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GEOMETRIC METHODS IN REPRESENTATION THEORY

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INTRODUCTION

The present notes are a more or less faithful reproduction of the lectures given at the Workshop on "Representations of Algebras" in Puebla, Mexico, 1980. The aim of this series of lectures was to describe some geometric methods which can be and have been used in representation theory, in particular methods from algebraic transformation groups and invariant theory. It turns out that from this geometric point of view there arise many questions and problems which seem to be quite interesting and which have not yet been studied in detail. On the other hand much material from representation theory can be understood in this geometric way and provides us with a big amount of exciting examples.

Rather than developing a general theory we have preferred to work out some of these examples, partly well known and elementary, in order to introduce the subject and to explain the main ideas. However it soon becomes clear that for the more advanced examples we also need more theory, some general facts from algebraic geometry, transformation groups and invariant theory.

So we start in the first chapter by describing three examples: "Conjugacy classes of matrices", "Modules over $\mathbb{C}\{X,Y\}$" and "Completely reachable pairs of matrices". (The last example originates from system and control theory.) Already here we sometimes use notations and facts from the following chapter, where we develop the foundations of algebraic geometry, transformation groups and invariant theory, again using many examples mainly from representation theory. Because of time constraints we include in this part only a few sample proofs, to convey some of the flavor of the subject. In the last chapter as an application of the methods we present a proof of a result of Gabriel which states that "finite representation type is open".

In order to explain the main ideas and also for convenience of the reader it seemed to us reasonable to concentrate on the most geometric situation, i.e. we are going to work over the field $\mathbb{C}$ of complex numbers. Of course we could replace $\mathbb{C}$ by any other algebraically closed field of characteristic zero. With slight modifications all results also hold in positive characteristic, but the proofs become more complicated and more technical. (We have to use Mumford's conjecture proved by Haboush.)

Finally I would like to thank Mrs. R. Wegmann for the perfect typing of the manuscript.

Chapter I SOME EXAMPLES

In the first chapter we describe three examples: "Conjugacy classes of matrices", "Modules over $\mathbb{C}\{X,Y\}$" and "Completely reachable pairs of matrices". In all three cases we have an important classification problem which we want to attack by geometric methods. It turns out that even in the first example where a complete classification is known this geometric point of view provides us with a deeper insight into the nature of the problem and a better understanding of some phenomena and also gives rise to new developments and quite interesting questions.

We have tried to keep this chapter as elementary as possible. As a consequence we often use ad hoc arguments in order to convince the reader, hoping that all this will become clear in the following chapter where we develop the general technical tools.

1. Conjugacy Classes of Matrices

1.1 Let $R := \mathbb{C}[X]$ be the polynomal ring in one variable. An R-module M is the same thing as a vectorspace V together with an endomorphism $A \in \mathrm{End}(V)$. If we fix a finite dimensional vectorspace V we may consider the set $\mathrm{mod}_{R,V}$ of all R-modules with underlying vectorspace V, i.e. the set of all R-module structures on V. By what we said above we have in a canonical way

$$\mathrm{mod}_{R,V} \xrightarrow{\sim} \mathrm{End}(V), \quad M \mapsto X_M. \tag{1}$$

In addition two R-modules $M,N \in \mathrm{mod}_{R,V}$ are isomorphic if and only if the corresponding endomorphisms X_M and X_N are conjugate in $\mathrm{End}(V)$ (i.e. there is a $g \in GL(V)$ such that $X_N = g X_M g^{-1}$) .

1.2 We may express this in a slightly different way. The group $GL(V)$ acts on $\mathrm{mod}_{R,V}$ by "transport of structure": If $M \in \mathrm{mod}_{R,V}$ and $g \in \mathrm{End}(V)$ there is a unique R-module structure ${}^gM$ on V such that $g : M \to {}^gM$ is a R-module homomorphism. (Clearly this is the action obtained via the isomorphism (1) from the adjoint action $Y \mapsto gYg^{-1}$ of $GL(V)$ on $\mathrm{End}(V)$.)
Now two R-modules $M,N \in \mathrm{mod}_{R,V}$ are isomorphic if and only if they belong to the same orbit under $GL(V)$.

1.3 In case $V = \mathbb{C}^n$ we simply write $\mathrm{mod}_{R,n}$ and identify this set with $M_n(\mathbb{C})$, the set of $n\times n$-matrices.

An R-module $M \in \mathrm{mod}_{R,V}$ is semisimple if and only if the corresponding endomorphism X_M is semisimple i.e. X_M is a diagonal matrix with respect to a suitable basis of V .

Similarly M is indecomposable if it corresponds to a matrix of the form

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

More generally the decomposition of an R-module M into indecomposable direct factors corresponds to the block decomposition of a matrix in Jordan normal form. (It's well known that in both cases the factors are uniquely determined but not the decomposition.)

1.4 We see now that any R-module M of dimension n uniquely determines an orbit C_M in $\mathrm{mod}_{R,n}$ (and also in $\mathrm{mod}_{R,V}$ if $\dim V = n$) and we have $C_M = C_N$ if and only if the R-modules M and N are isomorphic.

Therefore the set of isomorphism classes of R-modules of dimension n is given by the "orbit space"

$$\mathrm{mod}_{R,n} / GL_n(\mathbb{C}) \approx \{\text{conjugacy classes in } M_n(\mathbb{C})\} . \tag{2}$$

A conjugacy class is not necessarily closed. E.g. $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is conjugate to $\left(\begin{smallmatrix} 0 & \varepsilon \\ 0 & 0 \end{smallmatrix}\right)$ for all $\varepsilon \neq 0$, hence the closure of the conjugacy class of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ contains the zero matrix. This implies that the orbit space $\mathrm{mod}_{R,n} / GL_n(\mathbb{C})$ has a very unpleasant topological structure: it contains non closed points!

1.5 There is another way to attack this "geometric" classification problem, using invariant functions. Consider the characteristic polynomial of a matrix $A \in M_n(\mathbb{C})$:

$$\det(t\cdot 1 - A) = t^n + \sum_{i=1}^{n} (-1)^i \sigma_i(A) t^{n-i}. \qquad (3)$$

$\sigma_i(A)$ is the i^{th} elementary symmetric function of the eigenvalues of A. We see from the expression above that it depends polynomially on the entries of the matrix A, hence σ_i is an invariant polynomial function on $M_n(\mathbb{C})$ (i.e. it is constant on the conjugacy classes).

We use these functions to define the following map:

$$\pi : \mathrm{mod}_{R,n} \to \mathbb{C}^n ,$$

$$\pi(M) := (\sigma_1(X_M), \sigma_2(X_M), \ldots, \sigma_n(X_M)).$$

It is easy to see that π is surjective. Since $\pi(M)$ determines the characteristic polynomial of X_M, hence its eigenvalues and their multiplicities, each fibre of π contains exactly one orbit consisting in semisimple modules (i.e. the orbit corresponding to the conjugacy class of a diagonal matrix with the given eigenvalues). Furthermore $\pi^{-1}(x)$ is a single orbit for "almost all" $x \in \mathbb{C}^n$. More precisely the discriminant of the polynomial (3) defines a hypersurface $D \subset \mathbb{C}^n$,

$$D := \{(a_1, \ldots, a_n) \mid t^n + \Sigma(-1)^i a_i t^{n-i} \text{ has a multiple root}\} ,$$

and $\pi^{-1}(x)$ is a single orbit if (and only if) x belongs to the dense open set $\mathbb{C}^n - D$ of $\mathbb{C}^n$.
On the other hand $\pi^{-1}(0)$ is the union of the orbits C_M of R-modules M of the form

$$M = \bigoplus_{i=1}^{s} R/X^{n_i}R , \quad \sum_{i=1}^{s} n_i = n .$$

Clearly M is determined up to isomorphism by the (unordered) tupel $(n_1, n_2, \ldots, n_s)$. It follows that the orbits in $\pi^{-1}(0)$ are in 1-1 correspondence with the partitions of n.

A similar argument shows that in any fibre $\pi^{-1}(x)$ the number of orbits is finite. Furthermore $\pi^{-1}(x)$ always contains a dense orbit.

Remark: One can show that $\pi : \mathrm{mod}_{R,n} \to \mathbb{C}^n$ is "the best continous approximation" to the orbit space in the following sense: Every continuous invariant function on $\mathrm{mod}_{R,n}$ factors through π.

1.6 At this point we may ask the following question:

Given an R-module M and its orbit $C_M \subset \mathrm{mod}_{R,n}$, what is the meaning, in module theoretic terms, of the closure $\overline{C_M}$?

To understand this we need the concept of an algebraic family $\{M_\lambda\}_{\lambda\in S}$ of R-modules. We will give a precise definition in the next chapter (II.2.4).
The idea is that all M_λ have the same underlying vectorspace and that the module structure depends algebraically on $\lambda \in S$, S a subvariety of some $\mathbb{C}^m$.

Example: $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, $f_\lambda := X^n - \sum_{i=1}^{n} \lambda_i X^{n-i} \in R$; then

$M_\lambda := R/f_\lambda R$, $\lambda \in \mathbb{C}^n$, is an algebraic family of R-modules. (The image of $\{1, X, X^2, \ldots, X^{n-1}\} \subset R$ in M_λ is a basis for all λ; with respect to this basis we have

$$X_{M_\lambda} = \begin{pmatrix} 0 & & & & \lambda_1 \\ 1 & 0 & & & \lambda_2 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & \\ & & & 1 & \lambda_n \end{pmatrix}$$

which depends algebraically on λ.)

1.7 *Definition*: An R-module N is called a *degeneration* of an R-module M, if there is an algebraic family $\{M_\lambda\}_{\lambda \in S}$ of R-modules such that $M_\lambda \simeq N$ for some $\lambda \in S$ and $M_\lambda \simeq M$ for almost all $\lambda \in S$.

(Here "almost all" means for all λ in a dense subset of S.)

We shortly write $N \leq M$ for this ordering.

Example: If $M' \subset M$ is a submodule then $M' \oplus M/M'$ is a degeneration of M. (Consider $M[T] := \mathbb{C}[T] \otimes M$ and the submodule $\tilde{M} := T \cdot M[T] + M'[T] \subset M[T]$. Define

$$M_\lambda := \tilde{M}/(T-\lambda)\tilde{M}, \quad \lambda \in \mathbb{C}.$$

Then $\{M_\lambda\}_{\lambda \in \mathbb{C}}$ is an algebraic family of R-modules, $M_0 \simeq M' \oplus M/M'$ and $M_\lambda \simeq M$ for $\lambda \neq 0$.)

The following proposition gives a first answer to the question 1.6 . It easily follows from the definitions (cf. II.3.5).

Proposition: *Let* M,N *be two R-modules of the same dimension and* C_M, C_N *the corresponding orbits. Then* N *is a degeneration of* M *if and only if* $C_N \subset \overline{C_M}$. In particular $N \leq M$ *and* $M \leq N$ *implies that* N *and* M are isomorphic.

As a consequence we find:

Corollary: a) *An R-module* M *is semisimple if and only if* C_M *is closed*.

b) *Every R-module* M *has a semisimple degeneration, namely the direct sum of its Jordan-Hölder factors*.

(b) follows by induction from the example above. For a) we first remark that $\overline{C_M}$ is contained in the fibre $\pi^{-1}(\pi(C_M))$, which contains exactly one semisimple orbit by 1.5 . Using b) and the second part of the proposition this implies the claim.)

1.8 The degeneration problem can be solved in a purely combinatonial way. We describe it for the R-moduls M with *nilpotent* X_M ; the general case can easely be deduced from this.

Proposition: *If* N *and* M *are two R-modules of the same dimension with nilpotent* X_N *and* X_M, *then* $N \leq M$ *if and only if* $\operatorname{rk} X_N^i \leq \operatorname{rk} X_M^i$ *for all* i .

(cf. [H1],[KP1])

1.9 *Example*: Let A be a finite dimensional commutative algebra generated by one element, i.e. $A = R/fR$ with some polynomial f of positive degree. Then the set $\operatorname{mod}_{A,V}$ of A-module structures on V becomes in a natural way a closed subset of $\operatorname{mod}_{R,V}$:

$$\operatorname{mod}_{A,V} = \{M \in \operatorname{mod}_{R,V} \mid f \cdot M = 0\}$$
$$\simeq \{Y \in \operatorname{End} V \mid f(Y) = 0\} \subset \operatorname{End} V$$

If k denotes the number of simple A-modules (i.e. the number of maximal ideals of A) there exists exactly $\binom{n+k-1}{n}$ isomorphism classes of semisimple A-modules of dimension n .

This implies that $\mathrm{mod}_{A,n}$ has $\binom{n+k-1}{n}$ connected components: Two A-modules belong to the same component if and only if they have the same Jordan-Hölder factors (counted with multiplicity). It is not hard to see that each component is the closure of an orbit. It follows therefore from recent results on the geometry of conjugacy classes([KP1],[PK]) that $\mathrm{mod}_{A,n}$ is a <u>normal variety</u>.

E.g. for $A = R/X^3R$ we have one simple A-module and three indecomposable A-modules (up to isomorphism), of dimensions one, two and three.

The following diagram gives the isomorphism classes of A-modules of dimension 7 and their degenerations (the symbol describes the decomposition into indecomposables, the degenerations go from top to the bottom, cf. [H1]).

$$\begin{tikzcd}[column sep=small]
 & (3,3,1) \arrow[d, dash] & \\
 & (3,2,2) \arrow[d, dash] & \\
 & (3,2,1,1) \arrow[dl, dash] \arrow[dr, dash] & \\
(2,2,2,1) \arrow[dr, dash] & & (3,1^4) \arrow[dl, dash] \\
 & (2,2,1^3) \arrow[d, dash] & \\
 & (2,1^5) \arrow[d, dash] & \\
 & (1^7) &
\end{tikzcd}$$

1.10 For any $M \in \mathrm{mod}_{R,V}$ we have

$$\mathrm{End}_R(M) = \{g \in \mathrm{End}(V) \mid gX_M = X_M g\} .$$

In particular the <u>stabilizer</u> $\mathrm{Stab}_{GL(V)}\, X_M$ of X_M is the group of units of the endomorphismring $\mathrm{End}_R(M)$, and so

$$\dim \mathrm{End}_R(M) = \dim \mathrm{Stab}_{GL(V)}\, X_M$$

(cf. II. 3.6). On the other hand the orbit C_M is isomorphic to the conjugacy class of X_M, hence to the homogeneous space

$GL(V)/\mathrm{Stab}_{GL(V)}\, X_M$, which implies the following result.

