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# Hanspeter Kraft <br> Claudio Procesi <br> Graded morphisms of $G$-modules 

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# GRADED MORPHISMS OF G-MODULES 

by H. KRAFT and C. PROCESI

## 1. Introduction.

During the 1987 meeting in honor of J. K. Koszul, Steve Halperin explained to us the following conjecture (motivated by the study of the spectral sequence associated to a homogeneous space).
1.1. Conjecture. - If $f_{1}, f_{2}, \ldots, f_{n}$ is a regular sequence in the polynomial ring $\mathrm{C}\left[x_{1}, x_{2}, \ldots, x_{n}\right]$, the connected component of the automorphism group of the (finite dimensional) algebra $\mathbf{C}\left[x_{1}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{n}\right)$ is solvable.

In this paper we prove a weak form of this (Corollary 4.3) which implies the conjecture at least when the $f_{i}^{\prime} s$ are homogeneous (Remark 4.4).

## 2. Preliminaries.

Our base field is $\mathbf{C}$, the field of complex numbers, or any other algebraically closed field of characteristic zero.
2.1. Definition. - A morphism $\varphi: \mathrm{V} \rightarrow \mathrm{W}$ between finite dimensional vector spaces V and W is called graded if there is a basis of W such that the components of $\varphi$ are all homogeneous polynomials.

Let us denote by $\mathcal{O}(\mathrm{V}), \mathcal{O}(\mathrm{W})$ the ring of regular functions on V and W . These $\mathbf{C}$-algebras are naturally graded by degree: $\mathcal{O}(\mathrm{V})=\oplus \mathcal{O}(\mathrm{V})_{i}$. A subspace $\mathrm{S} \subset \mathcal{O}(\mathrm{V})$ is called graded if $\mathrm{S}=\underset{i}{\oplus} \mathrm{~S} \cap \mathcal{O}(\mathrm{~V})_{i}$.

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If $\varphi: \mathrm{V} \rightarrow \mathrm{W}$ is a morphism and $\varphi^{*}: \mathcal{O}(\mathrm{W}) \rightarrow \mathcal{O}(\mathrm{V})$ the corresponding comorphism we have the following equivalence:

$$
\varphi \text { is graded } \Leftrightarrow \varphi^{*}\left(\mathrm{~W}^{*}\right) \text { is a graded subspace of } \mathcal{O}(\mathrm{V}) .
$$

2.2. Lemma. - For any graded morphism $\varphi: \mathrm{V} \rightarrow \mathrm{W}$ there is a unique decomposition $\mathrm{W}=\oplus \mathrm{W}_{v}$ and homogeneous morphisms $\varphi_{v}: \mathrm{V} \rightarrow \mathrm{W}_{\mathrm{v}}$ of degree $v$ such that $v \geqslant 0$

$$
\varphi=\left(\varphi_{0}, \varphi_{1}, \varphi_{2}, \ldots\right): \mathrm{V} \rightarrow \mathrm{~W}_{0} \oplus \mathrm{~W}_{1} \oplus \mathrm{~W}_{2} \oplus \cdots
$$

(This is clear from the definitions.)
2.3. Remark. - Let G be an algebraic group. Assume that V and W are G -modules and that $\varphi: \mathrm{V} \rightarrow \mathrm{W}$ is graded and G -equivariant. Then in the notations of lemma 2.2 all $W_{v}$ are submodules and all components $\varphi_{v}$ are G-equivariant.
2.4. Remark. - If $\varphi: \mathrm{V} \rightarrow \mathrm{W}$ is graded and dominant with $\varphi^{-1}(0)=\{0\}$, then $\varphi$ is a finite surjective morphism. In fact given a finitely generated graded algebra $A=\underset{i \geqslant 0}{\oplus} \mathrm{~A}_{i}$ with $\mathrm{A}_{0}=\mathbf{C}$ and a graded subspace $S \subset A$ such that the radical $\operatorname{rad}(S)$ of the ideal generated by $S$ is the homogeneous maximal ideal $\oplus A_{i}$ of $A$, then $A$ is a finitely generated module over the subalgebra $C[S]$ generated by $S$ (see [1, II.4.3 Satz 8]).

## 3. The Main Theorem.

3.1. Theorem. - Let $G$ be a connected reductive algebraic group and let V, W be two G-modules. Assume that V and W do not contain 1-dimensional submodules. Then any graded G-equivariant dominant morphism with finite fibres is a linear isomorphism.

