V_1 = W_2$ and $V_2 = W_m$, $m \geq 1$, m odd. Then $r = (m-1)/2$. The corresponding non-linearizable actions of $\mathrm{O}_2 \times \mathbb{C}^*$ on $\mathbb{A}^4$ are also non-linearizable as O_2-actions.*

REFERENCES

[BH] H. Bass and W. Haboush, *Linearizing certain reductive group actions*, Trans. Amer. Math. Soc. **292** (1985), 463–482.

[Bb] A. Bialynicki-Birula, *Remarks on the action of an algebraic torus on k^n, II*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **15** (1967), 123–125.

[Br] G. Bredon, "Introduction to Compact Transformation Groups," Academic Press, New York, 1972.

[Ka] T. Kambayashi, *Automorphism group of a polynomial ring and algebraic group action on an affine space*, J. Algebra **60** (1979), 439–451.

[Kr] H. Kraft, "Geometrische Methoden in der Invariantentheorie," Vieweg-Verlag, Braunschweig, 1985.

[KPR] H. Kraft, T. Petrie and J. Randall, *Quotient varieties*, Adv. math., to appear.

[Lu] D. Luna, *Slices étales*, Bull. Soc. Math. France, Mémoire **33** (1973), 81–105.

[S] G. Schwarz, *Exotic algebraic group actions*, to appear.

[St] R. Steinberg, *Regular elements of semisimple algebraic groups*, Publ. Math. IHES **25** (1965), 281–312.

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Equivariant Completions and Tensor Products

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ABSTRACT. The coordinate ring of an affine algebraic variety with reductive algebraic group action is acted on by the group, as is its completion at any stable ideal. The largest submodule of the completion on which the group acts rationally is the equivariant completion. It is shown that the equivariant completion is the tensor product of the coordinate ring with the completion of the ring of invariants. Applications include the linearization of (non-singular) one fix pointed actions.

Let k be an algebraically closed field of characteristic zero, let G be a connected linearly reductive algebraic k-group and let X be an affine k variety on which G acts. The equivariant completion ${}^*k[X]_Q$ of $k[X]$ at a G-stable ideal Q is by definition the largest G-rational submodule of the Q-adic completion $\text{proj lim}(k[X]/Q^i)$ [4, Defn. 2.5, p. 179]. If $\{V\}$ is a set of representatives for the isomorphism classes of simple G-modules and $(\cdot)_V$ denotes V-isotypic component, then ${}^*k[X]_Q = \oplus(\text{proj lim}(k[X]/Q^i)_V)$ [4, Prop. 2.3, p.178]. Lex Renner has asked about the relation between the equivariant completion of $k[X]$ at Q and the tensor product of $k[X]$ with the completion of the ring of invariants at Q^G. The purpose of this note is to show that these are the same and to record some consequences of this fact.

It will turn out that a key point is the comparison of the topologies on $k[X]_V$ given by $\{(Q^i)_V | i = 1,2,\cdots\}$ and $\{(Q^G)^i k[X] | i = 1,2,\cdots\}$. They are the same, as the following lemmas imply.

Lemma 1. (Luna) With G, X and Q as above, there exists an integer $M > 0$ such that $(Q^{M\ell})^G \subseteq (Q^G)^\ell$ for all integers $\ell > 0$.

Lemma 2. With G, X, Q and V as above, there exists an integer $N > 0$ such that $(Q^i)_V \subseteq$

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