<u>Lemma</u>: <u>For any</u> R-<u>module</u> M <u>of dimension</u> n <u>we have</u>

$$\dim \mathrm{End}_R(M) + \dim C_M = n^2 .$$

<u>Remark</u>: For any strict degeneration $N < M$ (i.e. $N \le M$ and $N \not\cong M$) we have $\dim \mathrm{End}_R(N) > \dim \mathrm{End}_R(M)$.

(This follows from the fact that C_N is contained in the boundary $\partial C_M = \overline{C_M} - C_M$ which is a closed subset of $\overline{C_M}$ of strictly smaller dimension, cf. II.2.6.)

<u>Example</u>: a) Let M be a semisimple module of dimension n ,
$M \simeq \bigoplus_{i=1}^{t} (R/(X-\lambda_i)R)^{n_i}$ with pairwise different λ_i and $\sum_{i=1}^{t} n_i = n$. We find

$$\mathrm{End}_R(M) \simeq \prod_{i=1}^{t} M_{n_i}(\mathbb{C}),$$

hence

$$\dim C_M = n^2 - \sum_{i=1}^{t} n_i^2$$

b) For $M \simeq R/X^nR$ we find $\dim \mathrm{End}_R M = n$ and $\dim C_M = n^2 - n$.

1.11 In order to get a general dimension formula let us recall that every finite dimensional R-module M can be written in the form

$$M \simeq \bigoplus_{i=1}^{s} R/f_iR \tag{5}$$

with $f_{i+1} \mid f_i$ for $i=1,2,\dots,s-1$. The polynomials f_i are <u>uniquely determined</u> (up to a constant factor) and are called the <u>invariant factors</u> of M (or of X_M; f_1 is the minimal polynomial of X_M).

The degrees $p_i = \deg f_i$ form a partition $\underline{p}_M = (p_1, p_2, \ldots, p_s)$ of n (i.e. $p_1 \geq p_2 \geq \ldots \geq p_s$, $\Sigma p_i = n$). The decomposition (5) implies the following dimension formula:

$$\dim \mathrm{End}_R(M) = \sum_{i,j} \min(p_i, p_j) = \sum_j q_j^2 \tag{6}$$

where $(q_1, \ldots, q_t) = \hat{\underline{p}}_M$ is the dual partition to $\underline{p}_M$ (i.e. $q_j = \#\{i \mid p_i \geq j\}$).

In the example 1.10a we may assume $n_1 \geq n_2 \geq \ldots \geq n_s$; then the invariant factors f_j, $j=1,2,\ldots,n_1$, are given by

$$f_j = \prod_{i=1}^{r} (X-\lambda_i) \quad \text{if} \quad n_{r+1} < j \leq n_r,$$

hence $(n_1, \ldots, n_t)$ is the dual partition $\hat{\underline{p}}_M$.

1.12 Let us come back now to the orbit space $\mathrm{mod}_{R,n} / GL_n$ and the problem of a geometric description and a parametrization of the isomorphism classes.
We want to decompose the space $\mathrm{mod}_{R,n}$ into subsets consisting in orbits of a fixed dimension. For this purpose we use the partition $\underline{p}_M$ defined by the invariant factors of the matrix X_M. (1.11)

For any partition $\underline{p}$ of n, we put

$$S_{\underline{p}} := \{M \in \mathrm{mod}_{R,n} \mid \underline{p}_M = \underline{p}\} .$$

This subsets are called the sheets of $\mathrm{mod}_{R,n}$. They define a finite stratification of $\mathrm{mod}_{R,n}$ into locally closed subsets consisting in orbits of a fixed dimension. In particular all orbits in a given sheet S are closed, hence we may hope that the orbit space $S/GL_n(\mathbb{C})$ has a nice structure.

In the following proposition we collect the main results in this direction (cf. [K], [Pe], [Pe']).

Proposition: a) The sheets are the connected components of the subsets

$$\mathrm{mod}_{R,n}^{(d)} := \{M \in \mathrm{mod}_{R,n} \mid \dim \mathrm{End}_R(M) = d\} .$$

b) Every sheet is a smooth submanifold of $\mathrm{mod}_{R,n}$.

c) The orbit space $S_{\underline{p}}/GL_n$ is, in a natural way, an affine space of dimension p_1.

Summary:

The "geometric" classification of finite dimensional R-modules rises two problems, a "vertical" one - degenerations of modules and orbit closures - and a "horizontal" one - description of the sheets and parametrization. It will turn out that the same situation occurs in a much more general setting (e.g. for any finitely generated algebra R or for representations of quivers). In the present situation where $R = \mathbb{C}[X]$ the two problems are solved; here we have a good knowledge of the geometry of finite dimensional R-modules.

Problems:

1) If M is an R-module and $N \leq M$ a degeneration, is it true that there is a filtration

$$M = M_o \supset M_1 \supset \dots \supset M_s = 0$$

s. t. $$N \cong \bigoplus_{i=1}^{s} M_{i-1}/M_i \quad ?$$

2) Assume $N = P \oplus N' \leq M = P \oplus M'$. Then $N' \leq M'$ and conversely.

3) If $N \leq M$, the number of indecomposable direct factors of N is greater or equal than that of M.

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Degenerations of conjugacy classes and the geometry of closures of conjugacy classes are studied in [H1], [KP1] and [PK]. In development of an idea of Dixmier the notion of a sheet is introduced in [BK]. The description of the sheets in $M_n(\mathbb{C})$, their geometry and their parametrization can be found in [K] and [P1], [P2].

2. Representations of $\mathbb{C}\{X,Y\}$

2.1 Consider the non-commutative polynomial ring $R := \mathbb{C}\{X,Y\}$ in two variables X and Y. It is well known that the classification of R-modules is a hopeless problem. Nevertheless we may try to study R-modules in a more geometric way as indicated in the first section.

Clearly an R-module M is a vectorspace V together with a pair X_M, Y_M of endomorphisms of V. Hence we may identify the set $\mathrm{mod}_{R,V}$ of R-module structures on the finite dimensional vectorspace V with $\mathrm{End}(V) \times \mathrm{End}(V)$:

$$\mathrm{mod}_{R,V} \cong \mathrm{End}(V) \times \mathrm{End}(V) \ , \ M \mapsto (X_M, Y_M) \ .$$

In case $V = \mathbb{C}^n$ we simply write $\mathrm{mod}_{R,n}$.

Again the isomorphism classes of n-dimensional R-modules are canonically identified with the orbits of $GL_n(\mathbb{C})$ in $\mathrm{mod}_{R,n}$ under the obvious action (i.e. transport of structure) which corresponds to simultaneous conjugation of GL_n on $M_n(\mathbb{C}) \times M_n(\mathbb{C})$.

For small n there is some chance to obtain a complete description/classification of the orbits, but in general this is an impossible task. (Proof: try it!)

2.2 For $n = 2$ we consider the following map (given by invariant functions):

$$\pi : \mathrm{mod}_{R,2} \to \mathbb{C}^5$$

$$\pi(A,B) := (\mathrm{tr}\,A, \mathrm{tr}\,B, \mathrm{tr}\,AB, \det A, \det B) \ .$$

It is not hard to see that π is surjective. In fact π is an algebraic quotient in the sense that any polynomial map $\mu : \mathrm{mod}_{R,2} \to \mathbb{C}^m$ which is constant on the isomorphism classes factors through π (cf. II.5).

The zero fibre ("nullfibre") $\pi^{-1}(0)$ consists in the origine 0 and a one-parameter family of 2-dimensional orbits C_λ, $\lambda \in \mathbb{P}^1(\mathbb{C})$, corresponding to the modules

$$M_\lambda := \left(\begin{pmatrix} 0 & 0 \\ \lambda' & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \lambda'' & 0 \end{pmatrix} \right) \ , \ \lambda = (\lambda', \lambda'') \in \mathbb{P}^1(\mathbb{C}) \ .$$

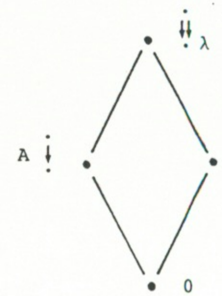
It may be represented by the following picture:

$M_\lambda \ , \ \lambda \in \mathbb{P}^1$

2.3 There is another way to describe the zero fibre. We symbolize the modules M_λ by

$A \downdownarrows B$	$A \downarrow$	$B \downarrow$
$\lambda \neq 0,\infty$	$\lambda = 0$	$\lambda = \infty$

and obtain the following picture:



Here the dot on top collects the orbits C_λ for $\lambda \neq 0,\infty$ i.e. the modules $M = (A,B) \in \pi^{-1}(0)$ with $A \neq 0 \neq B$, and the lines indicate the behavior of the closure of the corresponding orbit or family of orbits (as in the example 1.9). We also remark that any module M_λ degenerates into the trivial module, i.e. $\overline{C}_\lambda \ni 0$ (cf. 1.7 and II. 3.5).

2.4 A module $M = (A,B) \in \mathrm{mod}_{R,2}$ is simple if and only if A and B generate the algebra $M_2(\mathbb{C})$. This defines an open set $\mathrm{mod}_{R,2}^{\mathrm{simple}}$ of $\mathrm{mod}_{R,2}$. In fact the non-simple modules $M = (A,B)$ are defined by

$$(\operatorname{tr} AB)^2 - (\operatorname{tr} A)(\operatorname{tr} B)(\operatorname{tr} AB) + (\operatorname{tr} A)(\det B) + (\operatorname{tr} B)(\det A) - 4(\det A)(\det B) = 0$$

(For a proof remark that the non-simple modules are those which can be represented by pairs of upper triangular matrices. It follows that $\operatorname{tr} AB$ is either equal to $\alpha_1\beta_1 + \alpha_2\beta_2$ or to $\alpha_1\beta_2 + \alpha_2\beta_1$, where α_i resp. β_i are the eigenvalues of A resp. B. Now the expression $(\operatorname{tr} AB - \alpha_1\beta_1 - \alpha_2\beta_2)(\operatorname{tr} AB - \alpha_1\beta_2 - \alpha_2\beta_1)$ is easily seen to be equal to the left hand side of the equation above.)

Furthermore a simple module M is completely determined (up to isomorphism) by its invariants $\pi(M) \in \mathbb{C}^5$; its orbit C_M is closed. It follows that $U := \pi(\mathrm{mod}_{R,2}^{\mathrm{simple}})$ is an open set in $\mathbb{C}^5$, namely the complement of the hypersurface $Y \subset \mathbb{C}^5$ defined by the equation

$$x_3^2 - x_1x_2x_3 + x_1^2x_5 + x_2^2x_4 - 4x_4x_5 = 0 ,$$

and we have a canonical isomorphism

$$\mathrm{mod}_{R,2}^{\mathrm{simple}}/GL_2 \xrightarrow{\sim} U .$$

More precisely $\pi : \mathrm{mod}_{R,2}^{\mathrm{simple}} \to U$ is a (locally trivial) fibration whose fibres are orbits isomorphic to $PGL_2 = GL_2/\mathbb{C}^*$.

2.5 Up to now we have only seen two types of fibres of the map $\pi : \mathrm{mod}_{R,2} \to \mathbb{C}^5$, the fibre over a point of U (generic fibre), which is a single orbit isomorphic to PGL_2, and the nullfibre $\pi^{-1}(0)$ which contains a one-parameter family of orbits. This second type occurs also over the surface

$$F = \{(2\alpha, 2\beta, 2\alpha\beta, \alpha^2, \beta^2) \mid \alpha, \beta \in \mathbb{C}\} \subset Y$$

which is the image under π of the pairs of scalar matrices:

$$\pi : \mathbb{C}E \times \mathbb{C}E \xrightarrow{\sim} F .$$

Over the remaining part $Y - F$ the fibres have two components, each one containing a dense orbit of dimension 3.

(For a proof use the decomposition $M_2 = \mathbb{C}E \oplus M_2'$, $M_2' := \{A \in M_2 \mid \operatorname{tr} A = 0\}$, and replace π by the map

$$\pi' : M_2' \times M_2' \to \mathbb{C}^3 , \quad (A,B) \mapsto (\operatorname{tr} AB, \det A, \det B) .$$

which has the same fibre types. Furthermore we have an action of GL_2 on $M_2' \times M_2'$:

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (A,B) \mapsto (\alpha A + \gamma B, \beta A + \delta B) ,$$

which commutes with conjugation and induces an action on $\mathbb{C}^3$ with three orbits corresponding to the three fibre types.)

2.6 The three fibre types have the following module theoretic interpretation. The generic fibre represents the simple modules, the dense orbits in the two components of the fibres over $Y - F$ consists in indecomposable modules with two non-isomorphic simple factors and the family of 2-dimensional orbits in the fibres over F corresponds to indecomposable modules with isomorphic

1-dimensional simple factors.

2.7 Some of the statements above are of general nature. In particular the simple modules $\mathrm{mod}_{R,n}^{\mathrm{simple}}$ always form an open dense set consisting in closed orbits. The orbit space

$$U := \mathrm{mod}_{R,n}^{\mathrm{simple}}/GL_n$$

is a smooth algebraic manifold and the projection $\mathrm{mod}_{R,n}^{\mathrm{simple}} \to U$ is a locally trivial fibration with fibres $\cong PGL_n$ (cf. [P1], or more generally [L]). For the invariant theory of $\mathrm{mod}_{R,n}$ we refer the reader to [P2].

2.8 . On the other hand the modules which degenerate into the trivial module, i.e. those $M \in \mathrm{mod}_{R,n}$ with $0 \in \overline{C_M}$ (cf. II.3.5), form a closed set $\mathrm{mod}_{R,n}^{0}$ of $\mathrm{mod}_{R,n}$, again called the nullfibre. These modules can be represented by pairs of nilpotent upper triangular matrices. (This follows from Hilbert's Criterion, see II.4.4.)

Putting

$$N := \{\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \in M_n\} \quad \text{and} \quad B := \{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in GL_n\}$$

we obtain the following diagram:

$$\begin{tikzcd} GL_n \times^B N^2 \arrow[r, two heads, "\mu"] \arrow[d, "p"] & \mathrm{mod}_{R,n}^{0} \\ GL_n/B & \end{tikzcd}$$

Here $GL_n \times^B N^2$ is the orbit space of $GL_n \times N^2$ under the free action of B given by $b(g,(A,B)) = (gb^{-1},(bAb^{-1},bBb^{-1}))$, p is the projection onto the first factor and μ is the obvious map

$$(g,(A,B)) \longmapsto (gAg^{-1},gBg^{-1}) \quad .$$

It is easy to see that $GL_n \times^B N^2$ is a vector bundle over the flag variety GL_n/B and that μ is birational (i.e. an isomorphism between dense open subsets) and proper (i.e. $\mu^{-1}(\mathbb{C}\text{-compact}) = \mathbb{C}\text{-compact}$) ; such a map is sometimes called a desingularisation. As a consequence we have that $\mathrm{mod}_{R,n}^{0}$ is irreducible of dimension $3\binom{n}{2} = \frac{3n(n-1)}{2}$.