We first prove this for $\mathrm{G}=\mathrm{SL}_{2}$ and then reduce to this situation.
For any $\mathrm{C}^{*}$-module V we have the weight decomposition

$$
\mathrm{V}=\underset{j}{\oplus} \mathrm{~V}_{j}, \quad \mathrm{~V}_{j}:=\left\{v \in \mathrm{~V} \mid t(v)=t^{j} \cdot v\right\}
$$

We say that V has only positive weights if $\mathrm{V}=\underset{j>0}{\oplus} \mathrm{~V}_{j}$.
3.2. Lemma. - Let $\mathrm{V}, \mathrm{W}$ be two $\mathrm{C}^{*}$-modules with only positive weights, and let $\varphi: \mathrm{V} \rightarrow \mathrm{W}$ be a $\mathrm{C}^{*}$-equivariant graded morphism with finite fibres. For all $k \geqslant 0$ we have

$$
\varphi^{-1}\left(\underset{j \leqslant k}{\oplus} \mathrm{~W}_{j}\right) \subseteq \underset{j \leqslant k}{\oplus} \mathrm{~V}_{j}
$$

and the inclusion is strict for at least one $k$ in case $\varphi$ is not linear.
Proof. - By lemma 2.2 and remark 2.3 we have $\varphi=\sum_{v \geqslant 1} \varphi_{v}$ where $\varphi_{v}: V \rightarrow W_{v}$ is homogeneous of degree $v$ and $C^{*}$-equivariant. Let $v=\sum_{j=1}^{k} v_{j} \in \underset{j>0}{\oplus} \mathrm{~V}_{j}=\mathrm{V}$ with $v_{k} \neq 0$. Then

$$
\lim _{\lambda \rightarrow 0} \lambda^{k} \cdot t_{\lambda}^{-1}(v)=v_{k}
$$

(Here $t_{\lambda}$ denotes the action of $\mathbf{C}^{*}$.) Since $\varphi_{v}$ is homogeneous of degree $v$ and $\mathbf{C}^{*}$-equivariant we obtain

$$
\begin{equation*}
\lim _{\lambda \rightarrow 0} \lambda^{v k} \cdot t_{\lambda}^{-1}\left(\varphi_{v}(v)\right)=\varphi_{v}\left(v_{k}\right) \tag{1}
\end{equation*}
$$

This implies that $\varphi_{v}(v) \in \underset{j \leqslant v k}{\oplus} W_{i}$ for all $v$, proving the first claim.
If $\varphi$ is not linear, i.e. $\varphi \neq \varphi_{1}$, then there is a $v>1$, an index $k$ and an element $v \in \mathrm{~V}_{k}$ such that $\varphi_{v}(v) \neq 0$. But $\varphi_{v}(v) \in \mathrm{W}_{v k}$ by (1) and so $v \notin \varphi^{-1}\left(\sum_{j \leq k} \mathrm{~W}_{j}\right)$.
3.3. Corollary. - Under the assumptions of lemma 3.2 suppose that $\varphi$ is surjective. Put $\lambda_{j}:=\operatorname{dim} \mathrm{V}_{j}$ and $\mu_{j}:=\operatorname{dim} \mathrm{W}_{j}$. Then for all $k \geqslant 1$ we have

$$
\begin{equation*}
\lambda_{1}+\lambda_{2}+\cdots+\lambda_{k} \geqslant \dot{\mu}_{1}+\mu_{2}+\cdots \mu_{k} . \tag{2}
\end{equation*}
$$

If $\varphi$ is not linear the inequality is strict for at least one $k$.
(This is clear.)
3.4. Proposition. - Let $\mathrm{V}, \mathrm{W}$ be two $\mathrm{SL}_{2}$-modules containing no fixed lines. Let $\varphi: V \rightarrow \mathrm{~W}$ be a graded $\mathrm{SL}_{2}$-equivariant morphism, which is dominant and has finite fibres. Then $\varphi$ is a linear isomorphism.

Proof. - Consider the maximal unipotent subgroup

$$
\mathrm{U}:=\left\{\left(\begin{array}{ll}
1 & * \\
0 & 1
\end{array}\right)\right\} \subset \mathrm{SL}_{2}
$$

and the maximal torus

$$
\mathrm{T}:=\left\{\left.\left(\begin{array}{cc}
\lambda & 0 \\
0 & \lambda^{-1}
\end{array}\right) \right\rvert\, \lambda \in \mathbf{C}^{*}\right\} \simeq \mathbf{C}^{*}
$$