Remark: $\mathrm{mod}_{R,n}^{0}$ contains an interesting closed subset given by the modules $M = (A,B)$ with $AB = BA = 0$. These modules have been classified by Gelfand and Ponomarev [GP] ; it should be an interesting task to determine the degeneration properties of these modules, in particular the number of components and the generic structures.

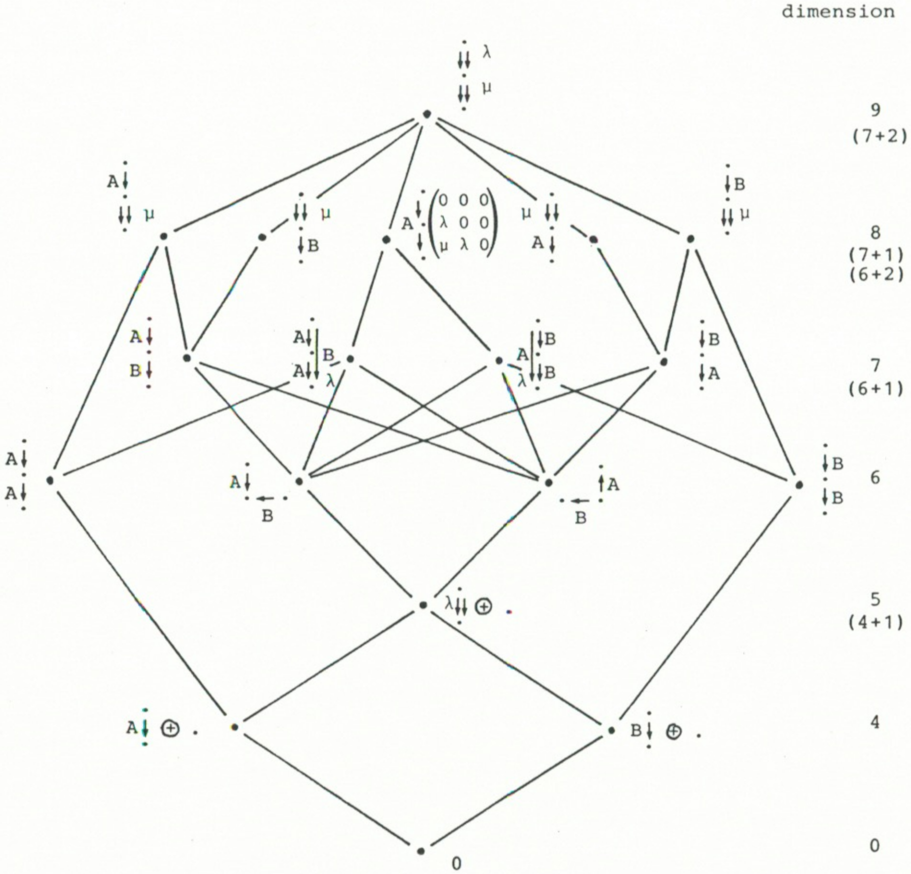
Another closed subset of $\mathrm{mod}_{R,n}$ is formed by the modules $M = (A,B)$ with $AB = BA$, i.e. the modules over $\mathbb{C}[X,Y]$:

$$\mathrm{mod}_{\mathbb{C}[X,Y],n} \subset \mathrm{mod}_{R,n}$$

Not much is known about this "commuting variety" in general except that it is irreducible (Gerstenhaber, cf. [R]).

2.9 To finish this section we give the picture of $\mathrm{mod}_{R,3}^{0}$ using similar notations as in 2.3.

We see that the "commutative" part coincides with the modules of Gelfand-Ponomarev and has two irreducible components.



<u>Remark</u>: The picture contains an interesting degeneration, namely $\begin{smallmatrix}A\downarrow\\ B\downarrow\end{smallmatrix} > A\downarrow\overset{\cdot}{\underset{B}{\leftarrow}}\cdot$, where both modules are <u>indecomposable</u>. (To see this degeneration, take the pairs $\left(\begin{pmatrix}0&0&1\\ &0&\varepsilon\\ &&0\end{pmatrix},\begin{pmatrix}0&1&0\\ &0&0\\ &&0\end{pmatrix}\right)$ for $\varepsilon \to 0$.)

<u>Summary:</u>

The classification of modules over $R = \mathbb{C}\{X,Y\}$ is equivalent to the classification of pairs of matrices under simultaneous conjugation and is known to be a hopeless problem. From a more geometric point of view the "module variety" $\mathrm{mod}_{R,n}$ of R-modules with (fixed) underlying vectorspace $\mathbb{C}^n$ seems to be the right object. Here the simple modules form an open dense subset $\mathrm{mod}_{R,n}^{\mathrm{simple}}$ consisting in closed orbits, and the isomorphism classes of simple modules form the "orbit space" $\mathrm{mod}_{R,n}^{\mathrm{simple}}/GL_n$ which has the structure of a smooth algebraic variety. On the other hand the "null-modules", i.e. those which degenerate into the trivial module, form an interesting irreducible closed subset $\mathrm{mod}_{R,n}^{0}$. Not much is known neither about the orbit space $\mathrm{mod}_{R,n}^{\mathrm{simple}}/GL_n$ nor about the nullfibre $\mathrm{mod}_{R,n}^{0}$, except for small n where a complete description of the module variety and its orbits can be obtained.

<u>Problems:</u>

1) Problem 1 of the first section has a negative answer for $R = \mathbb{C}\{X,Y\}$ by remark 2.9. What about problem 2 and 3? Is a degeneration of a decomposable module always decomposable?

2) Describe the sheets in $\mathrm{mod}_{R,2}$ and their parametrization. Give a description of $\mathrm{mod}_{R,3}$. Determine the nullmodules in $\mathrm{mod}_{R,4}$ and their degenerations.

3) Describe the subvariety of "Gelfand-Ponomarev-modules" (cf. remark 2.8), the number of irreducible components and the generic structures (i.e. the type of modules which form the dense families in the components).

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3. Completely Reachable Pairs of Matrices

The problem we are going to consider in this section arises from system and control theory. For a more detailed investigation of the whole subject and further references we refer the reader to the Lecture notes [T] of A. Tannenbaum. (See also the survey article [H] of M. Hazewinkel.)

3.1 Consider a linear time-invariant dynamical system Σ given by the differential equations

$$\Sigma \ : \ \begin{array}{l} \dot{x} = Bx + Au \\ y = Cx \end{array}$$

where u,x,y are vector variables, $u(t) \in \mathbb{C}^m$, $x(t) \in \mathbb{C}^n$, $y(t) \in \mathbb{C}^p$, and A,B,C real or complex matrices of size $n \times m$, $n \times n$, $p \times n$ respectively;

$$\Sigma \ : \ \xrightarrow{u} \boxed{x} \xrightarrow{y}$$

$u(t)$ is the input or control, $y(t)$ the output and $x(t)$ the state vector at time t.

Clearly the system Σ is determined by the tripel A,B,C of matrices; we shortly write $\Sigma = (A,B,C)$.

3.2 If the system $\Sigma = (A,B,C)$ is at the time t_0 in the state x_0 we obtain from elementary theory of linear differential equations the following solution:

$$y(t) = C e^{(t-t_0)B} x_0 + \int_{t_0}^{t} C e^{(t-\tau)B} A u(\tau) d\tau$$

We remark that the main part of the solution depends only on the matrices CB^iA, $i = 0,1,\ldots$. This follows also directly from the differential equations and leads to the following definition.

Definition: Given a system $\Sigma = (A,B,C)$ and $g \in GL_n$ we put

$$^g\Sigma := (gA, gBg^{-1}, Cg^{-1}) ;$$

two systems Σ' and Σ are called equivalent if $\Sigma' = {}^g\Sigma$ for some $g \in GL_n$.

Clearly equivalent systems define the same input-output operator $u \mapsto y$; a convers of this will follow under suitable assumptions (see 3.4 below).

3.3 There is the important notion of reachability which comes from the question whether a system Σ reaches any state in finite time with a suitable input starting from the zero state. Another notion is the observability; it's related to the problem whether any state can be detected from the outputs of the system. We give the definitions in terms of the matrices A,B,C .

Definition: A system $\Sigma = (A,B,C)$ is called completely reachable if the matrix

$$R(A,B) := (A, BA, B^2A, \dots, B^nA)$$

of size $n \times (n+1)m$ is of maximal rank (i.e. of rank n). It is called completely observable if the matrix

$$Q(B,C) := \begin{pmatrix} C \\ CB \\ CB^2 \\ \vdots \\ CB^n \end{pmatrix}$$

of size $(n+1)p \times n$ is of maximal rank.

We shortly write cr and co respectively.

3.4 Now the first result states that a cr and co system is determined up to equivalence by the input-output operators.

Proposition: Consider two systems Σ and Σ' which define the same input-output operator. If Σ is cr and co then Σ' is equivalent to Σ .

In terms of matrices this means the following. Given two tripels (A,B,C) and (A',B',C') of matrices of size $n \times m$, $n \times n$, $p \times n$ respectively with $CB^iA = C'B'^iA'$ for all i and $\operatorname{rk} R(A,B) = \operatorname{rk} Q(B,C) = n$, then there is $g \in GL_n$ such that $A' = gA$, $B' = gBg^{-1}$ and $C' = Cg^{-1}$.

For the proof we need the following result

3.5 Lemma: Put $R_k(A,B) = (A, BA, \dots, B^kA)$ and assume $\operatorname{rk} R_n(A,B) = n$.

a) We have $\operatorname{rk} R_k(A,B) = n$ for $k \geq n-1$.

b) If $R_n(A,B) = R_n(A',B')$ then $A = A'$ and $B = B'$.

(By Cayley-Hamilton we have $\sum_{i=0}^{\infty} B^i(\operatorname{Im} A) = \sum_{i=0}^{n-1} B^i(\operatorname{Im} A)$. This implies

a) since $\operatorname{rk} R_k(A,B) = \dim \sum_{i=0}^{k} B^i(\operatorname{Im} A)$. Now $R_n(A,B) = R_n(A',B')$ means that $A = A'$ and $B^iA = B'^iA'$ for $i = 1,\dots,n$. This implies by induction $B|_{B^i(\operatorname{Im} A)} = B'|_{B^i(\operatorname{Im} A)}$ for $i = 0,1,\dots,n-1$.

Now the claim follows since $\sum_{i=0}^{n-1} B^i(\operatorname{Im} A) = \mathbb{C}^n$ by a).)

Proof of proposition 3.4 : We have

$$Q(B,C)\ R(A,B) = \begin{pmatrix} CA & CBA & CB^2A & \cdots \\ CBA & CB^2A & CB^3A & \\ CB^2A & CB^3A & CB^4A & \\ \vdots & & & \ddots \end{pmatrix}$$

hence by assumption

$$Q(B,C)\cdot R(A,B) = Q(B',C')\cdot R(A',B') .$$

Since $Q(B,C)$ and $R(A,B)$ are of maximal rank there exists $g \in GL_n$ with

$$Q(B',C') = Q(B,C)g^{-1} = Q(gBg^{-1},Cg^{-1})$$

$$R(A',B') = gR(A,B) = R(gA,gBg^{-1})$$

Now the Lemma implies the claim.

3.6 The systems $\Sigma = (A,B,C)$ with fixed dimensions of input, output and state space form a vector space

$$L = L_{m,n,p} := M_{n,m}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{p,n}(\mathbb{C})$$

or in coordinate-free notation

$$L = L(U,V,W) := \mathrm{Hom}(U,V) \times \mathrm{End}(V) \times \mathrm{Hom}(V,W) .$$

Symbolically we may write:

$$\bullet \longrightarrow \overset{\circlearrowleft}{\bullet} \longrightarrow \bullet$$

The group GL_n (or $GL(V)$) operates linearly on L in the usual way :

$$g : (A,B,C) \longmapsto g(A,B,C) = (gA,gBg^{-1},Cg^{-1}) .$$

The cr and / or co systems form <u>open subsets</u> L^{cr}, L^{co}, $L^{cr,co} = L^{cr} \cap L^{co}$ which are stable under GL_n. The proposition 3.4 states that the equivalence classes of cr and co systems are given by the orbit space

$$L^{cr,co}_{m,n,p} / GL_n \quad ;$$

we may ask for a description of this space and try to investigate its structure.

3.7 In order to simplify the problem we concentrate on the "<u>input part</u>" of our system, i.e. we consider the space

$$L(U,V) := \mathrm{Hom}(U,V) \times \mathrm{End}(V)$$

with the linear $GL(U)$-action $g(A,B) := (gA,gBg^{-1})$; symbolically

$$\bullet \longrightarrow \overset{\circlearrowleft}{\bullet}$$

For the general problem we refer to the literature cited above.

First we have the following characterisation of completely reachable elements.

<u>Proposition:</u> <u>An element</u> $\alpha = (A,B) \in L(U,V)$ <u>is completely reachable if and only if the stabilizer</u> $\mathrm{Stab}_{GL(U)}\alpha := \{g \in GL(U) \mid g\alpha = \alpha\}$ <u>is trivial</u>.

(One implication is easy : If α is cr then $V = \sum_i B^iA(U)$. If g stabilizes α , then $g|_{B^iA(U)} = \mathrm{Id}_{B^iA(U)}$, hence $g = \mathrm{Id}$. For the other implication see [T] IV. 1.4)

<u>Remark:</u> The stabilizer of any element $\alpha \in L(U,V)$ is <u>connected</u>. In fact it is easy to see that $\mathrm{Stab}_{GL(U)}\alpha$ is isomorphic to an open set of the endomorphism algebra $\mathrm{End}\,\alpha := \{X \in \mathrm{End}(V) \mid XB = AB, XA = 0\}$ via the map $g \mapsto g - \mathrm{Id}$.

This shows that $L(U,V)^{cr}$ is the <u>open sheet</u> in $L(U,V)$, i.e. <u>the union of orbits of maximal dimension</u>.

3.8 We are going to give now a first description of the orbit space $L^{cr}/GL(V)$. Consider the map

$$\Psi : L(U,V) \to \mathrm{Hom}(U^{n+1},V) \ , \quad (A,B) \mapsto (A, BA, \dots, B^n A) \ ,$$

where $n = \dim V$. By definition α is cr if and only if $\Psi(\alpha) : U^{n+1} \to V$ is surjective. Furthermore by Lemma 3.5b $\Psi|_{L^{cr}}$ is injective.

Using again the lemma one shows that Ψ is of maximal rank on L^{cr}, i.e. the differential $(d\Psi)_\alpha$ is injective for all $\alpha \in L^{cr}$. This implies that Ψ induces an <u>isomorphism</u> Ψ' <u>of</u> L^{cr} <u>with a locally closed subset of</u> $\mathrm{Sur}(U^{n+1},V)$ (= the surjective linear maps $U^{n+1} \to V$).

$$\Psi' : L^{cr} \xrightarrow{\sim} \Psi(L^{cr}) \subset \mathrm{Sur}(U^{n+1},V) \ .$$

With respect to the obvious action of $GL(V)$ on $\mathrm{Hom}(U^{n+1},V)$ by "left multiplication" $\lambda \mapsto g \cdot \lambda$, the maps Ψ and Ψ' are equivariant. But clearly two surjective maps $\lambda, \mu : U^{n+1} \to V$ are equivalent under this action if and only if $\mathrm{Ker}\,\lambda = \mathrm{Ker}\,\mu$, hence the orbit space $\mathrm{Sur}(U^{n+1},V)/GL(V)$ is canonically identified with the <u>Grassmann-variety</u> of subspaces of U^{n+1} of co-dimension n, denoted by $Gr^n(U^{n+1})$. Thus our first structure theorem states:

<u>Proposition</u> (Kalman): <u>The orbit space</u> $L(U,V)^{cr}/GL(V)$ <u>of equivalence classes of completely reachable pairs of matrices is a locally closed submanifold of dimension</u> $\dim U \cdot \dim V$ <u>of the Grassmann-variety</u> $Gr^n(U^{n+1})$.

<u>Remark:</u> The classification of all equivalence classes of pairs (A,B) as above is a hopeless problem. It is therefore quite astonishing that one obtains such a nice geometric description of the orbit space of the open sheet of completely reachable pairs.

<u>3.9</u> We can even obtain more precise information on the structure of the orbit space $L^{cr}/GL(V)$. For this we consider the <u>invariant functions</u> σ_i on $\mathrm{End}(V)$ introduced in 1.5, $\sigma_i(B) :=$ the i^{th} elementary symmetric function of the eigenvalues of B, and define the map

$$\pi : L(U,V) \to \mathbb{C}^n \quad \text{by} \quad (A,B) \mapsto (\sigma_1(B), \dots, \sigma_n(B)) \ .$$

Since π is obviously constant on the equivalence classes, it induces a map

$$\bar{\pi} : L(U,V)^{cr}/GL(V) \to \mathbb{C}^n \ .$$

The following proposition collects the main properties of this map. (For proofs see [T] IV.4.)

<u>Proposition:</u> <u>The map</u> $\bar{\pi}$ <u>is surjective</u>, <u>flat and projective</u>, i.e. <u>the fibres are projective varieties all of the same dimension</u>, <u>namely</u> $n(m-1)$, <u>where</u> $n := \dim V$ <u>and</u> $m := \dim U$. <u>The generic fibre is isomorphic to</u> $(\mathbb{P}^{m-1}(\mathbb{C}))^n$. <u>For</u> $m = 1$ <u>the map</u> $\bar{\pi}$ <u>is an isomorphism</u>.

<u>3.10</u> <u>Remark:</u> The proposition gives a partial explanation of a result due to Hazewinkel which states the <u>non-existence of global canonical forms</u>, i.e. there is no family $\Sigma_t = (A_t, B_t)$ of systems depending continuously on a parameter t and containing for every equivalence class of completely reachable systems exactly one member. In more geometric terms this means that the quotient map $L^{cr} \to L^{cr}/GL(V)$ has no continuous section (except for $m = 1$ where it was known before). Now the proposition above implies that there is <u>no algebraic canonical form</u> i.e. no algebraic section except for $m = 1$, since an affine variety cannot contain a projective variety of positive dimension.

Summary:

Some questions in control and system theory coming for example from realization, base changes in state space or existence of canonical forms of linear dynamical systems can be formulated as "matrix problems" with respect to the linear action of GL_n on pairs (A,B) or triples (A,B,C) of matrices (given by $g(A,B,C) = (gA, gBg^{-1}, Cg^{-1}))$. In particular it turns out that the open sheet L' is formed by the completely reachable pairs, a notion coming from system theory, and that the orbit space L'/GL_n has a nice description via Grassmannians and invariant functions. As an application we obtain the non-existence of (algebraic) canonical forms.

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Chapter II TRANSFORMATION GROUPS AND ALGEBRAIC QUOTIENTS

In this chapter we develop the foundations of algebraic geometry, transformation groups and invariant theory. We have tried to introduce these subjects by giving examples strongly related to representation theory, like the module variety $\mathrm{mod}_{A,m}$ of m-dimensional A-modules or the variety alg_n of n-dimensional algebras. Because lack of time and space it was not possible to present complete proofs; in some cases we give an outline and indicate the main ideas, but in general we have to refer to the literature. This is easy in case of algebraic geometry (we recommend the excellent textbooks of R. Hartshorne, D. Mumford and I.R. Shafarevich), but a little problem for transformation groups and invariant theory (the reader may consult [F], [Kr], [Mu], [Sp]). It was not always possible to avoid technical difficulties; we have tried to concentrate on the main points which are necessary to get a better feeling for the examples in the first chapter and to understand the results in the last chapter.

1. Affine varieties

1.1 Let V be a finite dimensional vectorspace over $\mathbb{C}$ and denote by $\mathcal{O}(V)$ the $\mathbb{C}$-algebra of polynomial functions on V. These functions are also called regular functions on V. Since polynomials separate points, every basis $v_1, v_2, \ldots, v_n$ of V induces an isomorphism $\mathcal{O}(V) \xrightarrow{\sim} \mathbb{C}[X_1, X_2, \ldots, X_n]$, where $X_1, \ldots, X_n$ is the dual basis to $v_1, \ldots, v_n$.

For any subset $S \subset \mathcal{O}(V)$ we define the zero set of S ("Nullstellengebilde von S") by

$$\underline{V}(S) := \{x \in V \mid f(x) = 0 \text{ for all } f \in S\}.$$

Clearly we have

$$\underline{V}(S) = \underline{V}(\underline{a}) = \underline{V}(\sqrt{\underline{a}})$$

where $\underline{a} := (S)$ is the ideal generated by S and $\sqrt{\underline{a}} := \{f \in \mathcal{O}(V) \mid f^r \in \underline{a} \text{ for some } r \in \mathbb{N}\}$ its radical. Furthermore

$$\underline{V}(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} \underline{V}(S_i) \quad \text{and} \quad \underline{V}(S \cdot T) = \underline{V}(S) \cup \underline{V}(T)$$

This shows that the zero sets are the closed sets in some topology on V, the so called Zariski-topology.

1.2 Remark: In the sequel the expressions closed, open, dense, continuous,... are always used with respect to the Zariski-topology. In addition every subset of V will be provided with the induced topology of the Zariski-topology. If we want to consider the usual topology on V and its subsets we write $\mathbb{C}$-closed, $\mathbb{C}$-open, $\mathbb{C}$-continuous,... Clearly the Zariski-topology is weaker than the $\mathbb{C}$-topology; points are closed in the Zariski-topology, but the Zariski-topology is not Hausdorff.

1.3 Nullstellensatz (Hilbert): If $\underline{a} \subset \mathcal{O}(V)$ is an ideal then

$$\{f \in \mathcal{O}(V) \mid f \equiv 0 \text{ on } \underline{V}(\underline{a})\} = \sqrt{\underline{a}}.$$

Given a closed subset $Z \subset V$ we define the regular functions on Z by

$$\mathcal{O}(Z) := \{f|_Z \text{ with } f \in \mathcal{O}(V)\}.$$

$\mathcal{O}(Z)$ is called the coordinate ring of Z. Clearly $\mathcal{O}(Z) \xrightarrow{\sim} \mathcal{O}(V)/\underline{a}$ with $\underline{a} := \{f \in \mathcal{O}(V) \mid f \equiv 0 \text{ on } Z\}$.

Definition: A pair $(Y, \mathcal{O}(Y))$ of a set Y and a $\mathbb{C}$-algebra $\mathcal{O}(Y)$ of $\mathbb{C}$-valued functions on Y is an affine variety if it is isomorphic to a pair $(Z, \mathcal{O}(Z))$, Z a closed subset of a vectorspace.

As a main consequence of the Nullstellensatz we have that an affine variety Y is completely determined by the coordinate ring $\mathcal{O}(Y)$ and that any finitely generated commutative $\mathbb{C}$-algebra R without nilpotent elements $\neq 0$ occurs in this way.

Another consequence is the following: If $f \in \mathcal{O}(Z)$ and $f(z) \neq 0$ for all $z \in Z$ then $\frac{1}{f} \in \mathcal{O}(Z)$.

Remark: For any ideal $\underline{a} \subset \mathcal{O}(Z)$ we put

$$\underline{V}_Z(\underline{a}) = \underline{V}(\underline{a}) := \{z \in Z \mid f(z) = 0 \text{ for all } f \in \underline{a}\}.$$

These sets form the closed sets of the Zariski-topology on Z. If Z is a closed subset of some vectorspace V the Zariski-topology coincides with the induced topology from V. A similar argument shows that Z has also a natural $\mathbb{C}$-topology.

1.4 Example: Let A be a finitely generated (associative) $\mathbb{C}$-algebra and U a finite dimensional vectorspace. Define

$$\mathrm{mod}_{A,U} := \{A\text{-module-structures on } U\}$$
$$\cong \{\rho: A\times U\to U \mid \rho \text{ defines an A-module-structure on } U\}$$
$$\cong \{\rho: A\to \mathrm{End}(U) \mid \rho \text{ a } \mathbb{C}\text{-algebra-homomorphism}\}$$

If A is presented in the form

$$A = \mathbb{C}[X_1,\dots,X_m]/(P_i \mid i\in I)$$

we have a canonical identification

$$\mathrm{mod}_{A,U} \cong \{(S_1,\dots,S_m)\in \mathrm{End}(U)^m \mid P_i(S_j)=0 \text{ for all } i\in I\} .$$

Clearly the conditions $P_i(S_j)=0$ are polynomial equations in the coefficients of the matrices S_j, hence $\mathrm{mod}_{A,U}$ is identified with a closed subset of $\mathrm{End}(U)^m$. It is easy to see that this structure of an affine variety on $\mathrm{mod}_{A,U}$ is independent on the chosen presentation of A by generators and relations.

1.5 Let $h\in\mathcal{O}(Z)$ be a regular function $\neq 0$. Define

$$Z_h := Z-\underline{V}(h) = \{z\in Z \mid h(z)\neq 0\}$$

and consider the algebra $\mathcal{O}(Z_h)$ of functions on Z_h generated by $\frac{1}{h}$ and the restrictions $f|_{Z_h}$, $f\in\mathcal{O}(Z)$.

<u>Lemma</u>: $(Z_h,\mathcal{O}(Z_h))$ <u>is an affine variety and</u>

$$\mathcal{O}(Z_h) \xrightarrow{\sim} \mathcal{O}(Z)[t]/(th-1).$$

The open sets Z_h are called <u>special open sets</u> of Z; they form a basis of the topology.

<u>Example:</u> $GL_n = (M_n)_{\det}\subset M_n$ or more generally $GL(V) = (\mathrm{End}\,V)_{\det}$ is an affine variety with coordinate ring

$$\mathcal{O}(GL_n) = \mathbb{C}[X_{ij},\tfrac{1}{\det}].$$

<u>Definition:</u> A <u>linear algebraic group</u> G is a closed subgroup of

some GL_n.

E.g. the classical groups $SL_n\subset GL_n$, $O_n\subset GL_n$, $SO_n = O_n\cap SL_n\subset GL_n$, $Sp_n\subset GL_n$, and all finite groups $\subset GL_n$.

1.6 Given two affine varieties Y and Z consider the cartesian product $Y\times Z$ and the algebra $\mathcal{O}(Y\times Z)$ of functions on $Y\times Z$ generated by $f\cdot g$, $f\in\mathcal{O}(Y)$ and $g\in\mathcal{O}(Z)$, where $f\cdot g(y,z) := f(y)\cdot g(z)$.

<u>Lemma</u>: $(Y\times Z,\mathcal{O}(Y\times Z))$ <u>is an affine variety and</u>

$$\mathcal{O}(Y\times Z) \xrightarrow{\sim} \mathcal{O}(Y)\otimes_{\mathbb{C}}\mathcal{O}(Z).$$

<u>Example:</u> We have $\mathcal{O}(Z\times\mathbb{C}) \cong \mathcal{O}(Z)[t]$; consider the closed subset $Y := \underline{V}(th-1) \subset Z\times\mathbb{C}$ for some $h\in\mathcal{O}(Z)$, $h\neq 0$. Then the projection $Z\times\mathbb{C}\to Z$ identifies Y with Z_h and $\mathcal{O}(Y) = \mathcal{O}(Z)[t]/(th-1)$ with $\mathcal{O}(Z_h)$. (cf. 1.5)

1.7 <u>Example</u>: Let W be a finite dimensional vectorspace and define

$$\mathrm{alg}_W := \{\text{associative unitary } \mathbb{C}\text{-algebra-structures on } W\} .$$

Clearly alg_W may be considered as a subset of the vectorspace

$$\mathrm{bil}_W := \{\phi: W\times W\to W \text{ bilinear}\} .$$

Furthermore the <u>associative</u> algebra-structures form a closed subset $\mathrm{ass}_W\subset \mathrm{bil}_W$. Using the fact that an associative finite dimensional algebra A has a unit element if and only if there are elements $a,b\in A$ such that $aA = A = Ab$, one easily shows that alg_W is <u>open</u> in ass_W.

<u>Lemma:</u> alg_W <u>is an affine variety</u>.

The affine structure is obtained in the following way:

Take $\mathrm{ass}_W \times W$ and consider the closed subset

$$Z := \{(A,w) \mid w \text{ is a unit element of } A\} .$$

Then the projection $\mathrm{ass}_W \times W \to \mathrm{ass}_W$ identifies the affine variety Z with alg_W .

<u>1.8</u> <u>Definition:</u> An affine variety Z is <u>reducible</u>, if there is a decomposition $Z = Z_1 \cup Z_2$ with proper closed subsets $Z_i \subset Z$. Otherwise Z is <u>irreducible</u>.

<u>Proposition:</u> a) <u>Z is irreducible</u> $\Leftrightarrow$ <u>$\mathcal{O}(Z)$ is an integral domain</u> $\Leftrightarrow$ <u>Every non empty open subset of Z is dense</u>.

b) <u>There is a finite decomposition</u> $Z = \bigcup_{i=1}^{s} Z_i$ <u>with irreducible closed subsets</u> $Z_i \subset Z$. <u>If the decomposition is irredundant then the</u> Z_i <u>are the maximal irreducible subsets of</u> Z .