By assumption $\varphi$ is finite and surjective (Remark 2.4), and $\varphi^{-1}\left(\mathrm{~W}^{\mathrm{U}}\right)=\mathrm{V}^{\mathrm{U}}$. Hence the induced morphism

$$
\left.\varphi\right|_{\mathrm{VU}}: \mathrm{V}^{\mathrm{U}} \rightarrow \mathrm{~W}^{\mathrm{U}}
$$

is graded, T-equivariant, finite and surjective too. Furthermore all weights $\lambda_{j}$ of $\mathrm{V}^{\mathrm{U}}$ and $\mu_{j}$ of $\mathrm{W}^{\mathrm{U}}$ are positive. It follows from (2) that

$$
\lambda_{k}+\lambda_{k+1}+\cdots \leqslant \mu_{k}+\mu_{k+1}+\cdots
$$

for all $k$, because $\sum_{j} \lambda_{j}=\operatorname{dim} \mathrm{V}^{\mathrm{U}}=\operatorname{dim} \mathrm{W}^{\mathrm{U}}=\sum_{j} \mu_{j}$. From this we get

$$
\begin{aligned}
\operatorname{dim} \mathrm{V} & =2 \lambda_{1}+3 \lambda_{2}+\cdots+(n+1) \lambda_{n} \\
& \leqslant 2 \mu_{1}+3 \mu_{2}+\cdots+(n+1) \mu_{n}=\operatorname{dim} \mathrm{W}
\end{aligned}
$$

for all $n$ which are big enough. (Remember that an irreducible $\mathrm{SL}_{2}$-module of highest weight $j$ is of dimension $j+1$ ). If $\varphi$ is not linear this inequality is strict (Corollary 3.3), contradicting the fact that $\varphi$ is finite and surjective.
3.5. Proof of the Theorem. - Assume that $\varphi: \mathrm{V} \rightarrow \mathrm{W}$ is not linear, i.e. there is a $v_{0}>1$ such that the component $\varphi_{v_{0}}: V \rightarrow W_{v_{0}}$ is nonzero. Then there is a homomorphism $\mathrm{SL}_{2} \rightarrow \mathrm{G}$ and a non-trivial irreducible $\mathrm{SL}_{2}$-submodule $\mathrm{M} \subseteq \mathrm{V}$ such that $\left.\varphi_{j}\right|_{\mathrm{M}} \neq 0$. (In fact the intersection of the fixed point sets $\mathrm{V}^{\iota\left(\mathrm{SL}_{2}\right)}$ for all homomorphisms ı: $\mathrm{SL}_{2} \rightarrow \mathrm{G}$ is zero.) Now consider the G-stable decompositions $\mathrm{V}=\mathrm{V}^{\mathrm{SL}_{2}} \oplus \mathrm{~V}^{\prime}$ and $\mathrm{W}=\mathrm{W}^{\mathrm{SL}_{2}} \oplus \mathrm{~W}^{\prime}$ and the following morphism :

$$
\varphi^{\prime}: \mathrm{V}^{\prime} \subset \mathrm{V} \xrightarrow{\varphi} \mathrm{~W} \xrightarrow{\mathrm{pr}} \mathrm{~W}^{\prime} .
$$

Since $\mathrm{V}^{\prime}$ and $\mathrm{W}^{\prime}$ are sums of isotypic components the morphism $\varphi^{\prime}$ is again graded. Furthermore $\varphi^{-1}\left(W^{S L_{2}}\right)=V^{S L_{2}}$, hence $\varphi^{-1}(0)=\mathrm{V}^{\mathrm{SL}_{2}} \cap \mathrm{~V}^{\prime}=\{0\}$. This implies that $\varphi^{\prime}: \mathrm{V}^{\prime} \rightarrow \mathrm{W}^{\prime}$ is dominant
with finite fibres and satisfies therefore the assumptions of proposition 3.4. As a consequence $\varphi^{\prime}$ is linear. Since $\left.\varphi\right|_{v^{\prime}}: \mathrm{V}^{\prime} \rightarrow \mathrm{W}$ is graded too we have $\left.\varphi_{\mathrm{v}}\right|_{\mathrm{v}^{\prime}}=0$ for all $v>1$. This contradicts the facts that $\mathrm{M} \subseteq \mathrm{V}^{\prime}$ and $\left.\varphi_{v_{0}}\right|_{M} \neq 0$ (see the construction above).

## 4. Some Consequences.

We add some corollaries of the theorem. Let $G$ be a connected reductive group. For every G-module V we have the canonical G -stable decomposition $\mathrm{V}=\mathrm{V}^{\circ} \oplus \mathrm{V}^{\prime}$ where $\mathrm{V}^{\circ}$ is the sum of all 1-dimensional representations (i.e. $\mathrm{V}^{\circ}=\mathrm{V}^{(\mathrm{G}, \mathrm{G})}$ ) and $\mathrm{V}^{\prime}$ the sum of all others. The proof of the theorem above easily generalizes to obtain the following result :
4.1. Theorem. - Let $\varphi: \mathrm{V} \rightarrow \mathrm{W}$ be a graded G -equivariant dominant morphism with finite fibres. Then $\varphi$ induces a linear isomorphism

$$
\left.\varphi\right|_{\mathrm{v}^{\prime}}: \mathrm{V}^{\prime} \xrightarrow{\sim} \mathrm{W}^{\prime} .
$$

4.2. Corollary. - Let $\mathcal{O}(\mathrm{V})$ be the ring of regular functions on a G-module V , and let $f_{1}, \ldots, f_{n}$ be a regular sequence of homogenous elements of $\mathcal{O}(\mathrm{V})$ such that the linear span $\left\langle f_{1}, \ldots, f_{n}\right\rangle$ is G -stable. Then $\left\langle f_{1}, \ldots, f_{n}\right\rangle$ contains all non-trivial representations of $(\mathrm{G}, \mathrm{G})$ in $\mathcal{O}(\mathrm{V})_{1}$, the linear part of $\mathcal{O}(\mathrm{V})$.

Proof. - The regular sequence $f_{1}, \ldots, f_{n}$ defines a G-equivariant finite morphism $\varphi: \mathrm{V} \rightarrow \mathrm{W}, \mathrm{W}:=\left\langle f_{1}, \ldots, f_{n}\right\rangle^{*}$. By the theorem above the restriction $\left.\varphi^{\prime}\right|_{V^{\prime}}: \mathrm{V}^{\prime} \rightarrow \mathrm{W}^{\prime}$ is a linear isomorphism which means that every non-trivial $(\mathrm{G}, \mathrm{G})$-submodule of $\left\langle f_{1}, \ldots, f_{n}\right\rangle$ is contained in the linear part $\mathcal{O}\left(\mathrm{V}_{1}\right)$ of $\mathcal{O}(\mathrm{V})$.
4.3. Recall that a finite dimensional $\mathbf{C}$-algebra is called a complete intersection if it is of the form $\mathbf{C}\left[x_{1}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{n}\right)$ with a regular sequence $f_{1}, \ldots, f_{n}$.

Corollary. - Let A be a finite dimensional local C-algebra with maximal ideal m and let $\mathrm{gr}_{\mathrm{m}} \mathrm{A}$ be the associated graded algebra (with respect to the $\mathbf{m}$-adic filtration). If $\mathrm{gr}_{\mathrm{m}} \mathrm{A}$ is a complete intersection then the connected component of the automorphism group of A is solvable.

Proof. - Let $G$ and $\bar{G}$ be the connected components of the automorphism groups of A and of $\mathrm{gr}_{\mathrm{m}} \mathrm{A}$ respectively. Since the m-adic filtration of $A$ is $G$-stable we have a canonical homomorphism $\rho: G \rightarrow \bar{G}$. It is easy to see that $\operatorname{ker} \rho$ is unipotent, so it remains to show that $\bar{G}$ is solvable.

Assume that $\bar{G}$ is not solvable. Then $\bar{G}$ contains a (non-trivial) semisimple subgroup H . By assumption we have an isomorphism

$$
\operatorname{gr}_{\mathrm{m}} \mathrm{~A} \simeq \mathbf{C}\left[x_{1}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{n}\right)
$$

with a regular sequence $f_{1}, \ldots, f_{n}$ where all $f_{i}$ are homogeneous of degree $\geqslant 2$. Clearly the action of $\bar{G}$ on ${g r_{m} A \text { is induced from a }}$ (faithful) linear representation on $\mathbf{C}\left[x_{1}, \ldots, x_{n}\right]_{1} \subset \mathbf{C}\left[x_{1}, \ldots, x_{n}\right]$. Hence it follows from corollary 4.2 that $\left\langle f_{1}, \ldots, f_{n}\right\rangle$ contains all non-trivial H -submodules of $\mathbf{C}\left[x_{1}, \ldots, x_{n}\right]_{1}$, contradicting the fact that all $f_{i}$ have degree $\geqslant 2$.
4.4. Remark. - The corollary above implies that conjecture 1.1 is true in case all $f_{i}$ are homogeneous, i.e. if the algebra

$$
\mathrm{A}=\mathbf{C}\left[x_{1}, \ldots, x_{n}\right] /\left(f_{1}, \ldots, f_{n}\right)
$$

is finite dimensional and graded with all $x_{i}$ of degree 1 .
4.5. Remark. - Another formulation of our result is the following : Let V be a representation of a connected algebraic group G and $\mathrm{Z} \subset \mathrm{V}$ a G-stable graded subscheme, which is a complete intersection supported in $\{0\}$. Then $(\mathrm{G}, \mathrm{G})$ acts trivially on Z .

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