The maximal irreducible subsets Z_i are called the <u>irreducible components</u> of Z .

<u>Example:</u> If G is a linear algebraic group the irreducible components are the connected components. (Multiplication by an element $g \in G$ is a topological map, hence permutes the irreducible components of G . If $h \in G$ belongs to two components, also gh belongs to two components, and so every element of G does, which is a contradiction.)

<u>2.</u> <u>Morphisms</u>

<u>2.1</u> <u>Definition:</u> A map $\mu : Y \to Z$ between affine varieties is a <u>morphism</u> (or a <u>regular map</u> or an <u>algebraic map</u>) if $\mu^* f := f \circ \mu \in \mathcal{O}(Y)$ for all $f \in \mathcal{O}(Z)$. It is an <u>isomorphism</u> if it is bijective and its inverse is also a morphism.

We see that μ defines a $\mathbb{C}$-algebra-homomorphism

$$\mu^* : \mathcal{O}(Z) \to \mathcal{O}(Y) ;$$

as usual we have $(\sigma \circ \mu)^* = \mu^* \circ \sigma^*$.

<u>Proposition:</u> <u>The map</u>

$$?^* : \mathrm{Mor}(Y,Z) \to \mathrm{Alg}_{\mathbb{C}}(\mathcal{O}(Z),\mathcal{O}(Y))$$

<u>is bijective</u>.

(Here Mor denotes the set of morphisms and $\mathrm{Alg}_{\mathbb{C}}$ the set of $\mathbb{C}$-algebra-homomorphisms.)

<u>2.2</u> We remark that any morphism is <u>continuous</u> and also $\mathbb{C}$-<u>continuous</u>. In fact one easily proves the following result.

<u>Proposition:</u> <u>Let</u> $Y \subset \mathbb{C}^n$ <u>and</u> $Z \subset \mathbb{C}^m$ <u>be closed subsets and</u> $\mu : Y \to Z$ <u>a map</u>. <u>Then</u> μ <u>is a morphism if and only if there are polynomials</u> $\mu_i \in \mathbb{C}[X_1,\dots,X_n]$, $i = 1,\dots,m$, <u>such that</u>

$$\mu(y) = (\mu_1(y),\dots,\mu_m(y)) \quad \underline{\text{for all}} \quad y = (y_1,\dots,y_n) \in Y \subset \mathbb{C}^n .$$

The following proposition describes images and invers images under morphisms.

<u>Proposition:</u> <u>Let</u> $\mu : Y \to Z$ <u>be a morphism</u>.

a) <u>If</u> $Z' = \underline{V}_Z(\underline{a})$ <u>then</u> $\mu^{-1}(Z') = \underline{V}(\mu^*(\underline{a}))$.

b) <u>If</u> $Y' = \underline{V}_Y(\underline{b})$ <u>then</u> $\overline{\mu(Y')} = \underline{V}_Z(\mu^{*-1}(\underline{b}))$.

2.3 Examples: a) The product $Y \times Z$ of two affine varieties has the usual universal property: The two projections pr_Y and pr_Z are morphisms and for any two morphisms $\mu: X \to Y$ and $\rho: X \to Z$ the map $(\mu,\rho): X \to Y \times Z$, $x \mapsto (\mu(x),\rho(x))$, is also a morphism.

b) Notations of 1.4: If $U = U' \oplus U''$ is a direct sum we have a canonical morphism

$$\mathrm{mod}_{A,U'} \times \mathrm{mod}_{A,U''} \to \mathrm{mod}_{A,U}$$

given by $(M',M'') \mapsto M' \oplus M''$.

c) For any linear algebraic group G the multiplication $G \times G \to G$ and the inverse $G \to G$ are morphisms.

This example leads to the following definition.

Definition: An affine variety G with a group structure is an algebraic group if the multiplication $G \times G \to G$ and the inverse $G \to G$ both are morphisms.

Remark: One can show that every algebraic group is isomorphic to a linear algebraic group.

2.4 We consider again the "module-variety" $\mathrm{mod}_{A,U}$ (1.4). Let Z be an affine variety.

Definition: An indexed set $(M_z)_{z \in Z}$ of A-modules $M_z \in \mathrm{mod}_{A,U}$ is called an algebraic family of A-modules if for any $a \in A$ the map $Z \to \mathrm{End}(U)$, $z \mapsto a \cdot \mathrm{Id}_{M_z}$, is a morphism.

Remark: If $\mu: Z \to \mathrm{mod}_{A,U}$ is a morphism then $(\mu(z))_{z \in Z}$ is an algebraic family. Conversely if $(M_z)_{z \in Z}$ is an algebraic family of A-modules $M_z \in \mathrm{mod}_{A,U}$ the map $Z \to \mathrm{mod}_{A,U}$, $z \mapsto M_z$, is a morphism. This shows that $\mathrm{mod}_{A,U}$ is in a certain sense the universal family of A-modules of dimension $\dim U$.

2.5 Definition: Given two A-modules M and N we call N a degeneration of M if there is an algebraic family $(M_z)_{z \in Z}$ of A-modules in $\mathrm{mod}_{A,U}$ such that $M_z \tilde{\to} M$ on an open dense set of Z and $M_{z'} \tilde{\to} N$ for some $z' \in Z$.

We simply write $N \leq M$ in this case.

Remark: We will see in the next section that "$\leq$" defines an ordering on the isomorphism classes of A-modules.

The following lemma shows that any degeneration can be obtained along an irreducible curve i.e. we may assume that Z is irreducible of dimension 1 (cf. 2.6).

Lemma: Any two points on an irreducible affine variety can be connected by an irreducible curve.

2.6 If Z is an irreducible variety we denote by $\mathbb{K}(Z)$ the field of fractions of $\mathcal{O}(Z)$. We call $\mathbb{K}(Z)$ the field of rational functions on Z.

Remark: The elements of $\mathbb{K}(Z)$ may be regarded as "functions defined almost everywhere on Z". In fact if $r \in \mathbb{K}(Z)$, $r = \frac{p}{q}$ with $p,q \in \mathcal{O}(Z)$ then r is a well defined function on the dense open set $Z - \underline{V}(q)$ of Z.

Definition: The transcendence degree of $\mathbb{K}(Z)$ over $\mathbb{C}$ is the dimension of Z :

$$\dim Z := \operatorname{trdeg} K(Z) .$$

If Z is reducible and $Z = \bigcup_i Z_i$ the decomposition into irreducible components we put

$$\dim Z := \operatorname*{Max}_i \dim Z_i ;$$

In addition we define the local dimension in a point $z \in Z$ by

$$\dim_z Z := \operatorname*{Max}_{Z_i \ni z} \dim Z_i .$$

Examples: $\dim \mathbb{C}^n = n$.

$\dim Z = 0 \Leftrightarrow Z$ is a finite set .

A variety of dimension 1 resp. 2 is called a curve resp. a surface.

Lemma: If Z is irreducible and $Y \subset Z$ closed, $Y \neq Z$, then $\dim Y < \dim Z$.

2.7 The following is the main result on dimensions of fibres of a morphism.

Proposition: Let $\mu: Y \to Z$ be a dominant morphism between irreducible affine varieties (i.e. $\overline{\mu(Y)} = Z$) . Then for all $z \in Z$ and every irreducible component C of $\mu^{-1}(z)$ we have

$$\dim C \geq \dim Y - \dim Z$$

with equality on a dense open set of Z .

Remark: A special case of the result above is Krull's "Hauptidealsatz" : Given regular functions $f_1, \dots, f_t$ on a vectorspace V and an irreducible component C of the zero set $\underline{V}(f_1, \dots, f_t)$ (assumed to be non empty) we have

$$\dim C \geq \dim V - t .$$

3. Group actions and orbit spaces

For any algebraic group G we denote by $e \in G$ the unit element.

3.1 Definition: An action of an algebraic group G on an affine variety Z is a morphism $\rho: G \times Z \to Z$ with

(i) $\rho(e,z) = z$ and

(ii) $\rho(g, \rho(h,z)) = \rho(gh,z)$

for all $z \in Z$ and $g,h \in G$.

We shortly write gz for $\rho(g,z)$, and we call Z a G-variety. The conditions (i) and (ii) have the usual meaning: $ez = z$ and $g(hz) = (gh)z$ for all $z \in Z$ and all $g,h \in G$.

3.2 A special case of a group action occurs in the following way.

Definition: A linear representation of an algebraic group G is a regular group homomorphism

$$\rho : G \to GL(V) .$$

A linear representation is the same thing as a linear action of G on a vectorspace V , i.e. an action $\rho: G \times V \to V$ such that $\rho(g,?)$ is a linear automorphism of V for all $g \in G$.

We shortly say that V is a G-module. The notions of simple or semisimple modules or equivalently of irreducible or completely reducible representations are defined in the usual way.

A one dimensional representation $\rho: G \to GL_1 = \mathbb{C}^*$ is called a character of G . The characters form a group $X(G)$, the character group of G .

3.3 We use the following notations:

$Gz := \{gz \mid g \in G\}$ is the orbit of $z \in Z$,

$Z^G := \{z \in Z \mid gz = z \text{ for all } g \in G\}$ is the fixed point set of G in Z,

$\mathrm{Stab}_G z = G_z := \{g \in G \mid gz = z\}$ is the stabilizer of z in G,

$Z' \subset Z$ is G-stable if $gz \in Z'$ for all $z \in Z'$.

A morphism $\mu: Y \to Z$ between G-varieties is G-equivariant or a G-morphism if $\mu(gy) = g\mu(y)$ for all $g \in G$ and $y \in Y$. A linear G-equivariant map between G-modules is a G-homomorphism.

Proposition: a) The fixed point set Z^G is a closed subset of Z, the stabilizer $\mathrm{Stab}_G z$ is a closed subgroup of G.

b) An orbit Gz is open in its closure $\overline{Gz}$. The closure $\overline{Gz}$ contains always a closed orbit.

(For the first part of b) one uses 2.7, the second follows by induction on the dimension.)

3.4 Example (notations 1.4): On the module variety $\mathrm{mod}_{A,U}$ we have a natural action of $GL(U)$ by "transport of structure": If $g \in GL(U)$ and $M \in \mathrm{mod}_{A,U}$ is given by $\rho: A \to \mathrm{End}(U)$ then ${}^gM \in \mathrm{mod}_{A,U}$ is defined by $g\rho: A \to \mathrm{End}(U)$, $a \mapsto g\rho(a)g^{-1}$. This is exactly that module structure for which the linear map $g: M \to {}^gM$ is an A-module homomorphism.

It is easy to see that two modules $M, N \in \mathrm{mod}_{A,U}$ are isomorphic if and only if they belong to the same orbit. In particular the orbit space $\mathrm{mod}_{A,U}/GL(U)$ is canonically identified with the isomorphism classes of n-dimensional A-modules, $n := \dim U$. If M is any n-dimensional A-module we denote by C_M the corresponding orbit in $\mathrm{mod}_{A,U}$ or in $\mathrm{mod}_{A,n}$.

3.5 Proposition: Let M, N be two A-modules of dimension n. Then N is a degeneration of M (2.5) if and only if $C_N \subset \overline{C_M}$.

(Use the definition 2.5 and remark 2.4)

Remark: The proposition shows that the relation "$\leq$" defines an ordering on the isomorphism classes of A-modules (cf. 2.5).

3.6 The next proposition gives a module theoretic interpretation of the stabilizer of a point of $\mathrm{mod}_{A,U}$.

Proposition: For any $M \in \mathrm{mod}_{A,U}$ we have

$$\mathrm{Stab}_{GL(U)}(M) = \mathrm{Aut}_A(M)$$

and this group is connected.

(The connectedness follows from the fact that $\mathrm{Aut}_A(M)$ is an open subset of the vectorspace $\mathrm{End}_A(M)$.)

3.7 In a similar way as above we have an action of $GL(W)$ on alg_W (and also on ass_W and bil_W, cf. 1.7) by "transport of structure": If $A \in \mathrm{alg}_W$ is given by the multiplication $\alpha: W \times W \to W$, the map $g\alpha: W \times W \to W$, $(w, w') \mapsto g(\alpha(g^{-1}w, g^{-1}w'))$, defines a new algebra structure ${}^gA \in \mathrm{alg}_W$, which is again associative and has a unit element.

Again the orbits correspond to the isomorphism classes and the stabilizer of $A \in \mathrm{alg}_W$ is equal to the automorphism group:

$$\mathrm{Stab}_{GL(W)}(A) = \mathrm{Aut}_{alg}(A) \ .$$

We also have the notion of degenerations of algebras with a similar result as proposition 3.5

Proposition: alg_W is connected and contains exactly one closed orbit, namely the orbit of the commutative algebra $A_0 = \mathbb{C} \oplus I$ with $I^2 = 0$.

(If (γ_{ij}^k) are the structure constants of a n-dimensional algebra B with respect to a basis $e_1 = 1, e_2, \ldots, e_n$ then the constants

$$\gamma_{ij}^k(t) = \begin{cases} t\cdot\gamma_{ij}^k & \text{for } i,j,k \neq 1 \ , \\ t^2\cdot\gamma_{ij}^k & \text{for } i,j \neq 1 \ , \ k = 1 \ , \\ \gamma_{ij}^k & \text{otherwise} \end{cases}$$

define algebras $B_t \in alg_W$ for $t \in \mathbb{C}$ with $B_t \in C_B$ for $t \in \mathbb{C}^*$ and $B_0 \xrightarrow{\sim} A_0$. Hence $A_0 \leq B$ for any algebra $B \in alg_W$.)

3.8 It's an interesting but difficult problem to determine the number of irreducible components of alg_n and the "generic structures", i.e. those algebras which are not degenerations of other structures.

E.g. n = 3 :

$$\begin{tikzcd} & & \mathbb{C}\times\mathbb{C}\times\mathbb{C} \arrow[d, dash] \\ \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix} \arrow[ddr, dash] & & \mathbb{C}\times\mathbb{C}[t]/(t^2) \arrow[d, dash] \\ & & \mathbb{C}[t]/(t^3) \arrow[dl, dash] \\ & \mathbb{C}[s,t]/(s^2,st,t^2) & \end{tikzcd}$$

Here we have two components, one of dimension 9 (the closure of the orbit of $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$) and one of dimension 7 (the closure of the orbit of $\begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$) .

The known cases are those of dimension ≤ 5 : alg_4 has 5 irredu ible components (Gabriel [G]) and alg_5 has 10 irreducible components (Mazzola [M 1]).

Remark: One component of alg_n is always the closure of the orbit of $\mathbb{C}\times\ldots\times\mathbb{C}$. Mazzola has shown that this is the subset of commutative algebras for $n \leq 7$, but there is a 10-dimensional commutative algebra which is not a degeneration of $\mathbb{C}\times\ldots\times\mathbb{C}$ ([M 2]; cf. 6.7).

3.9 The following proposition explains the experimental fact that all degenerations of modules and algebras known up to now have been obtained with a "one parameter family".

Proposition: Each degeneration of modules or algebras can be obtained along the affine line $\mathbb{C}$.

(Since $\mathbb{K}(GL_n)$ is a rational function field any orbit C_M or C_A and its closure is a unirational variety. Hence we have to show that two points on a unirational variety X can be connected by a rational curve. Using Hironaka's result on resolution of singularities one reduces to the case where X is obtained from a projective space $\mathbb{P}^n$ by blowing up a number of times, from which the required result can be deduced.)

3.10 Example: Consider the set $algmod_{W,U}$ of pairs (A,M) where A is an algebra structure on the vectorspace W and M an A-module structure on U . It is easy to see that $algmod_{W,U}$ is in an obvious way a locally closed subset of $\mathrm{bil}_W \times \mathrm{Hom}(W,\mathrm{End}\,U)$. By a similar argument as in 1.7 one can show that $\mathrm{algmod}_{W,U}$ is in fact an affine variety.

The group $GL(W) \times GL(U)$ operates in a natural way on $\mathrm{algmod}_{W,U}$:

$$(g,h)(A,M) = ({}^gA, {}^{(g,h)}M) ,$$

where the ${}^gA$-module ${}^{(g,h)}M$ is obtained from the A-module M via the map

$${}^gA \xrightarrow{g^{-1}} A \longrightarrow \mathrm{End}\,M \xrightarrow{\mathrm{Int}\,h} \mathrm{End}\,{}^{(g,h)}M .$$

We see from the construction that the projection $(A,M) \mapsto A$ defines a morphism

$$\mu : \mathrm{algmod}_{W,U} \longrightarrow \mathrm{alg}_W$$

with the fibres

$$\mu^{-1}(A) = \mathrm{mod}_{A,U} .$$

Clearly μ is $GL(W)$-equivariant and the fibres are stable under $GL(U)$.

4. Linearly reductive groups and the Hilbert-Criterion

4.1 Definition: An algebraic group G is called linearly reductive if any linear representation of G is completely reducible.

Examples: A finite group is linearly reductive (Theorem of Maschke).

$\mathbb{C}^*$ is linearly reductive; in fact for any $\mathbb{C}^*$-module V we have the decomposition $V = \bigoplus_i V_i$, $V_i := \{v \in V \mid t(v) = t^i \cdot v \text{ for all } t \in \mathbb{C}^*\}$. (The subgroup $\mu_\infty \subset \mathbb{C}^*$ of elements of finite order is dense in $\mathbb{C}^*$. Furthermore any commutative subgroup $F \subset GL(V)$ consisting in elements of finite order is simultanously diagonalisable. This two facts imply the result.)

The same result holds for any group isomorphic to a product $\mathbb{C}^* \times \ldots \times \mathbb{C}^*$; such a group is called a torus.

GL_n is linearly reductive. (Consider the $\mathbb{C}$-compact subgroup $K := \{g \in GL_n \mid g \cdot \bar{g}^t = e\}$ of unitarian matrices. It is well known that $GL_n = K \cdot D \cdot K$ where D are the diagonal matrices in GL_n . Since $D \cap K$ are the diagonal matrices whose entries are roots of unity we have $D \subset \bar{K}$ and hence $\bar{K} = GL_n$, i.e. K is (Zariski-)dense in GL_n . Now let $\rho : GL_n \to GL(V)$ be a linear representation. The existence of a Haar measure on the $\mathbb{C}$-compact group K implies that the restriction $\rho|_K$ is completely reducible. Since K is dense in GL_n any K-stable decomposition of V is also GL_n-stable.)

4.2 Theorem: All classical groups, the tori and the finite groups are linearly reductive. Furthermore products and extensions of linearly reductive groups are linearly reductive.

(The first part follows like for GL_n from the fact that these groups contain $\mathbb{C}$-compact subgroups which are (Zariski-) dense, the so called "maximal compact subgroups".)

Remark: A connected algebraic group G is called semisimple if the maximal solvable normal subgroup is finite. (E.g. SL_n, SO_n, Sp_n) The classification of these groups is known. One shows that semisimple groups are linearly reductive and that a connected linearly reductive group is an extension of a semisimple group with a torus. Furthermore a semisimple group has no non-trivial character.

4.3 A group homomorphism $\lambda:\mathbb{C}^*\to G$ is called a one parameter subgroup of G (shortly : 1-PSG).

If Z is a G-variety and $z \in Z$, a 1-PSG λ defines a morphism

$$\varphi : \mathbb{C}^* \to Z \ , \ t \mapsto \lambda(t)z \ .$$

If φ can be extended in a $\mathbb{C}$-continuous way to a map $\tilde{\varphi}:\mathbb{C}\to Z$ then $\tilde{\varphi}$ is automatically a morphism. In this case we shortly write

$$\lim_{t\to 0} \lambda(t)z = z_0$$

where $z_0 := \tilde{\varphi}(0)$. Clearly we have $z_0 \in \overline{Gz}$.

In the example of the module variety $\mathrm{mod}_{A,U}$ we have the following interpretation of the 1-PSGs .

Proposition: Given two A-modules $M,N \in \mathrm{mod}_{A,U}$ there is a 1-PSG $\lambda : \mathbb{C}^* \to GL(U)$ with $\lim\limits_{t\to 0} {}^{\lambda(t)}M = N$ if and only if there is a filtration on M such that the associated graded module is isomorphic to N .

Proof: For any 1-PSG $\lambda : \mathbb{C}^* \to GL(U)$ we have the decomposition $U = \bigoplus_i U_i$ into eigenspaces $U_i = \{u \in U \mid \lambda(t)u = t^i u \text{ for all } t \in \mathbb{C}^*\}$, $i \in \mathbb{Z}$. Given $M \in \mathrm{mod}_{A,U}$ and the corresponding decomposition $M = \bigoplus_i M_i$ it is not hard to see that ${}^{\lambda(t)}M$ has a limit for $t \to 0$ if and only if the subspaces $M_{(j)} := \bigoplus_{i>j} M_i$ are A-submodules, and that $\lim\limits_{t\to 0} {}^{\lambda(t)}M$ is isomorphic to $\bigoplus_j M_{(j)}/M_{(j+1)}$. From this the proposition follows easily.

4.4 One-parameter subgroups can be used for the study of orbit closures. One of the main results in this direction goes back to Hilbert; it is a partial inverse of the fact mentioned above that $z_0 \in \overline{Gz}$ if $z_0 = \lim\limits_{t\to 0} \lambda(t)z$.

Hilbert-Criterion: Let V be a GL_n-module and $v \in V$ a "nullvector", i.e. $\overline{GL_n v} \ni 0$. Then there is a 1-PSG $\lambda : \mathbb{C}^* \to GL_n$ with $\lim\limits_{t\to 0} \lambda(t)v = 0$.

Proof: We only indicate the main steps of Hilbert's proof.

a) Consider the ring $\mathbb{C}[[t]]$ of power series and its quotient field $\mathbb{C}((t))$ of Laurent series. Then there is a matrix $g(t) \in GL_n(\mathbb{C}((t)))$ such that $(g(t)v)_{t=0} = 0$.

b) The theorem of elementary divisors implies that every matrix $g(t) \in GL_n(\mathbb{C}((t)))$ can be written in the form

$$g(t) = g_1(t) \cdot \lambda(t) \cdot g_2(t)$$

with $g_i(t) \in GL_n(\mathbb{C}[[t]])$ and $\lambda(t) = \begin{pmatrix} t^{r_1} & & \\ & \ddots & \\ & & t^{r_n} \end{pmatrix}$, $r_i \in \mathbb{Z}$ (i.e. λ is a 1-PSG!) From a) we get

$$0 = (g(t)v)_{t=0} = (\lambda(t)g_2(t)v)_{t=0} = (\lambda(t)g_2(0)v)_{t=0} \ .$$

Hence $\lim_{t\to 0}\lambda(t)v' = 0$ for $v' := g_2(0)v \in V$. Replacing λ by the conjugate $\lambda' = g_2(0)^{-1}\lambda g_2(0)$ we have the required result.

4.5 For many applications the following generalization of Hilbert's Criterion is very useful; a nice proof may be found in [Bi] .

Theorem (Hilbert-Mumford-Birkes): Let G be linearly reductive, Z a G-variety and Gz an orbit in Z . If $C \subset \overline{Gz}$ is a closed orbit then there is a 1-PSG $\lambda : \mathbb{C}^* \to G$ with $\lim_{t\to 0}\lambda(t)z \in C$.

In case of the module variety this result and proposition 4.3 imply the following. (For b) use the theorem of Jordan-Hölder.)

Proposition: Let M be an A-module of dimension n and $C_M \subset \mathrm{mod}_{A,n}$ the corresponding orbit.

a) C_M is closed if and only if M is semisimple.

b) Each orbit closure $\overline{C_M}$ contains exactly one closed orbit, namely $C_{\mathrm{gr}M}$ where $\mathrm{gr}\,M$ is the direct sum of the simple factors of M (i.e. the associated graded with respect to a composition series).

Corollary: Assume A finite dimensional.

a) The connected components of $\mathrm{mod}_{A,n}$ are in one-to-one correspondence with the semisimple A-modules of dimension n .

b) A is semisimple if and only if $\mathrm{mod}_{A,n}$ is a (disjoint) union of closed orbits for all n .

4.6 Remarks: 1) The second statement of the proposition above holds in a more general situation (cf. 5.3): The closure of an orbit under a linearly reductive group contains exactly one closed orbit.

2) One may ask whether every degeneration of A-modules can be obtained via a 1-PSG. More precisely, given $M \in \mathrm{mod}_{A,U}$ and a degeneration $N < M$, does there exist a 1-PSG $\lambda : \mathbb{C}^* \to GL(U)$ with $\lim_{t\to 0}{}^{\lambda(t)}M \in C_N$?

By proposition 4.3 this would imply that N must be decomposable. But we have seen in the first chapter an example of an indecomposable degeneration (II.2.9 Remark).

3) If G is a torus, Gz an orbit and $y \in \overline{Gz}$ there is always a 1-PSG λ with $\lim_{t\to 0}\lambda(t)z \in Gy$. In connection with the question above this implies the following: If the simple factors of $M \in \mathrm{mod}_{A,n}$ all have multiplicity one, any degeneration of M can be obtained via a 1-PSG and is in particular decomposable.

(For a proof one has to use Luna's slice theorem [Lu]).

4) Recently G. Kempf has developed the theory of optimal 1-PSGs [Ke] : Given a G-module V and a "nullvector" $v \in V$ he attaches to v the "best" 1-PSG λ with $\lim_{t\to 0}\lambda(t)v = 0$.

This has interesting applications to rationality questions and to the study of the geometry of orbit closures (cf. [H2]).

5. *Invariants and algebraic quotients*

In the first chapter we have seen in some examples how invariant functions may help to distinguish non equivalent objects and to attack the classification problem. In this section we want to formulate the relevant general results about invariants and there geometric interpretation.

5.1 If G is an algebraic group and Z a G-variety the subring

$$\mathcal{O}(Z)^G := \{f \in \mathcal{O}(Z) \mid f(gz) = f(z) \quad \text{for all } g \in G, z \in Z\}$$

of $\mathcal{O}(Z)$ is called the *ring of invariant functions* on Z with respect to G or shortly the *invariant ring*.

Theorem (Hilbert, Noether, Weyl, ...): *If G is linearly reductive and Z a G-variety then the invariant ring $\mathcal{O}(Z)^G$ is finitely generated.*

(For a short proof see [Mu] or [Kr].)

Remark: It was an open question for a long time whether such a finiteness result holds in general (Hilbert's fourteenth problem, 1900 International Congress, Paris). A counterexample was constructed only in 1959 by M. Nagata. It implies that for any non linearly reductive group G there is a G-variety Z such that $\mathcal{O}(Z)^G$ is not finitely generated. Clearly for special varieties the finiteness result may be true, as in the case of linear actions of the additive group $\mathbb{C}^+$ (Theorem of Weitzenböck, 1932).

5.2 We now use the result above to make the following important *construction*. Choose generators $f_1, \dots, f_r$ of $\mathcal{O}(Z)^G$ and consider

the morphism

$$\eta : Z \to \mathbb{C}^r \;, \quad z \mapsto (f_1(z), \dots, f_r(z)) \;.$$

Putting $Y := \overline{\eta(Z)} \subset \mathbb{C}^r$ we get the diagram

$$\begin{tikzcd} Z \arrow[r, "\eta"] \arrow[dr, "\pi"'] & \mathbb{C}^r \\ & Y \arrow[u, phantom, "\cup"] \end{tikzcd}$$

and an isomorphism $\pi^* : \mathcal{O}(Y) \xrightarrow{\sim} \mathcal{O}(Z)^G$.

Definition: A morphism $\pi : Z \to Y$ such that π^* induces an isomorphism $\mathcal{O}(Y) \xrightarrow{\sim} \mathcal{O}(Z)^G$ is called an *algebraic quotient of Z by G* or shortly a *quotient map*.

By definition an algebraic quotient is uniquely determined (up to isomorphism) by G-variety Z (use 1.3 and proposition 2.1); it will be denoted by $\pi_Z : Z \to Z\tilde{/}G$. (This notation refers to the fact that the quotient has something to do with the orbit space Z/G ; cf. below.)

5.3 In the following proposition we collect the three main properties of a quotient map $\pi : Z \to Y$ by a linearly reductive group G . (For a proof we refer to the literature cited above.)

Proposition: *Let G be linearly reductive, Z a G-variety and $\pi : Z \to Y$ an algebraic quotient.*

(U) π *is constant on orbits and universal with this property.*

(C) *If $X \subset Z$ is a closed G-stable subset then $\pi(X)$ is closed in Y and $\pi|_X : X \to \pi(X)$ is an algebraic quotient.*

(S) π *separates disjoint closed G-stable subsets of Z* .

Remark: The universal property (U) of an algebraic quotient shows that it is the best "algebraic approximation" to the orbit space. It is even the best continuous approximation, as shown by the following result.

Corollary: Each fibre $\pi^{-1}(y)$ contains exactly one closed orbit C and we have

$$\pi^{-1}(y) = \{z \in Z \mid \overline{Gz} \supset C\} .$$

As a consequence we see that the quotient $Z\tilde{/}G$ parametrizes the closed orbits in Z. In particular we have $Z\tilde{/}G = Z/G$ for a finite group G.

5.4 In case of a linear action of G on a vector space V we find

$$\pi^{-1}(\pi(0)) = \{v \in V \mid \overline{Gv} \ni 0\} .$$

This explains the notations nullvector (4.4) and nullfibre and shows in particular that the set of nullvectors is closed.

We have already seen that the Hilbert-Criterion is the right tool to determine the nullfibre (4.4). From the knowledge of the nullfibre one obtains informations on the invariant ring by the following result due to Hilbert (cf. [Kr]).

Proposition: Let $f_1,\dots,f_s$ be invariant functions defining the nullfibre, i.e. $f_1,\dots,f_s \in \mathcal{O}(V)^G$ with $\underline{V}(f_1,\dots,f_s) = V^0$. Then $\mathcal{O}(V)^G$ is a finite module over its subring $\mathbb{C}[f_1,\dots,f_s]$.

5.5 Examples: 1) The quotient $\mathrm{moss}_{A,U} := \mathrm{mod}_{A,U}\tilde{/}GL(U)$ describes the semisimple modules. If A is finite dimensional then $\mathrm{moss}_{A,U}$ is finite (cf. Proposition 4.5).

2) Let A be commutative and choose an affine variety Y with coordinate ring $\mathcal{O}(Y) \cong A/\sqrt{0}$. Then there is a bijection

$$Y^{(n)} \to \mathrm{moss}_{A,n}$$

where $Y^{(n)}$ is the symmetric product Y^n/σ_n (the symmetric group σ_n operates by permuting the factors).

In particular if Y is connected then $\mathrm{moss}_{A,n}$ and $\mathrm{mod}_{A,n}$ are both connected. Furthermore if $A/\sqrt{0}$ is not normal then $\mathrm{moss}_{A,n}$ and $\mathrm{mod}_{A,n}$ are both not normal (cf. proposition 5.6a).

3) The subset $S := \mathrm{mod}_{A,U}^{\mathrm{simple}}$ of simple modules is open in $\mathrm{mod}_{A,U}$, its image under $\pi : \mathrm{mod}_{A,U} \to \mathrm{moss}_{A,U}$ is open ~~and smooth~~ smooth and $\pi|_S : S \to \pi(S)$ is a fibration whose fibres are orbits, i.e. $\pi(S) = S/GL(U)$ (cf. [P1]; a more general result follows from Luna's slice theorem [Lu]).

5.6 For a quotient map $\pi : Z \to Y := Z\tilde{/}G$ by a reductive group G we have the following transition properties. (Recall that Z is called normal if $\mathcal{O}(Z)$ is normal, i.e. integrally closed in its quotient field.)

Proposition: a) If Z is connected, irreducible or normal then $Z\tilde{/}G$ has the same property.

b) If Z is factorial and if G has no non-trivial character then Y is also factorial.

Remark: Concerning the smoothness of the quotient G.Kempf has shown the following result conjectured by V.L. Popov: If G is semisimple and V a G-module with $\dim V\tilde{/}G = d \le 2$, then $V\tilde{/}G \cong \mathbb{C}^d$.

6. Semicontinuity results

It is sometimes important to know whether a certain naturally defined subset of an algebraic variety is open. A typical example is the set of points where the variety is normal (or smooth). Other examples occur in the case of our module variety $\mathrm{mod}_{A,U}$ or of alg_W, e.g. the subset of projective (or injective) modules in $\mathrm{mod}_{A,U}$, the subset of non-commutative algebras in alg_W or of semisimple algebras (cf. below). Many of these problems can be reduced to the following result.

6.1 Theorem (Chevalley): Let $\eta : Z \to Y$ be a morphism between affine varieties. Then the function $z \mapsto \dim_z \eta^{-1}(\eta(z))$ is upper semicontinuous.

(A function $d : Z \to \mathbb{N}$ is upper semicontinuous if for all n the set $\{z \in Z \mid d(z) < n\}$ is open in Z.)

Remark: In general it is not true that $y \mapsto \dim \eta^{-1}(y)$ is upper semicontinuous. Nevertheless this holds for quotient maps $\pi : Z \to Y$ as a consequence of the property (C) (proposition 5.3).

6.2 Example: The function $A \mapsto \dim(\mathrm{zent}\,A)$ on alg_W is upper semicontinuous. In particular the commutative algebras form a closed subset (cf. remark 3.8).

To see this take the closed subset $Z := \{(w,A) \mid w \in \mathrm{zent}\,A\}$ of $W \times \mathrm{alg}_W$ and consider the map $\eta : Z \to \mathrm{alg}_W$ induced by the projection and the section $\sigma : \mathrm{alg}_W \to Z$, $A \mapsto (0,A)$. Clearly $\eta^{-1}(A) = \mathrm{zent}\,A \times \{A\}$, hence

$$\dim \mathrm{zent}\,A = \dim \eta^{-1}(A) = \dim_{\sigma(A)} \eta^{-1}(\eta(\sigma(A))) ,$$

and the claim follows from Chevalley's theorem.

6.3 Proposition: Let Z be a G-variety. Then the function $z \mapsto \dim Gz$ is lower semicontinuous.

The proof is similar to that of the example above and follows from the diagram

$$\begin{tikzcd} G \times Z \supset \{(g,z) \mid gz = z\} \arrow[d, "pr"'] \arrow[d, shift left=4ex, "\eta"] \\ Z \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

where η is induced by the projection and σ is the section $z \mapsto (e,z)$.

This result leads to the concept of sheets (Dixmier, Borho-Kraft [BK]). Assume G connected.

Definition: A sheet in Z is a maximal irreducible subset consisting in orbits of a fixed dimension.

It follows that a sheet is locally closed and G-stable, hence we obtain a finite stratification of Z into locally closed G-stable subsets.

6.4 As an application let us prove the following.

Proposition: Let Z be a G-variety, G linearly reductive, and $\pi : Z \to Y = Z/\!\!/G$ the quotient map. Then the points $y \in Y$ where the fibre $\pi^{-1}(y)$ contains only finitely many orbits form an open set of Y.

Proof: Put $Y' := \{y \in Y | \pi^{-1}(y)/G \text{ finite}\}$ and consider the closed G-stable subsets

$$Z_k := \{z \in Z | \dim Gz \leq k\}$$

of Z and the quotient maps $\pi_k := \pi|_{Z_k} : Z_k \to \pi(Z_k)$. It follows that $\pi_k^{-1}(y)$ contains ∞-many orbits if $\dim \pi_k^{-1}(y) > k$. Define

$$Y_k := \{y \in \pi(Z_k) | \dim \pi_k^{-1}(y) > k\} .$$

This subset is closed in $\pi(Z_k)$ (cf. remark 6.1), hence closed in Y, and we have $Y_k \subset Y'$ for all k. On the other hand if $\pi^{-1}(y)$ contains ∞-many orbits of dimension k then $y \in Y_k$. This proves $Y' = \bigcup_k Y_k$, hence Y' is closed in Y.

6.5 Example: The projective resp. the injective modules $M \in \mathrm{mod}_{A,U}$ form an open set.

To see this fix a free resolution

$$\ldots \to A^{i_2} \to A^{i_1} \to A \to A/\mathrm{rad}\,A \to 0$$

of $A/\mathrm{rad}\,A$. From the canonical identification $\mathrm{Hom}_A(A^i, M) = U^i$ for all $A \in \mathrm{mod}_{A,U}$ we get a sequence

$$0 \to U \xrightarrow{\varphi_0} U^{i_1} \xrightarrow{\varphi_1} U^{i_2} \to \ldots$$

where the linear maps $\varphi_i = \varphi_i(M)$ depend regularly on $M \in \mathrm{mod}_{A,U}$. It follows that the function $M \mapsto \dim \mathrm{Ext}^i(A/\mathrm{rad}\,A, M)$ $= \dim \mathrm{Ker}\,\varphi_i - \dim \mathrm{Im}\,\varphi_{i-1}$ is upper semicontinuous. In particular the injective modules form an open set. For the projective modules we may use the isomorphism $\mathrm{mod}_{A,U} \xrightarrow{\sim} \mathrm{mod}_{A^{op},U^*}$ given by $M \mapsto M^* := \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ which identifies projective with injective modules.

6.6 Sometimes we may use another method based on the concept of parabolic subgroups.

Definition: A closed subgroup $P \subset G$ is called parabolic if G/P is $\mathbb{C}$-compact.

Example: A subgroup $P \subset GL_n$ is parabolic if and only if it contains a conjugate of the group of upper triangular matrices. More precisely the parabolic subgroups of $GL(V)$ are the stabilizers of flags in V. (It's easy to see that GL_n/P is $\mathbb{C}$-compact for any subgroup P containing the group B of upper triangular matrices, since $GL_n = K \cdot B$ with the $\mathbb{C}$-compact subgroups K of unitarian matrices.)

Proposition: Let Z be a G-variety and $Y \subset Z$ a closed subset. If Y is stable under some parabolic subgroup of G then $GY = \{gy | g \in G, y \in Y\}$ is closed in Z.

The proof is based on the following construction. Consider the free action of P on $G \times Z$ given by $p(g,z) := (gp^{-1}, pz)$. The orbit space $G \times^P Z$ is a fibre bundle over G/P and in fact the trivial bundle $G/P \times Z$, the trivialization being induced by the map $G \times Z \xrightarrow{\sim} G \times Z$, $(g,z) \mapsto (g, gz)$. Now $G \times Y$ is P-stable, hence defines a subbundle $G \times^P Y$ of $G \times^P Z \xrightarrow{\sim} G/P \times Z$. It follows that GY is the image of $G \times^P Y$ under the projection $G \times^P Z \xrightarrow{\sim} G/P \times Z \to Z$. Since G/P is $\mathbb{C}$-compact the subset GY is $\mathbb{C}$-closed. From this the claim can be deduced (cf. 7.5).

6.7 Example: The function $A \mapsto \dim(\mathrm{rad}\,A)$ on alg_W is upper semicontinuous. In particular the semisimple algebras $A \in \mathrm{alg}_W$ form an open set.

To see this fix a subspace $W' \subset W$ and consider the closed subset $Y := \{A \in alg_W | W' \subset \operatorname{rad} A\}$ of alg_W. Clearly the stabilizer of Y is a parabolic subgroup (namely the stabilizer of the flag $0 \subset W' \subset W$). It follows that

$$GL(W)Y = \{A \in alg_W | \dim(\operatorname{rad} A) \geq \dim W'\}$$

is closed in alg_W.

Remark: Using tangent space arguments one shows that $\{A \in alg_W | A$ semisimple$\}$ and more generally $\{A \in alg_W | \text{global} \dim A \leq 1\}$ are finite unions of open orbits.

A similar argument proves that the projective (resp. injective) modules $M \in mod_{A,V}$ are finite unions of open orbits (cf. [G]).

6.8 Example: An algebra A is called basic if $A/\operatorname{rad} A$ is commutative i.e. if $A/\operatorname{rad} A \xrightarrow{\sim} \mathbb{C} \times \ldots \times \mathbb{C}$. The subset of basic algebras in alg_W is closed.

Again fix subspaces $W_i \subset W$ of dimension $i = 0,1,\ldots,\dim W - 1$. Consider the subsets

$$Y_i := \{A \in alg_W | W_i \subset \operatorname{rad} A \text{ and } [A,A] \subset W_i\}$$

which are easily seen to be closed. Clearly any algebra in Y_i is basic. On the other hand a basic algebra A belongs to $GL(W)Y_i$ for $i = \dim(\operatorname{rad} A)$. As above Y_i is stable under a parabolic, hence the basic algebras form the closed subset $\bigcup_i GL(W)Y_i$.

7. Constructible subsets

7.1 In general a morphism $\mu : Z \to Y$ is neither open nor closed. But one can show that the image $\mu(Z)$ is a finite union of locally closed subsets of Y.

Definition: A finite union of locally closed subsets of a variety Y is called a constructible subset.

The constructible subsets of Y form a lattice, the lattice generated by the open and the closed subsets. The following examples will explain a little bit the term "constructible".

7.2 The main general result on the structure of images of morphisms is the following.

Proposition (Chevalley): If $\mu : Z \to Y$ is a morphism and $Z' \subset Z$ a constructible subset, then $\mu(Z')$ is also constructible.

For a proof we refer to the literature.

Remark: Clearly for more special morphisms we get more precise results. E.g. a finite morphism is always closed ($\mu : Z \to Y$ is finite, if the coordinate ring $\mathcal{O}(Z)$ is a finitely generated module over $\mathcal{O}(Y)$), or a flat morphism $\mu : Z \to Y$ is always open (μ is flat if $\mathcal{O}(Z)$ is a flat module over $\mathcal{O}(Y)$).

7.3 Example: Define the affine variety $algmod_{W,U}$ to be the set of pairs (A,M), where A is an algebra with underlying vectorspace W and M an A-module with underlying vectorspace U (cf. 3.10). We claim that the pairs (A,P) where P is a projective

A-module form a constructible subset.

To see this consider the variety

$$\mathrm{algmod}_{W,U} \times \mathrm{Hom}(W^n,U) \times \mathrm{Hom}(U,W^n)$$

and the closed subset Z of 4-tupels (A,M,σ,τ), where (a) $\sigma\circ\tau = \mathrm{Id}_U$ and (b) $\sigma : A^n \to M$ and $\tau : M \to A^n$ are A-linear; i.e. M is via τ a direct summand of A^n. Choosing n big enough (e.g. $n = \dim U$) the image of Z in $\mathrm{algmod}_{W,U}$ under the projection $(A,M,\sigma,\tau) \mapsto (A,M)$ is the required subset.

7.4 Example: The subset of algebras $A \in \mathrm{alg}_W$ of global homological dimension $\leq s$ is constructible for any s.

Define $\mathrm{algmod}_{W,U_1,U_2,\ldots,U_s}$ as above to be the set of tupels $(A,M_1,\ldots,M_s)$, A an algebra on W and M_i an A-module on U_i. Now consider the variety

$$\mathrm{algmod}_{W,U_1,\ldots,U_s} \times \mathrm{Hom}(U_s,U_{s-1}) \times \ldots \times \mathrm{Hom}(U_1,W)$$

and the subset Z of tupels $(A,P_1,\ldots,P_s,\mu_s,\ldots,\mu_1)$ defined by the following conditions:

(a) $P_1,\ldots,P_s$ are projective A-modules,

(b) the maps $\mu_i : P_i \to P_{i-1}$ and $\mu_1 : P_1 \to A$ are A-linear,

(c) the sequence $0 \to P_s \xrightarrow{\mu_s} P_{s-1} \to \ldots \to P_1 \xrightarrow{\mu_1} A$ is exact,

(d) $\mu_1(P_1) = \mathrm{rad}\, A$.

It is easy to see that (b) and (c) define closed subsets, (d) a locally closed subset and (a) a constructible subset (7.3), which implies that Z is constructible. Projecting onto alg_W we obtain a constructible subset of algebras of global homological

dimension $\leq s$. The claim follows now by varying the dimensions of the vectorspaces $U_1,\ldots,U_s$ in a suitable finite range.

7.5 It easily follows from the definition that any constructible subset $Y \subset Z$ contains a set U which is open and dense in $\bar{Y}$. A consequence of this "thickness" property of constructible sets is the following result which is useful in comparing the Zariski- and the $\mathbb{C}$-topology.

Lemma: If a constructible set $Y \subset Z$ is $\mathbb{C}$-closed then it is also closed in the Zariski-topology.

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Chapter III ALGEBRAS OF FINITE REPRESENTATION TYPE

A finite dimensional algebra A is called of finite representation type if there is only a finite number of equivalence classes of indecomposable finite dimensional representations. It is important to know whether this condition is "open" in the following sense: Given a family (A_t) of algebras, is it true that in a neighbourhood of an algebra A_{t_0} of finite representation type all algebras are of finite representation type? P. Gabriel has given a positive answer to this question.

Theorem (Gabriel [G]): *The algebras $A \in \mathrm{alg}_n$ of finite representation type form an open set.*

One could express this in a slightly different way: There exist polynomials $p_1,\ldots,p_s$ (in n^3 variables) such that an n-dimensional algebra A is of finite representation type if and only if some of the p_i's do not vanish on the structure constants of A.

In this last chapter we are going to present a proof of Gabriel's theorem, following his original ideas with some small modifications.

1. Auslander's construction

1.1 Definition: A finite dimensional algebra Γ is an Auslander algebra if (a) Γ is basic, (b) Γ is of global homological dimension ≤ 2 and (c) there is an exact sequence $0 \to \Gamma \to I_0 \to I_1$ of Γ-modules with I_i injective and projective.

We use the results of II.6 and II.7 to prove the following proposition.

Proposition: The Auslander algebras $\Gamma \in \mathrm{alg}_m$ form a constructible subset.

Proof: As in II.7 we denote by $\mathrm{algmod}_{U,V_0,V_1}$ the affine variety of triples (Γ, I_0, I_1), Γ an algebra structure on U, I_0 and I_1 Γ-module structures on V_0 and V_1. Now consider the product

$$Z := \mathrm{algmod}_{U,V_0,V_1} \times \mathrm{Hom}(U_1,V_0) \times \mathrm{Hom}(V_0,V_1)$$

and the subset Y of 5-tupels $(\Gamma, I_0, I_1, \alpha_0, \alpha_1)$ satisfying the following conditions:

(i) Γ is basic,

(ii) Γ has global homological dimension ≤ 2,

(iii) I_0 and I_1 are injective and projective Γ-modules,

(iv) $\alpha_0 : \Gamma \to I_0$ and $\alpha_1 : I_0 \to I_1$ are Γ-linear,

(v) the sequence $0 \to \Gamma \to I_0 \to I_1$ is exact.

We have already seen that (i) defines a closed subset (II.6.8), (ii) and (iii) both constructible subsets (II. 7.4 and II. 7.3) of Z. It is easy to see that the last two conditions determine a locally closed subset (cf. II. 7.4). Alltogether shows that Y is

constructible. By definition its image in alg_U consists in Auslander algebras. Varying the dimensions of V_0 and V_1 in a big enough but finite range we see that the Auslander algebras in alg_U form a constructible subset.

1.2 Now Auslander has shown the following [A] : Given an Auslander algebra Γ and a projective and injective Γ-module M then the algebra $\mathrm{End}_\Gamma(M)$ is of finite representation type, and every algebra of finite representation type occurs in this way.

We use this construction to show the following result which is a first step in the proof of the main theorem. Let us denote by $\mathrm{alg}_n^{\mathrm{fin}}$ the subset of algebras $A \in \mathrm{alg}_n$ of finite representation type.

Proposition: $\mathrm{alg}_n^{\mathrm{fin}}$ is a countable union of constructible subsets of alg_n.

Proof: Consider the variety $\mathrm{bimod}_{U,V,W}$ of triples (Γ, M, A), where Γ and A are $\mathbb{C}$-algebras with underlying vectorspaces U and W and M is a Γ-A-bimodule structure on the vectorspace V. Let Y be the subset of triples (Γ, M, A) satisfying the following 3 conditions:

(i) Γ is an Auslander algebra,

(ii) M is projective and injective as a Γ-module,

(iii) The canonical map $A \to \mathrm{End}_\Gamma(M)$ is bijective.

Using the projections

$$\begin{tikzcd} \mathrm{bimod}_{U,V,W} \arrow[r] \arrow[dr] & \mathrm{algmod}_{U,V} \arrow[d] \\ & \mathrm{alg}_U \end{tikzcd}$$

it follows from proposition 1.1 and example II.7.3 that the first two conditions define a constructible subset of $\mathrm{bimod}_{U,V,W}$. For the last condition we first remark that the function $(\Gamma,M,A) \mapsto \dim \mathrm{End}_\Gamma(M)$ is upper semicontinuous (since $\dim \mathrm{End}_\Gamma(M) = \dim \mathrm{Aut}_\Gamma(M)$ and $\mathrm{Aut}_\Gamma(M) = \mathrm{Stab}_{GL(V)}(\Gamma,M)$, cf. II.6.3). Furthermore $(\Gamma,M,A) \mapsto \dim \mathrm{Ann}_A M$ is also upper semicontinuous (cf. example II. 6.2).

As a consequence condition (iii) defines a locally closed subset of $\mathrm{bimod}_{U,V,W}$. Via the projection $\mathrm{bimod}_{U,V,W} \to \mathrm{alg}_W$ we get a constructible subset of alg_W and the result follows from Auslander's construction by varying the dimensions of U and V.

2. A first openness result

2.1 We have already considered the variety $\mathrm{algmod}_{W,U}$ of pairs (A,M), A an algebra structure on W and M an A-module structure on U (cf. II. 3.10). The group $GL(W) \times GL(U)$ operates on $\mathrm{algmod}_{W,U}$ in a natural way. Furthermore the fibres of the canonical map $\mu : \mathrm{algmod}_{W,U} \to \mathrm{alg}_W$ are of the form $\mu^{-1}(A) \cong \mathrm{mod}_{A,U}$, hence stable under $GL(U)$. Therefore we obtain a commutative diagram $(n := \dim U)$

$$\begin{tikzcd} \mathrm{algmod}_{W,U} \arrow[r, "\pi"] \arrow[dr, "\mu"'] & \mathrm{algmoss}_{W,n} := \mathrm{algmod}_{W,U} \tilde{/} GL(U) \arrow[d, "\bar{\mu}"] \\ & \mathrm{alg}_W \end{tikzcd}$$

where the quotient $\mathrm{algmoss}_{W,n}$ can be seen as the variety of pairs (A,η), $A \in \mathrm{alg}_W$ and η an isomorphism class of a semisimple module of dimension n:

$$\bar{\mu}^{-1}(A) \cong \mathrm{mod}_{A,U} \tilde{/} GL(U) = \mathrm{moss}_{A,n}$$

(II. 5.5 example 1). In particular the fibres of $\bar{\mu}$ are <u>finite</u>.

2.2 <u>Proposition: The map $\bar{\mu} : \mathrm{algmoss}_{W,n} \to \mathrm{alg}_W$ is closed.</u>

<u>Proof:</u> We consider the variety $\mathrm{alg}_W \times \mathrm{Sur}(W^n,U)$ of pairs (A,α) of algebras $A \in \mathrm{alg}_W$ and surjective linear maps $\alpha : W^n \to U$, $n := \dim U$, and the closed subset Z of those pairs (A,α), where $\mathrm{Ker}\,\alpha \subset A^n$ is an A-submodule. We have an obvious surjective map $\psi : Z \to \mathrm{algmod}_{W,U}$, $(A,\alpha) \mapsto (A, \mathrm{Im}\,\alpha)$, which is algebraic (i.e. the pullback $\psi^* f = f \circ \psi$ of any regular function $f \in \mathcal{O}(\mathrm{algmod}_{W,U})$ is locally a quotient of regular functions on the affine variety

$\bar{Z} \subset \mathrm{alg}_W \times \mathrm{Hom}(W^n,U)$). Now the action of $GL(U)$ on $\mathrm{alg}_W \times \mathrm{Sur}(W^n,U)$ is free and the orbit space $Z/GL(U)$ is a closed subset of $\mathrm{alg}_W \times Gr^n(W^n)$ where $Gr^n(W^n) := \mathrm{Sur}(W^n,U)/GL(U)$ is the Grassmann variety of subspaces of codimension n of W^n. Using the fact that the grassmannian is $\mathbb{C}$-compact one can show that the map

$$\Phi : Z/GL(U) \to \mathrm{alg}_W$$

induced by the projection $\mathrm{pr} : \mathrm{alg}_W \times Gr^n(W^n) \to \mathrm{alg}_W$ is closed (with respect to both the Zariski- and the $\mathbb{C}$-topology, cf. II. 7.5). Furthermore the map $\Psi : Z \to \mathrm{algmod}_{W,U}$ clearly induces a map $\bar{\Psi} : Z/GL(U) \to \mathrm{algmoss}_{W,n}$:

$$\begin{tikzcd}
\mathrm{alg}_W \times \mathrm{Sur}(W^n,U) \supset Z \arrow[r, "\Psi"] \arrow[d] & \mathrm{algmod}_{W,U} \arrow[d, "\pi"] \\
\mathrm{alg}_W \times Gr^n(W^n) \supset Z/GL(U) \arrow[r, "\bar{\Psi}"] \arrow[d, "\Phi"] \arrow[d, bend right, "\mathrm{pr}"'] & \mathrm{algmoss}_{W,n} \arrow[dl, "\bar{\mu}"] \\
\mathrm{alg}_W &
\end{tikzcd}$$

In particular Φ decomposes in the form $\Phi = \bar{\mu}\cdot\bar{\Psi}$, hence $\bar{\mu}$ is a closed map too.

Remark: Along the same type of arguments one can show more generally that $\bar{\mu}$ is a finite morphism, i.e. the algebra $\mathcal{O}(\mathrm{algmoss}_{W,n})$ is a finitely generated module over $\mathcal{O}(\mathrm{alg}_W)$. (It's well known that finite morphisms are closed, cf. II. Remark 7.2.)

2.3 For any finite dimensional algebra A we denote by $\nu_A(n)$ the number of isomorphism classes of A-modules of dimension n .

Proposition: For a fixed $n \in \mathbb{N}$ the set $\{A \in \mathrm{alg}_W \mid \nu_A(n) < \infty\}$ is open in alg_W .

Proof: This is a consequence of the result above and proposition 6.4 of the second chapter. In fact we have the factorization (2.1)

$$\begin{tikzcd}
\mathrm{algmod}_{W,U} \arrow[d, "\mu"'] \arrow[r, "\pi"] & \mathrm{algmoss}_{W,n} = \mathrm{algmod}_{W,U}\tilde{/}GL(U) \arrow[dl, "\bar{\mu}"] \\
\mathrm{alg}_W &
\end{tikzcd}$$

with a quotient map π and a closed map $\bar{\mu}$ (proposition 2.2) , $n := \dim U$. Clearly $\nu_A(n) < \infty$ means that the fibre $\mu^{-1}(A)$ contains only finitely many orbits (since $\mu^{-1}(A) \cong \mathrm{mod}_{A,U}$) . We have seen in chapter II. 6.4 that the set

$$Y := \{y \in \mathrm{algmoss}_{W,n} \mid \#\,\text{orbits in } \pi^{-1}(y) < \infty\}$$

is open in $\mathrm{algmoss}_{W,n}$. By definition we have

$$\{A \in \mathrm{alg}_W \mid \nu_A(n) = \infty\} = \bar{\mu}(\mathrm{algmoss}_{W,n} - Y)$$

and the claim follows from proposition 2.2.

3. Proof of the main theorem

3.1 Let us denote by S_r the set of all $A \in alg_W$ having only a finite number of isomorphism classes of modules of dimension $\leq r$. The famous Brauer-Thrall conjecture proved by L.A. Nazarova and A.V. Roiter [NR] states that

$$alg_W^{fin} = \bigcap_{r=1}^{\infty} S_r \quad .$$

In other words if an algebra is not of finite representation type then for some dimension there are infinitely many non isomorphic modules.

3.2 Now we are ready to prove Gabriel's theorem.

Theorem: The algebras $A \in alg_n$ of finite representation type form an open set alg_n^{fin} of alg_n. More precisely there is a natural number d depending only on n such that

$$alg_n^{fin} = S_d = \{A \in alg_n \mid \nu_A(r) < \infty \text{ for } r \leq d\}$$

Proof: We have seen above that

$$alg_n^{fin} = \bigcap_{r=1}^{\infty} S_r$$

On the other hand proposition 1.2 shows that

$$alg_n^{fin} = \bigcup_{t=1}^{\infty} C_t$$

with a suitable increasing sequence $C_1 \subset C_2 \subset C_3 \subset \ldots$ of constructible subsets of alg_W. Now the claim follows from 3.3.

3.3 Lemma: Let Z be a variety, $C_1 \subset C_2 \subset \ldots$ an increasing sequence of constructible subsets and $S_1 \supset S_2 \supset \ldots$ a decreasing

sequence of open subsets with

$$\bigcup_{i=1}^{\infty} C_i = \bigcap_{j=1}^{\infty} S_j$$

Then we have $S_r = C_t$ for some r,t.

Proof: For any closed subset $Z' \subset Z$ we get a similar statement replacing C_i by $C_i \cap Z'$ and S_j by $S_j \cap Z'$. It follows that we can assume Z irreducible and furthermore that the statement is true for all proper closed subsets of Z. By assumption we have for any $j \geq 0$

$$Z = \bigcup_{i \geq 0} \overline{C_i} \cup (Z - S_j)$$

But an irreducible variety Z cannot be a countable union of proper closed subsets. (This is clear for $\dim Z = 1$ and follows in general by induction, since every hypersurface must be contained in one of these subsets.) As a consequence $Z = \overline{C_s}$ for some s (the case $S_j = \emptyset$ being trivial). Now C_s contains an open dense subset U of Z and by induction the claim holds for $Z' := Z - U$. This means that $(Z-U) \cap S_r = (Z-U) \cap C_t$ for big enough r,t, hence $S_r = C_t$ since both contain U.

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